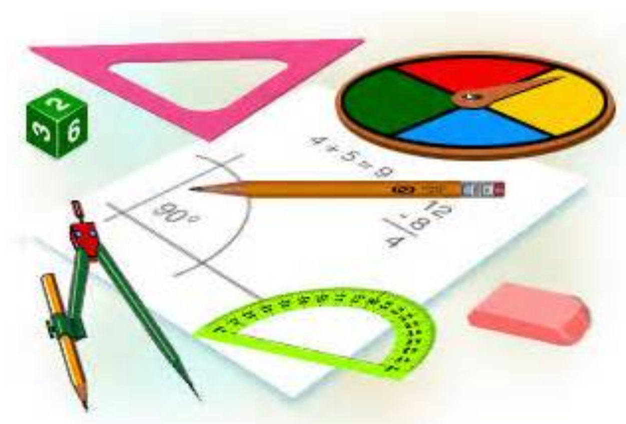
Using Manipulatives and Investigations in Geometry



For more information about the materials you find in this packet, contact:

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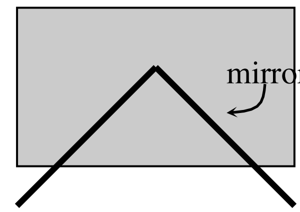
1-37. BUILDING A KALEIDOSCOPE

How does a kaleidoscope create the complicated, colorful images you see when you look inside? A hinged mirror and a piece of colored paper can demonstrate how a simple kaleidoscope creates its beautiful repeating designs.

Your Task: Place a hinged mirror on a piece of colored, unlined paper so that its sides extend beyond the edge of the paper as shown at right. Explore what shapes you see when you look directly at the mirror, and how those shapes change when you change the angle of the mirror. Discuss the questions below with your team. Be ready to share your responses with the rest of the class.



Discussion Points

What do you notice?

What happens when you change the angle (opening) formed by the sides of the mirror?

How can you describe the shapes you see in the mirror?

1-38. To complete your exploration, answer these questions together as a team.

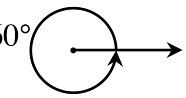
a. What happens to the shape you see as the angle formed by the mirror gets bigger (wider)? What happens as the angle gets smaller?

b. What is the smallest number of sides the shape you see in the mirror can have? What is the largest?

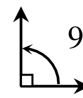
c. With your team, find a way to form a **regular hexagon** (a shape with six equal sides and equal angles).

d. How might you describe to another team how you set the mirrors to form a hexagon? What types of information would be useful to have?

1-39. A good way to describe an angle is by measuring how *wide* or *spread apart* the angle is. For this course, you can think of the **measure of an angle** as the measure of rotation of the two sides of the mirror from a closed position. The largest angle you can represent with a hinged mirror is 360°. This is formed when you open a mirror all the way so that the backs of the mirror touch. This is a called a **circular angle** and is represented by the diagram at right.



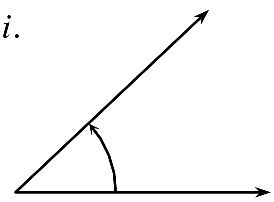
a. Other angles may be familiar to you. For example, an angle that forms a perfect "L" or a quarter turn is a 90° angle, called a **right angle** (shown at right). Four right angles can together form a circular angle.



What if the two mirrors are opened to form a straight line? What measure would that angle have? Draw this angle and label its degrees. How is this angle related to a circular angle?

b. Based on the examples above, estimate the measures of the angles shown below. Then confirm your answer using a **protractor**, a tool that measures angles.

i.



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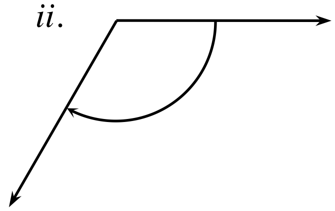
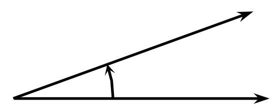
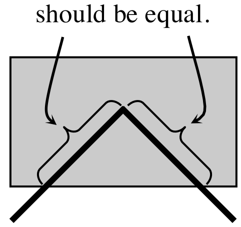


iii.



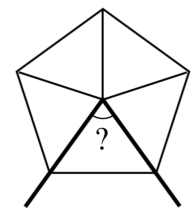
1-40. Now use your understanding of angle measurement to create some specific shapes using your hinged mirror. Be sure that both mirrors have the same length on the paper, as shown in the diagram at right.



a. Antonio says he can form an **equilateral triangle** (a triangle with three equal sides and three equal angles) using his hinged mirror. How did he do this? Once you can see the triangle in your mirror, place the protractor on top of the mirror. What is the measure of the angle formed by the sides of the mirror?

b. Use your protractor to set your mirror so that the angle formed is 90°. Be sure that the sides of the mirror intersect the edge of the paper at equal lengths. What is this shape called? Draw and label a picture of the shape on your paper.

c. Carmen's mirror shows the image at right, called a **regular pentagon**. She noticed that the five triangles in this design all meet at the hinge of her mirrors. She also noticed that the triangles must all be the same size and shape, because they are reflections of the triangle formed by the mirrors and the paper.



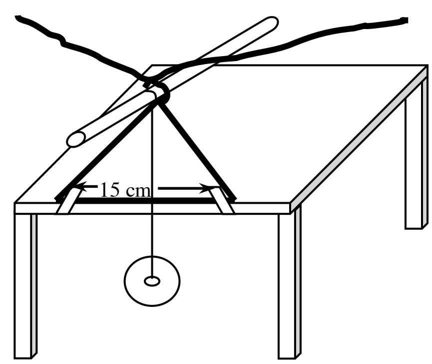
What must the sum of these five angles at the hinge be? And what is the angle formed by Carmen's mirrors? Test your conclusion with your mirror.

d. Discuss with your team and predict how many sides a shape would have if the angle that the mirror forms measures 40°. Explain how you made your prediction. Then check your prediction using the mirror and a protractor. Describe the shape you see with as much detail as possible.

2-91. HEIGHT LAB

What is the height of a triangle? Is it like standing at the highest point and looking straight down? Or is it like walking up a side of the triangle? Today your team will build triangles with string and consider different ways height can be seen for triangles of various shapes.

a. Use the materials given to you by your teacher to make a triangle like the one in the diagram below.



(1) Tie one end of the short string to the weight and the other to the end of a pencil (or pen).

(2) Tape a 15 cm section of the long string along the edge of a desk or table. Be sure to leave long ends of string hanging off each side.

(3) Bring the loose ends of string up from the table and cross them as shown in the diagram. Then put the pencil with the weight over the crossing of the string. Cross the strings again on top of the pencil.

b. Now, with your team, build and sketch triangles that meet the three conditions below. To organize your work, assign each team member one of the jobs described at right.

Student jobs:

- Hold the pencil (or pen) with the weight.
- Make sure that the weight hangs freely.
- Draw accurate sketches.
- Obtain and return materials as directed by the teacher.

(1) The height of the triangle is inside the triangle.

(2) The height of the triangle is a side of the triangle.

(3) The height of the triangle is outside of the triangle.

c. Now make sure that everyone in your team has sketches of the triangles that you made.

2-92. How can you find the height of a triangle if it is not a right triangle?

a. On the Lesson 2.2.4 Resource Page there are four triangles labeled (1) through (4). Assume you know the length of the side labeled "base." For each triangle, draw in the height that would enable you to find the area of the triangle. Note: You do not need to find the area.

b. Find the triangle for part (b) at the bottom of the same resource page. For this triangle, draw all three possible heights. First choose one side to be the base and draw in the corresponding height. Then repeat the process of drawing in the height for the other two sides, one at a time.

c. You drew in three pairs of bases and heights for the triangle in part (b). Using centimeters, measure the length of all three sides and all three heights. Find the area three times using all three pairs of bases and heights. Since the triangle remains the same size, your answers should match.

3-1. WARM-UP STRETCH

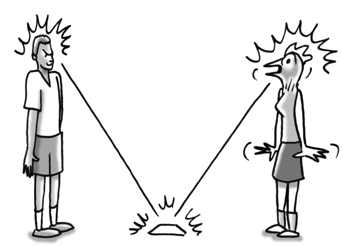
Before computers and copy machines existed, it sometimes took hours to enlarge documents or to shrink text on items such as jewelry. A pantograph device (like the one shown below) was often used to duplicate written documents and artistic drawings. You will now employ the same geometric principles by using rubber bands to draw enlarged copies of a design. Your teacher will show you how to do this. During this activity, discuss the following questions:

What do the figures have in common?
What do you predict?
What specifically is different about the figures?

3-105. YOU ARE GETTING SLEEPY…

Legend has it that if you stare into a person's eyes in a special way, you can hypnotize them into squawking like a chicken. Here's how it works.

Place a mirror on the floor. Your victim has to stand exactly 200 cm away from the mirror and stare into it. The only tricky part is that you need to figure out where you have to stand so that when you stare into the mirror, you are also staring into your victim's eyes.



If your calculations are correct and you stand at the *exact* distance, your victim will squawk like a chicken!

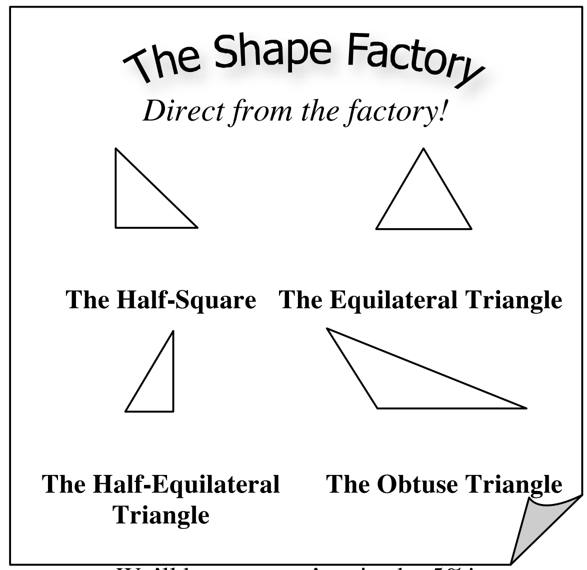
a. Choose a member of your team to hypnotize. Before heading to the mirror, first analyze this situation. Draw a diagram showing you and your victim standing on opposite sides of a mirror. Measure the heights of both yourself and your victim (heights to the eyes, of course), and label all the lengths you can on the diagram. (Remember, your victim will need to stand 200 cm from the mirror.)

b. Are there similar triangles in your diagram? Justify your conclusion. (Hint: Remember what you know about how light reflects off mirrors.) Then calculate how far you will need to stand from the mirror to hypnotize your victim.

c. Now for the moment of truth! Have your teammate stand 200 cm away from the mirror, while you stand at your calculated distance from the mirror. Do you make eye contact? If not, check your measurements and calculations and try again.

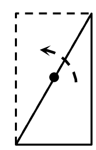
1-90. THE SHAPE FACTORY

The Shape Factory, an innovative new company, has decided to branch out to include new shapes. As Product Developers, your team is responsible for finding exciting new shapes to offer your customers. The current company catalog is shown at right.



Since your boss is concerned about production costs, you want to avoid buying new machines and instead want to reprogram your current machines.

The factory machines not only make all the shapes shown in the catalog, but they also are able to rotate or reflect a shape. For example, if the half-equilateral triangle is rotated 180° about the **midpoint** (the point in the middle) of its longest side, as shown at right, the result is a rectangle.



Your Task: Your boss has given your team until the end of this lesson to find as many new shapes as you can. Your team's reputation, which has suffered recently from a series of blunders, could really benefit by an impressive new line of shapes formed by a triangle and its transformations. For each triangle in the catalog, determine which new shapes are created when it is rotated or reflected so that the image shares a side with the original triangle. Be sure to make as many new shapes as possible. Use tracing paper or any other reflection tool to help.

7-12. IS THERE MORE TO THIS CIRCLE?

Circles can be folded to create many different shapes. Today, you will work with a circle and use properties of other shapes to develop a three-dimensional shape. Be sure to have reasons for each conclusion you make as you work. Each person in your team should start by obtaining a copy of a circle from your teacher and cutting it out.

a. Fold the circle in half to create a crease that lies on a line of symmetry of the circle. Unfold the circle and then fold it in half again to create a new crease that is perpendicular to the first crease. Unfold your paper back to the full circle. How could you convince someone else that your creases are perpendicular? What is another name for the line segment represented by each crease?
[When folded, each angle is reflected across a crease. Therefore, the adjacent angles are equal. Since they must add up to 180°, each is 90°; The line segments are each a diameter of the circle, because the line of symmetry must pass through the circle's center.]

b. On the circle, label the endpoints of one diameter A and B. Fold the circle so that point A touches the center of the circle and create a new crease. Then label the endpoints of this crease C and D. What appears to be the relationship between $\overline{AB}$ and $\overline{CD}$? Discuss and justify with your team. Be ready to share your reasons with the class. **[They are perpendicular.]**

c. Now fold the circle twice to form creases $\overline{BC}$ and $\overline{BD}$ and use scissors to cut out ΔBCD. What type of triangle is ΔBCD? How can you be sure? Be ready to convince the class. **[ΔBCD is equilateral. Students can demonstrate this by folding the equilateral triangle along each line of symmetry to show that adjacent sides are congruent.]**

7-13. ADDING DEPTH

Your equilateral triangle should now be flat (also called two-dimensional). **Two-dimensional** shapes have length and width, but not depth (or "thickness").

a. If you cut the labels off when creating your equilateral triangle, label the vertices of ΔBCD again. Then, with the unmarked side of the triangle facedown, fold and crease the triangle so that B touches the midpoint of $\overline{CD}$. Keep it in the folded position.

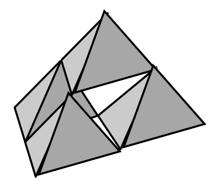
What does the resulting shape appear to be? What smaller shapes do you see inside the larger shape? Justify that your ideas are correct. For example, if you think that lines are parallel, you must provide evidence.
[Resulting shape: trapezoid, smaller shapes: three equilateral triangles; See the "Suggested Lesson Activity" for ways to justify these conclusions.]

b. Open your shape again so that you have the large equilateral triangle in front of you. How does the length of a side of the large triangle compare to the length of the side of the small triangle formed by the crease? How many of the small triangles would fit inside the large triangle? In what ways are the small and large triangles related? **[The sides of the large triangle are twice as long as the sides of the small triangles; 4; The small triangles are similar to the large triangle.]**

c. Repeat the fold in part (a) so that C touches the midpoint of $\overline{BD}$. Unfold the triangle and fold again so that D touches the midpoint of $\overline{BC}$. Create a three-dimensional shape by bringing points B, C, and D together. A **three-dimensional** shape has length, width, and depth. Use tape to hold your shape together.

d. Three-dimensional shapes formed with polygons have **faces** and **edges**, as well as **vertices** (plural of vertex). Faces are the flat surfaces of the shape, while edges are the line segments formed when two faces meet. Vertices are the points where edges intersect. Discuss with your team how to use these words to describe your new shape. Then write a complete description. If you think you know the name of this shape, include it in your description. **[Answers vary, but expect some students to call it a pyramid or a "3-D" triangle. You may want to introduce the term *tetrahedron* at this time.]**

7-14. Your team should now have 4 three-dimensional shapes (called tetrahedra). If you are working in a smaller team, you should quickly fold more shapes so that you have a total of four. **[Note: Have teams save their tetrahedrons for Lesson 9.1.5.]**

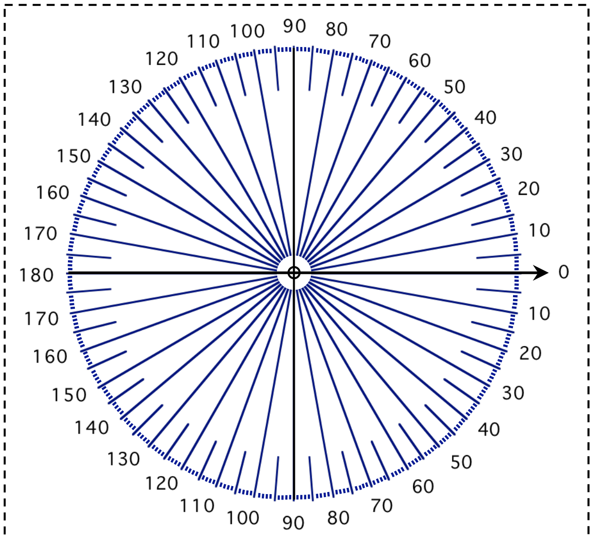


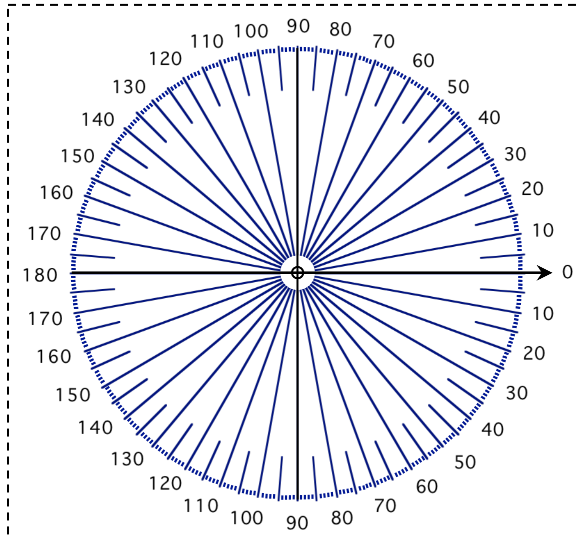
a. Put four tetrahedra together to make an enlarged tetrahedron like the one pictured at right. Is the larger tetrahedron similar to the small tetrahedron? How can you tell? **[Yes, they have the same shape.]**

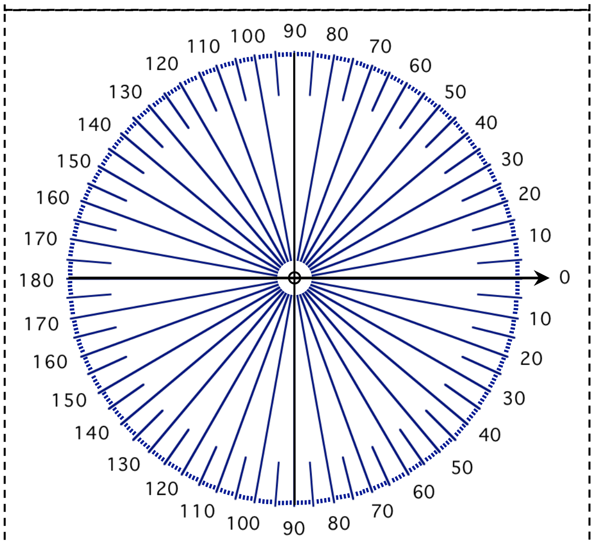
b. To determine the edges and faces of the new shape, pretend that it is solid. How many edges does a tetrahedron have? Are all of the edges the same length? How does the length of an edge of the team shape compare with the length of an edge of one of the small shapes? **[These tetrahedra have 6 edges of equal length. The edges of the large tetrahedron are twice as long as the length of the corresponding side of the small tetrahedron.]**

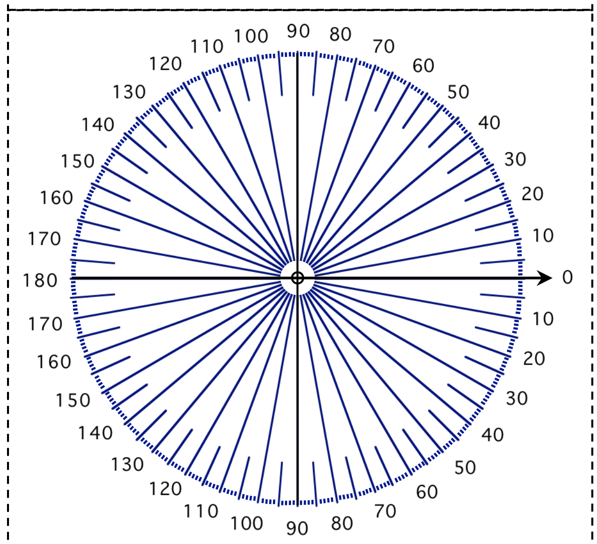
c. How many faces of the small tetrahedral would it take to cover the face of the large tetrahedron? Remember to count gaps as part of a face. Does the area of the tetrahedron change in the same way as the length? **[The area of each face of the large tetrahedron is four times as large as the area of the each face of a small tetrahedron.]**

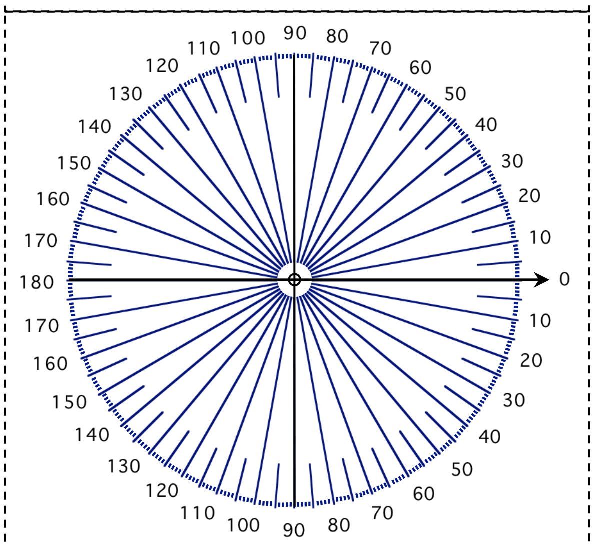
Protractors

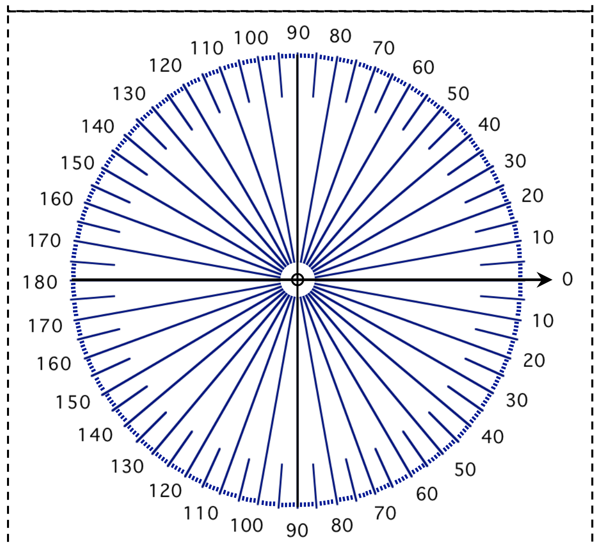








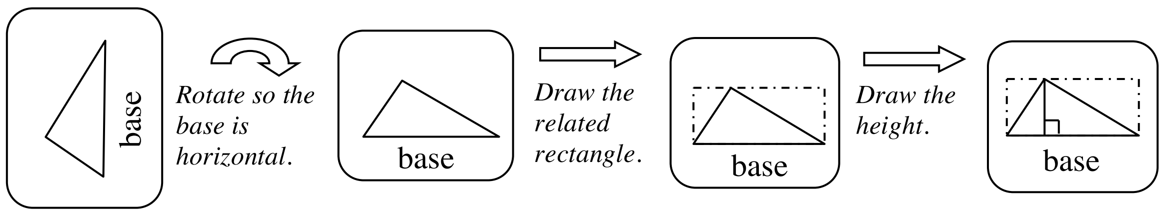




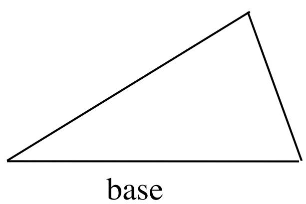
Problem 2-92.

Heights

For figures (1) through (4), draw a height to the side labeled “base.” One possible way to do this (there are many more!) is shown below.



(1)



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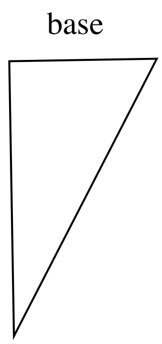

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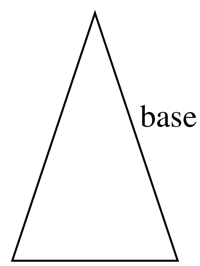

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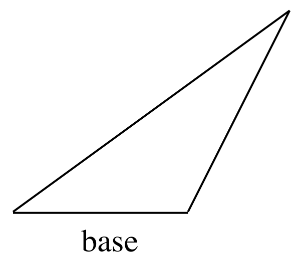
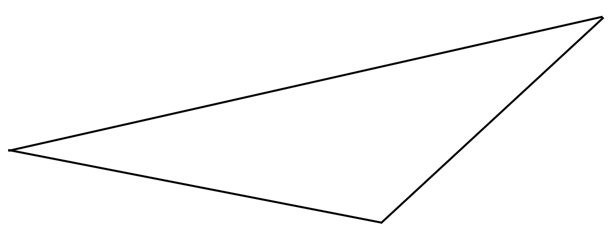


2-92, part (b)



Mathematics | Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1 **Make sense of problems and persevere in solving them.**
- **Find meaning in problems**
- **Look for entry points**
- **Analyze, conjecture and plan solution pathways**
- **Monitor and adjust**
- **Verify answers**
- **Ask themselves the question: "Does this make sense?"**

2 **Reason abstractly and quantitatively.**
- **Make sense of quantities and their relationships in problems**
- **Learn to contextualize and decontextualize**
- **Create coherent representations of problems**

3 **Construct viable arguments and critique the reasoning of others.**
- **Understand and use information to construct arguments**
- **Make and explore the truth of conjectures**
- **Recognize and use counterexamples**
- **Justify conclusions and respond to arguments of others**

4 **Model with mathematics.**
- **Apply mathematics to problems in everyday life**
- **Make assumptions and approximations to simplify a complicated situation**
- **Identify quantities in a practical situation**
- **Interpret results in the context of the situation and reflect on whether the results make sense**

5 **Use appropriate tools strategically.**
- **Consider the available tools when solving problems**
- **Are familiar with tools appropriate for their grade or course (pencil and paper, concrete models, ruler, protractor, calculator, spreadsheet, computer programs, digital content located on a website, and other technological tools)**
- **Make sound decisions of which of these tools might be helpful**

6 **Attend to precision.**
- **Communicate precisely to others**
- **Use clear definitions, state the meaning of symbols and are careful about specifying units of measure and labeling axes**
- **Calculate accurately and efficiently**

7 **Look for and make use of structure.**
- **Discern patterns and structures**
- **Can step back for an overview and shift perspective**
- **See complicated things as single objects or as being composed of several objects**

8 **Look for and express regularity in repeated reasoning.**
- **Notice if calculations are repeated and look both for general methods and shortcuts**
- **In solving problems, maintain oversight of the process while attending to detail**
- **Evaluate the reasonableness of their immediate results**