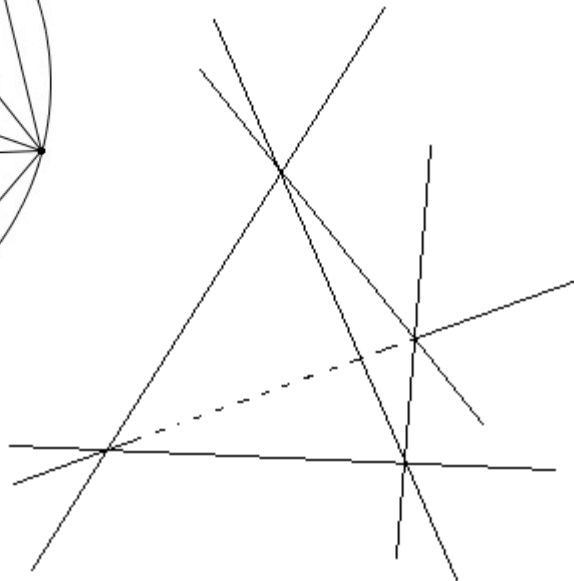
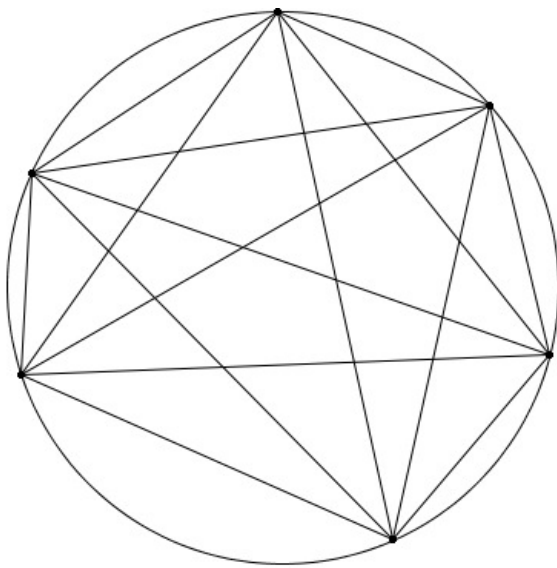


Once Upon A Circle 2014

... , 1, 2, 4, 8, 16, ...



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Once Upon A Circle

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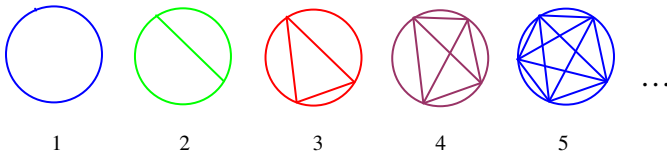
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There once was a sequence...

Guess the next number: 1, 2, 4, 8, 16, ?

Circles...

Not so fast! In order to make an intelligent guess, it helps to understand just how these first five numbers are generated. Here's one way:



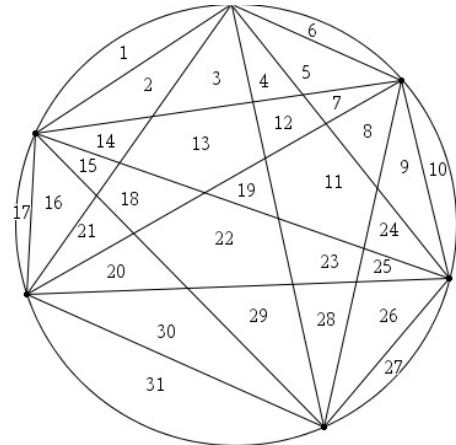
Going from one circle to the next above, we add another distinct point on the circle and connect it with all the previous points, forming the *maximum* number of regions inside the circle. So, when there are 4 points on the circle, there are 8 regions formed inside the circle (see circle #4).

The pattern thus generated is (*points, regions*): (1, 1), (2, 2), (3, 4), (4, 8), (5, 16), (6, ___). We'll call this sequence **S4** for reasons that will become apparent later.

Funny thing is, no matter how you place the *sixth* point on the circle, the most regions you can count is... 31. Hmmm. What's the pattern?

One method for determining the pattern of a sequence is to look at the differences between consecutive terms and see if there's a pattern. Let's examine the differences between consecutive terms of **S4**:

$$\begin{aligned}(2 - 1) &= 1 \\(4 - 2) &= 2 \\(8 - 4) &= 4 \\(16 - 8) &= 8 \\(31 - 16) &= 15\end{aligned}$$



We'll call this new sequence **S3**: {1, 2, 4, 8, 15}

Find the differences between consecutive terms of **S3** and get **S2**: {1, 2, 4, 7}

The differences in the sequence **S2** gives the sequence **S1**: {1, 2, 3}, which is the first 3 terms of an arithmetic sequence (the Natural numbers, in fact) with a common difference of 1 (**S0**).

A Parallelogram?

The following table builds these sequences from *left to right* as in *Pascal's Triangle*: each cell from row 2 on contains the sum of the number in the cell above it and the number to the left of the number above it (see the shaded $3+4=7$ portion of the table).

<i>N</i>	S0	S1	S2	S3	S4				
1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2
3	1	3	4	4	4	4	4	4	4
4	1	4	7	8	8	8	8	8	8
5	1	5	11	15	16	16	16	16	16
6	1	6	16	26	31	32	32	32	32
7	1	7	22	42	57	63	64	64	64

Our sequence **S4** is in the fifth column. So far, so good, but where's the technology? Well, here it comes...

S1 is the natural numbers, a linear sequence.

S2 is (1 + sum of the numbers in **S1**), denoted by $\mathbf{S2} = 1 + \sum_{i=1}^j i = 1 + \frac{j(j+1)}{2}$, a quadratic.

S3 is (1 + sum of the numbers in **S2**), so $\mathbf{S3} = 1 + \sum_{i=1}^n \mathbf{S2}(i) = 1 + \sum_{j=1}^n (1 + \sum_{i=1}^j i)$

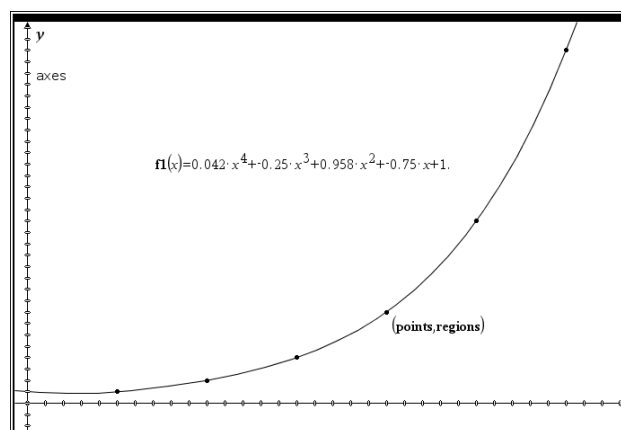
S4 is (1 + sum of the numbers in **S3**), so $\mathbf{S4} = 1 + \sum_{i=1}^n \mathbf{S3}(i) = 1 + \sum_{k=1}^n (1 + \sum_{j=1}^k (1 + \sum_{i=1}^j i))$

This is why I quit trying to figure out the function. Who cares to figure out *that* monster?! (see page 7 for some help from a CAS)

Regression

It occurred to me to try the **TI** graphing calculator's regression capabilities to try to find a function to fit **S4**. In L_1 enter the list $\{1, 2, 3, 4, 5, 6\}$. L_2 is the list $\{1, 2, 4, 8, 16, 31\}$. What kind of function best models this data? Well, since **S0** is a constant sequence and **S1** is a linear sequence and **S2** is a quadratic sequence, I figured that **S3** must be a cubic sequence and **S4** must be a quartic sequence (hence, their names!). So execute **QuartReg** L_1, L_2, Y_1 . Setup the **Table** beginning at 1 with an increment of 1. Lo and behold, our sequence (see figure *function* below)! The reasoning for the linear-quadratic-cubic-quartic pattern also follows from your experience with 'rates of change' in calculus. **S3** is the 'change' in **S4**, so if **S3** is cubic in nature, then **S4** is quartic.

points	regions					
					=QuartReg	
1	1	1	Title	Quartic R.		
2	2	2	RegEqn	a*x^4+b*...		
3	3	4	a	0.041667		
4	4	8	b	-0.25		
5	5	16	c	0.958333		
6	6	31	d	-0.75		
7			e	1.		
8			R ²	1.		
9			Resid	{-2.66E-1...		
10						
11						
E7="Quartic Regression"						



The values for a, b, c, d , and e look like rational numbers. It's easy to see that $b = -1/4, d = -3/4$, and $e = 1$. It turns out (after providing more decimal places at the end and using **[MATH]** **►Frac**) that $a = 1/24$ and $c = 23/24$, so our function can be written:

$$y = \frac{1}{24}x^4 - \frac{1}{4}x^3 + \frac{23}{24}x^2 - \frac{3}{4}x + 1$$

StatPlot of (L_1, L_2) with Y_1 in ZoomStat

Pascal's Triangle

We can find these sequences in Pascal's Triangle if we 'chop off' the right side of the triangle at the appropriate position and then look at the sums of the rows:

1					1
1	1				2
1	2	1			4
1	3	3	1		8
1	4	6	4	1	16
1	5	10	10	5	31
1	6	15	20	15	57
1	7	21	35	35	99
1	8	28	56	70	163
1	9	36	84	126	256
1	10	45	120	210	386
1	11	55	165	330	562
1	12	66	220	495	794

This discovery will lead us to other interesting representations of this sequence later.

Hyperspace

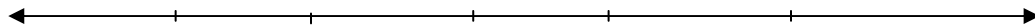
Here's our table again:

S0	S1	S2	S3	S4				
1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2
1	3	4	4	4	4	4	4	4
1	4	7	8	8	8	8	8	8
1	5	11	15	16	16	16	16	16
1	6	16	26	31	32	32	32	32
1	7	22	42	57	63	64	64	64

Another *geometric* pattern hides in this table:

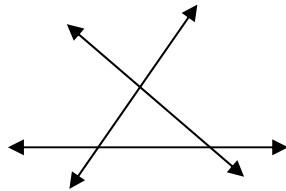
S0 represents the 'dividing' of a point. It cannot be divided so all the values are 1.

S1 is the number of regions into which points on a line divide the line:



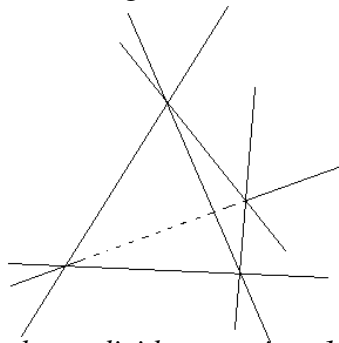
(5 points divide a line into 6 regions)

S2 is the number of regions into which lines on a plane divide the plane:



(3 lines divide a plane into 7 regions)

S3 is the number of regions into which planes divide space (remember Polya?):



(4 planes divide space into 15 regions)

So, is **S4** the number of regions that 3-space 'things' divide 4-space? (and so on?).

Can you explain why the "**circle's** regions pattern" is a *model* for the dividing of **hyperspace** by 3-space 'things'?

Computer Algebra Systems

And now for a little help from the **TI Nspire CAS**...

- Enter **1**
- then repeatedly compute

$$\sum_{x=1}^{n-1} (\text{ans}|n=x) + 1$$

The TI Nspire CAS interface shows the iterative computation of the sum of the first $n-1$ terms of the sequence defined by the recursive formula. The interface shows four steps of the calculation, with the final result being a polynomial in n .

$$\sum_{x=1}^{n-1} (1|n=x) + 1 = 1$$

$$\sum_{x=1}^{n-1} (n|n=x) + 1 = \frac{n^2}{2} - \frac{n}{2} + 1$$

$$\sum_{x=1}^{n-1} \left(\frac{n^2}{2} - \frac{n}{2} + 1|n=x \right) + 1 = \frac{n^3}{6} - \frac{n^2}{2} + \frac{4n}{3}$$

$$\sum_{x=1}^{n-1} \left(\frac{n^3}{6} - \frac{n^2}{2} + \frac{4n}{3}|n=x \right) + 1 = \frac{n^4}{24} - \frac{n^3}{4} + \frac{23n^2}{24} - \frac{3n}{4} + 1$$

“ $|n = x$ ”, reads “with $n=x$ ” and replaces the n ’s in the previous answer *ans* with x ’s and gives a polynomial in n . A ‘recursive’ program.

- *Try this:* paste each polynomial into a *sequence* function and look at the table of values to confirm that these are indeed the correct polynomials!

I was expecting that these polynomials would somehow converge to the Maclaurin Series for 2^n , but the alternating signs make it pretty clear that this will not happen.

Newton's Difference Theorem

Let's apply another *idea* (see Cuoco and Goldenberg, **Delving Deeper; Match Making: Fitting Polynomials to Tables**, *The Mathematics Teacher*, v 96, No. 3 March 2003, p 180)

Newton's Difference Theorem: Suppose we have a table whose inputs are the integers 0..m. A polynomial function that agrees with the table is:

$$f(x) = \sum_{k=0}^m a_k \binom{x}{k}$$

where

$$\binom{x}{k} = \frac{x(x-1)(x-2)(x-3)\cdots(x-k+1)}{k!}$$

The Table:

in	out	Δ_1	Δ_2	Δ_3	Δ_4
0	1	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	1
3	4	4	4	3	
4	8	8	7		
5	16	15			
6	31				

but we want to shift the function down 1 unit so x becomes $(x-1)$ and the a_k are all 1.

Then, according to the theorem,

$$f(x) = \binom{x-1}{0} + \binom{x-1}{1} + \binom{x-1}{2} + \binom{x-1}{3} + \binom{x-1}{4}$$

or

$$f(x) = \binom{x}{0} + \binom{x}{2} + \binom{x}{4} \quad \text{(see Pascal's Triangle again)}$$

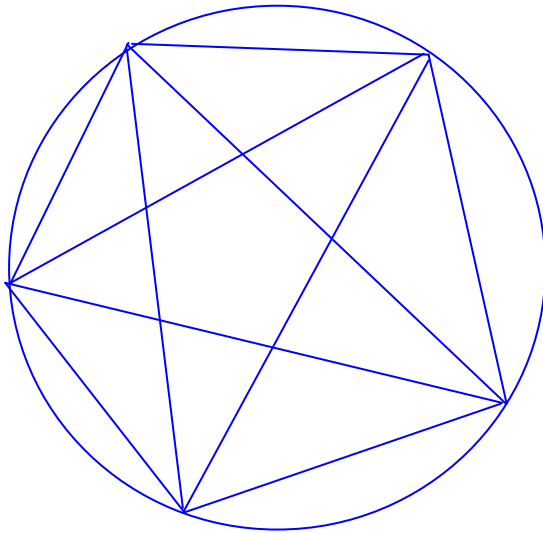
Both of which the **Ti CAS** evaluates as

$$f(x) = \frac{1}{24}x^4 - \frac{1}{4}x^3 + \frac{23}{24}x^2 - \frac{3}{4}x + 1$$

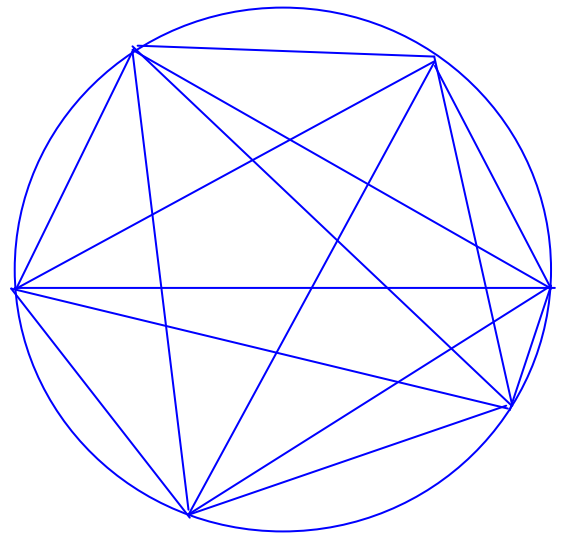
Making connections

$f(x) = \binom{x}{0} + \binom{x}{2} + \binom{x}{4}$ has a geometric interpretation as well:

- The first term is always 1, representing the circle: the *circle* contributes one region.
- The second term represents the number of chords: each *chord* contributes a region.
- The third term is the number of points of intersections of the chords inside the circle: each *intersection point* contributes a region.



$1 \text{ circle} + 10 \text{ chords} + 5 \text{ intersections} = 16 \text{ regions}$
 $1 + \text{nCr}(5,2) + \text{nCr}(5,4)$



$1 \text{ circle} + 15 \text{ chords} + 15 \text{ intersections} = 31 \text{ regions}$
 $1 + \text{nCr}(6,2) + \text{nCr}(6,4)$