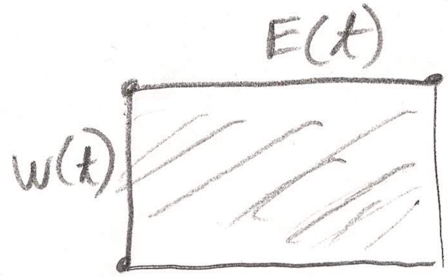
# Product Rule

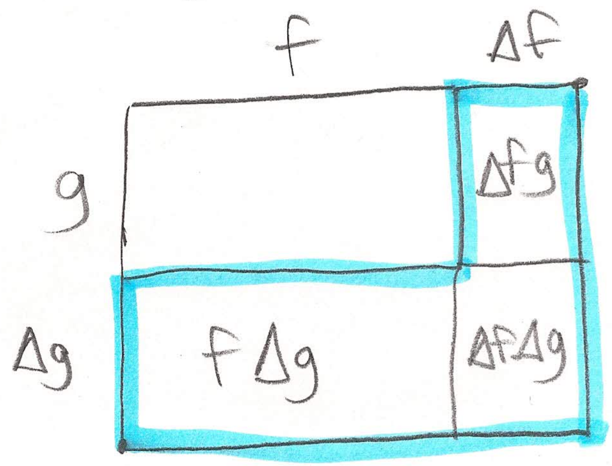
$W(t)$ = # of workers at a time $t$
$E(t)$ = toys produced per worker per hour at time $t$

$WE(t)$ = toys produced per hour

$\frac{d}{dt}\left[WE(t)\right]$ = how toy production changes over time

Relevant: $W'(t)$, $E'(t)$, $E(t)$, $W(t)$

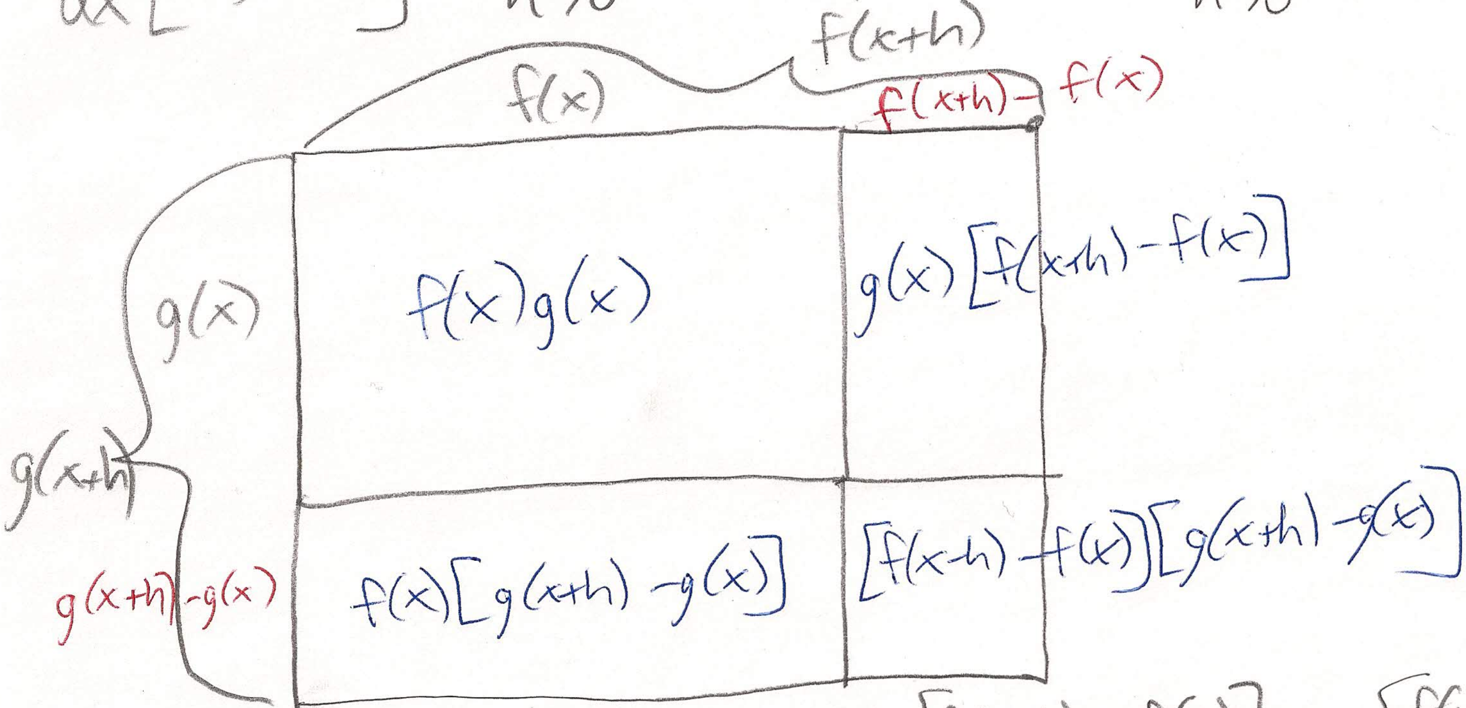




$$\boxed{\Delta(fg) = \Delta f \cdot g + f\Delta g} + \Delta f \Delta g$$

negligible

# Product Rule

$$\frac{d}{dx}\left[(fg)(x)\right] = \lim_{h\to0}\frac{fg(x+h)-fg(x)}{h} = \lim_{h\to0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$




$$= \lim_{h\to0}\frac{f(x)[g(x+h)-g(x)] + g(x)[f(x+h)-f(x)] + [f(x+h)-f(x)][g(x+h)-g(x)]}{h}$$

$$f(x)\cdot\lim_{h\to0}\frac{g(x+h)-g(x)}{h} + g(x)\lim_{h\to0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to0}\frac{f(x+h)-f(x)}{h}\cdot\lim_{h\to0}[g(x+h)-g(x)]$$

$$f(x)\cdot g'(x) + g(x)f'(x) + \cancel{f'(x)\cdot 0}$$

## Quotiend Rule

$$\frac{d}{dx}\left[\frac{f}{g}(x)\right] = \frac{d}{dx}\left[f(x)\cdot\frac{1}{g(x)}\right]$$

$$f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\frac{-g'(x)}{[g(x)]^2}$$

$$\boxed{\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$

## Reciprocal Rule

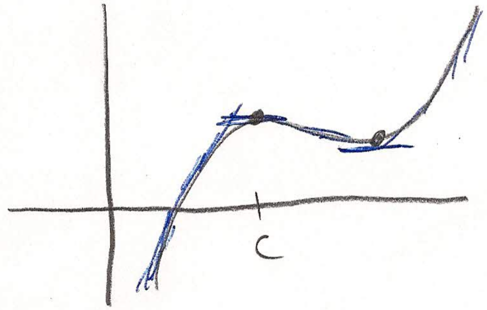
$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = \lim_{h\to 0}\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{g(x) - g(x+h)}{g(x)\,g(x+h)\,h}$$

$$= -\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\cdot\lim_{h\to 0}\frac{1}{g(x+h)g(x)}$$

$$-g'(x)\cdot\frac{1}{(g(x))^2}$$

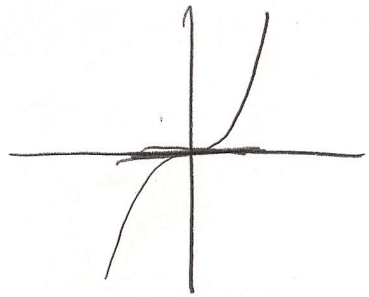
$$\boxed{\frac{d}{dx}\left[\frac{1}{g(x)}\right] = \frac{-g'(x)}{[g(x)]^2}}$$




$f'(c)=0 \not\Rightarrow f(c)$ is a rel extremum

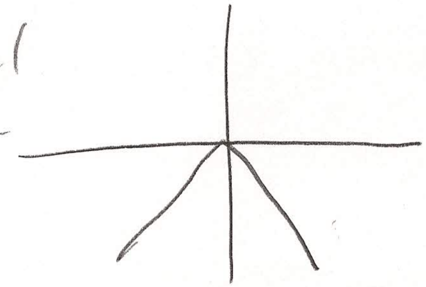
$f'(c)=0$

$f(c)$ is not a rel max or min



f does have a rel max @ $x=c$

$f'(c) \neq 0$



## Fermat's Theorem

If $f$ has a rel max @ $x=c$ & $f'(c)$ exists, then $f'(c)=0$

**Proof:** Given: $f$ has a rel max @ $x=c \implies f(c) \geq f(x)$ near $c$

$f'(c)$ exists $\implies \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists

Prove: $f'(c)=0 \implies \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 0$

$$\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c^-} \frac{(\text{something} \leq 0)}{(\text{something} < 0)} = \lim_{x\to c^-} (\text{something} \geq 0) \geq 0$$

$$\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c^+} \frac{(\text{something} \leq 0)}{(\text{something} > 0)} = \lim_{x\to c^+} (\text{something} \leq 0) \leq 0$$

so $f'(c)=0$

Thm: If $f'(x) = 0$ on $(a,b)$, then $f$ is constant on $(a,b)$

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Given: $f'(x) = 0$ on $(a,b)$

Prove: $f$ is constant on $(a,b)$

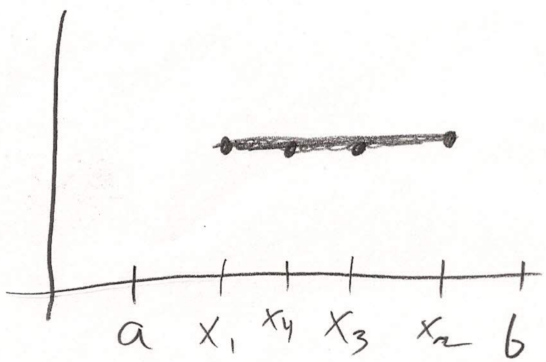
Prove: $\forall x_1, x_2 \in (a,b)$ $f(x_1) = f(x_2)$

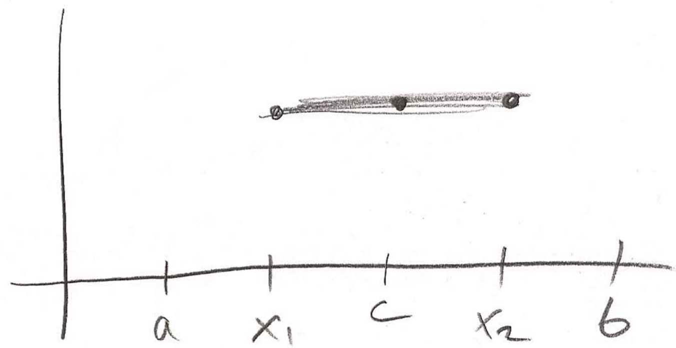
Apply MVT to $f$ on $[x_1, x_2]$

$\exists c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

$f(x_1) = f(x_2)$

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$f(x) = g(x)$

$f'(x) = g'(x)$

Given: $f'(x) = g'(x)$

?

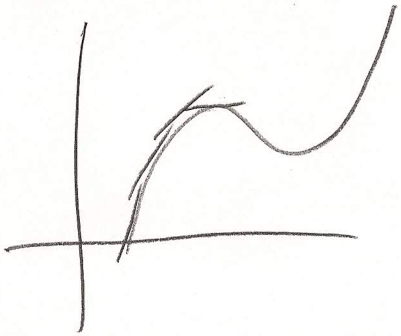
$h(x) = g(x) - f(x)$

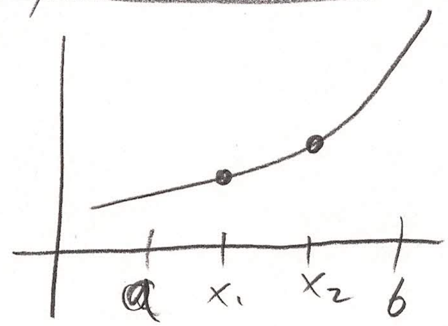
$h'(x) = g'(x) - f'(x) = 0$

$\Longrightarrow h(x) = c = g(x) - f(x)$

$\Longrightarrow g(x) = f(x) + c$

If $f'(x) > 0$ on $(a,b)$, then $f$ is inc on $(a,b)$

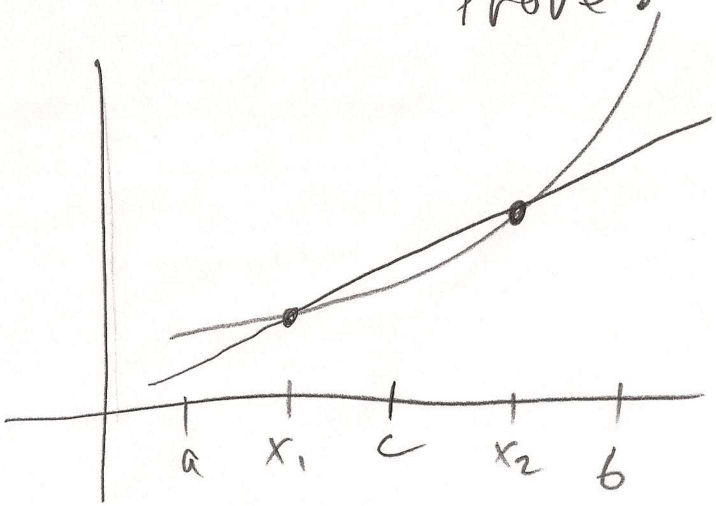





Inc on $(a,b)$ iff
$\forall x_1, x_2 \in (a,b)$
if $x_1 < x_2$, then $f(x_1) < f(x_2)$

Proof:

Given: $f'(x) > 0$ on $(a,b)$
Prove: $f$ is inc on $(a,b)$




Given: $x_1 < x_2$
Prove: $f(x_1) < f(x_2)$

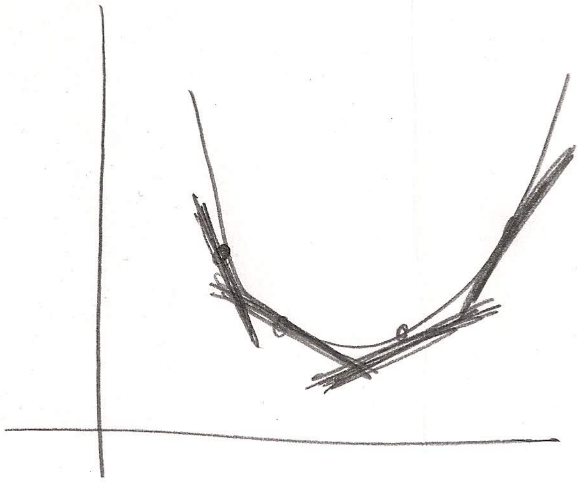
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Prove: this pos

pos

Prove: $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$

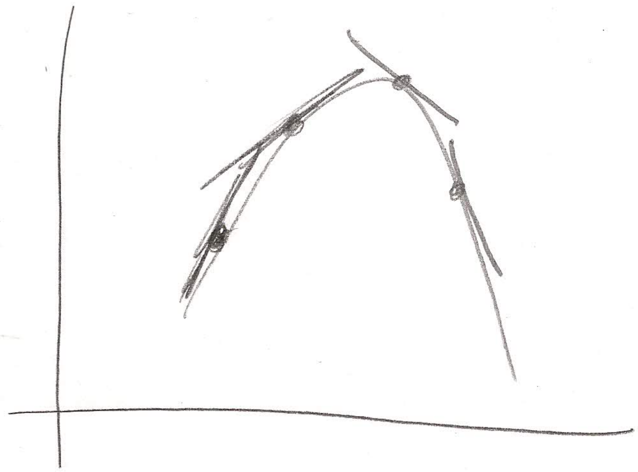
Apply MVT to $f$ on $[x_1, x_2]$
$\exists c \in (x_1, x_2)$ s.t.
$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$



f is Concave Up iff tangent lines to f lie below f

f' is increasing

$f'' > 0$



f is Concave Down iff tangent lines to f lie above f

f' is decreasing

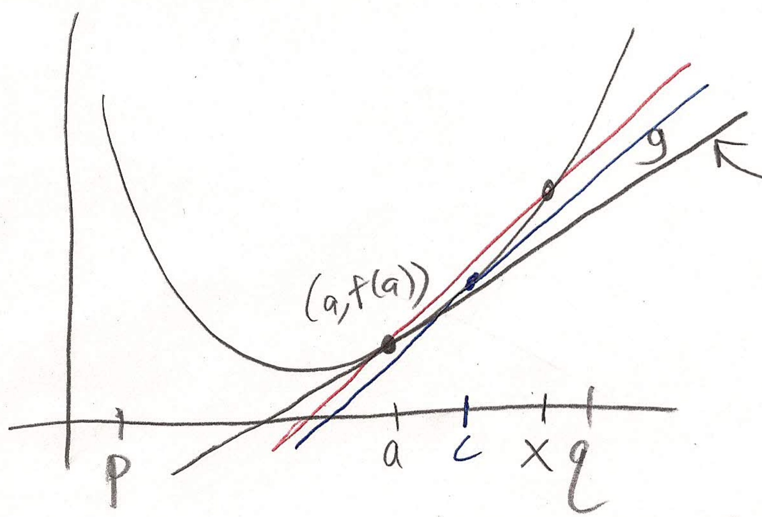
$f'' < 0$

Thm: If $f''(x) > 0$ on $(p, q)$; then $f$ is concave up.

Given: $f''(x) > 0 \implies f'$ is increasing

Prove: $f$ is CU $\implies$ tangent lines to $f$ lie below $f$

Prove: tangent line to $f$ at $x=a$ lies below $f$




$y - f(a) = f'(a)(x-a)$

$g(x) = f'(a)(x-a) + f(a)$

Show: $\forall_{x \neq a}\ x \quad f(x) > g(x)$

Prove: $f(x) > f'(a)(x-a) + f(a)$

Prove: $f(x) - f(a) > f'(a)(x-a)$

Case 1: $x > a \implies x - a > 0$

Prove: $\dfrac{f(x)-f(a)}{x-a} > f'(a)$

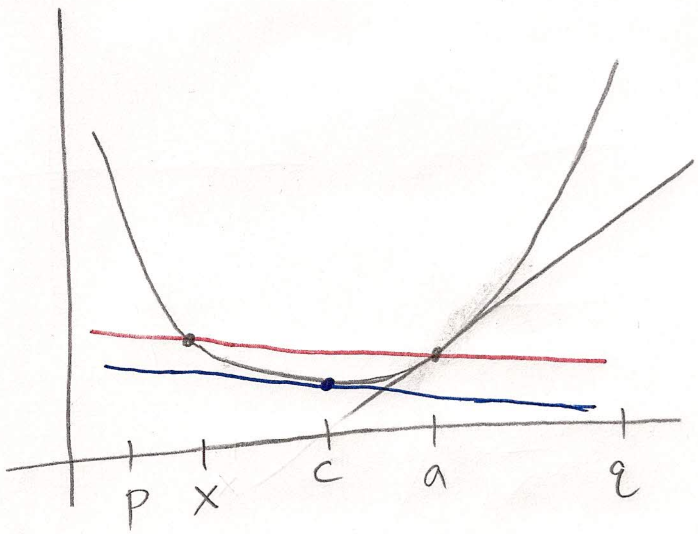
Apply MVT to $f$ on $[a, x]$

$\exists c \in (a, x)$ s.t. $f'(c) = \dfrac{f(x)-f(a)}{x-a}$

Prove: $f'(c) > f'(a)$ true b/c $f'$ is inc

Case 2: $x < a \Rightarrow x - a < 0$

Prove: $\dfrac{f(x)-f(a)}{x-a} < f'(a)$ $\left( \text{So then } f(x)-f(a) > f'(a)(x-a) \right)$




Apply MVT to f on $[x,a]$:

$\exists c \in (x,a)$ s.t.

$f'(c) = \dfrac{f(a)-f(x)}{a-x} = \dfrac{f(x)-f(a)}{x-a}$

Since $c < a$, $f'(c) < f'(a)$

b/c $f'$ is inc (since $f'' > 0$)

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So in both cases, $f(x) - f(a) > f'(a)(x-a)$

$f(x) > f'(a)(x-a) + f(a)$

$f(x) >$ tangent line to f @ $x = a$

Conclusion: If $f'' > 0$, then the tangent lines to f lie below f