Presentation 151
Shapes in Shapes
Building Number Sense and Formula Sense

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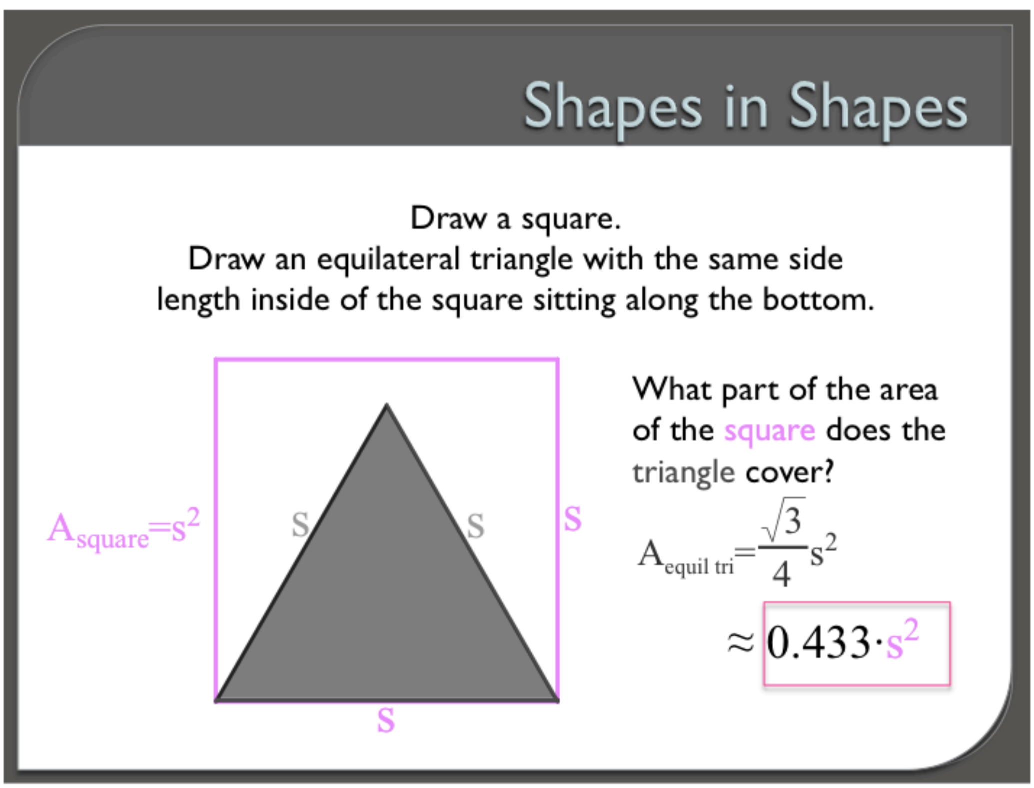
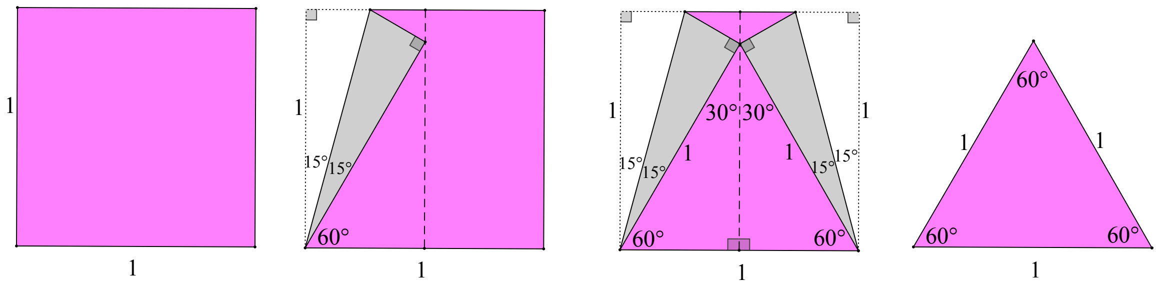
- Starting with Triangles, Squares, and Circles, fit shapes inside other shapes and compare relative Side Lengths and Areas
- Build Unit Tetrahedron out of envelopes
- Each table will collaborate to build a Stella Octangula
- Discuss further properties of these basic 3D shapes and how they fit within each other using the readily available Stella Octangula model

Question:
What part of the area of a Square does an Equilateral Triangle cover?



To fold an equilateral triangle from a square (patty paper works great):

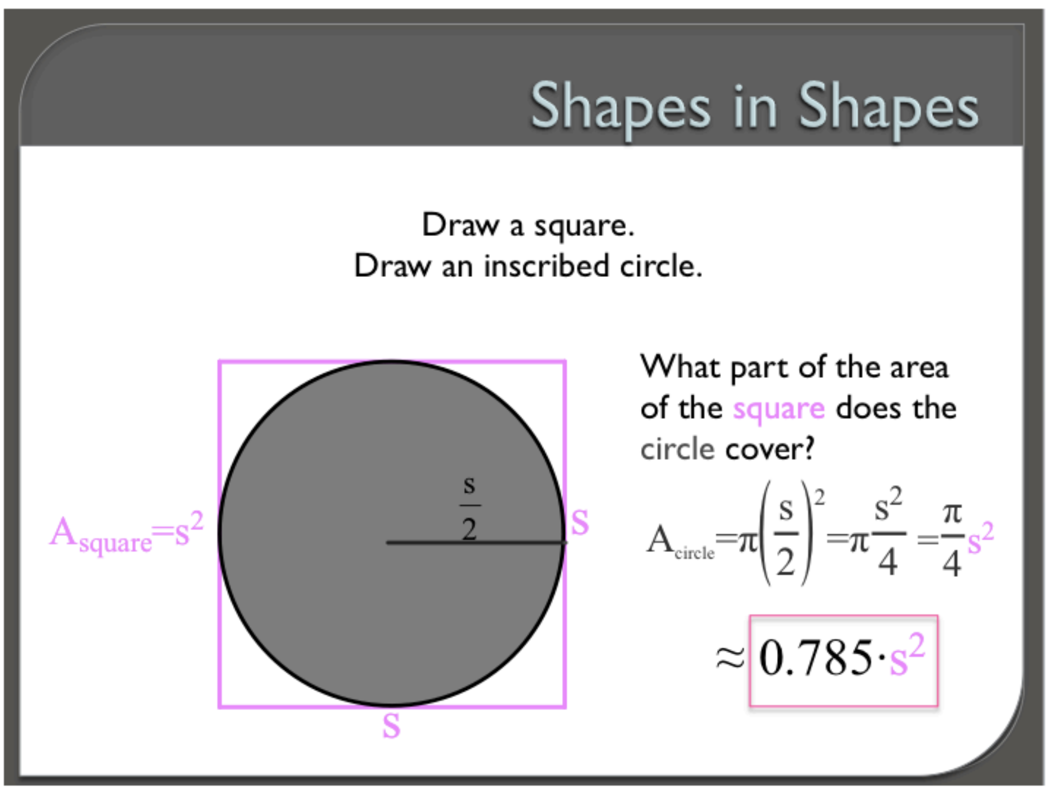


Answer:

The ratio of Areas of an Equilateral Triangle to a Square is $\frac{\sqrt{3}}{4}:1$.

A Unit Equilateral Triangle takes up 43.3% of the area of a Square.

Question:
What part of the area of a Square does a Circle cover?



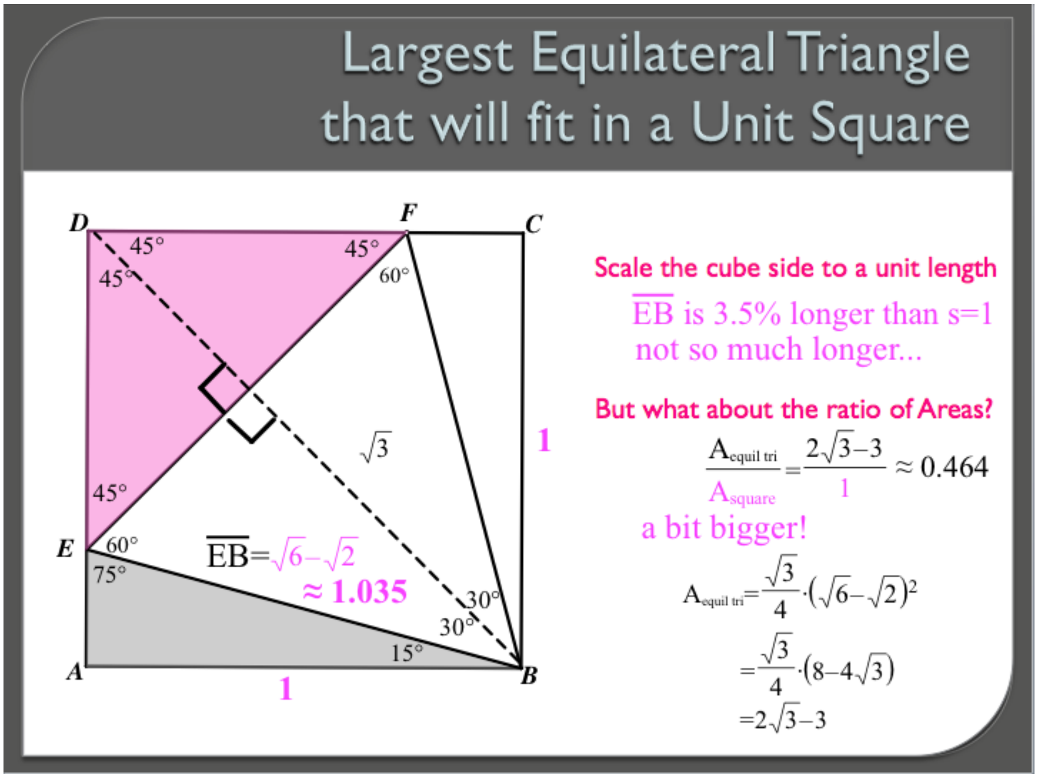
Answer:

The ratio of Areas of a Circle to a Square is $\frac{\pi}{4}:1$.

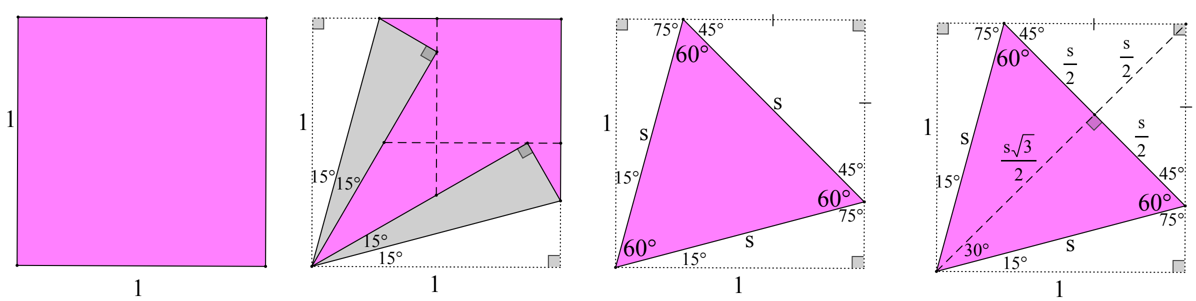
A Circle takes up 78.5% of the area of a Square.

Question:

What is the largest Equilateral Triangle that will fit in a Unit Square with $s = 1$?



To fold the largest equilateral triangle in a square (patty paper works great):



$$d = \frac{s\sqrt{3}}{2} + \frac{s}{2} = \sqrt{2}$$

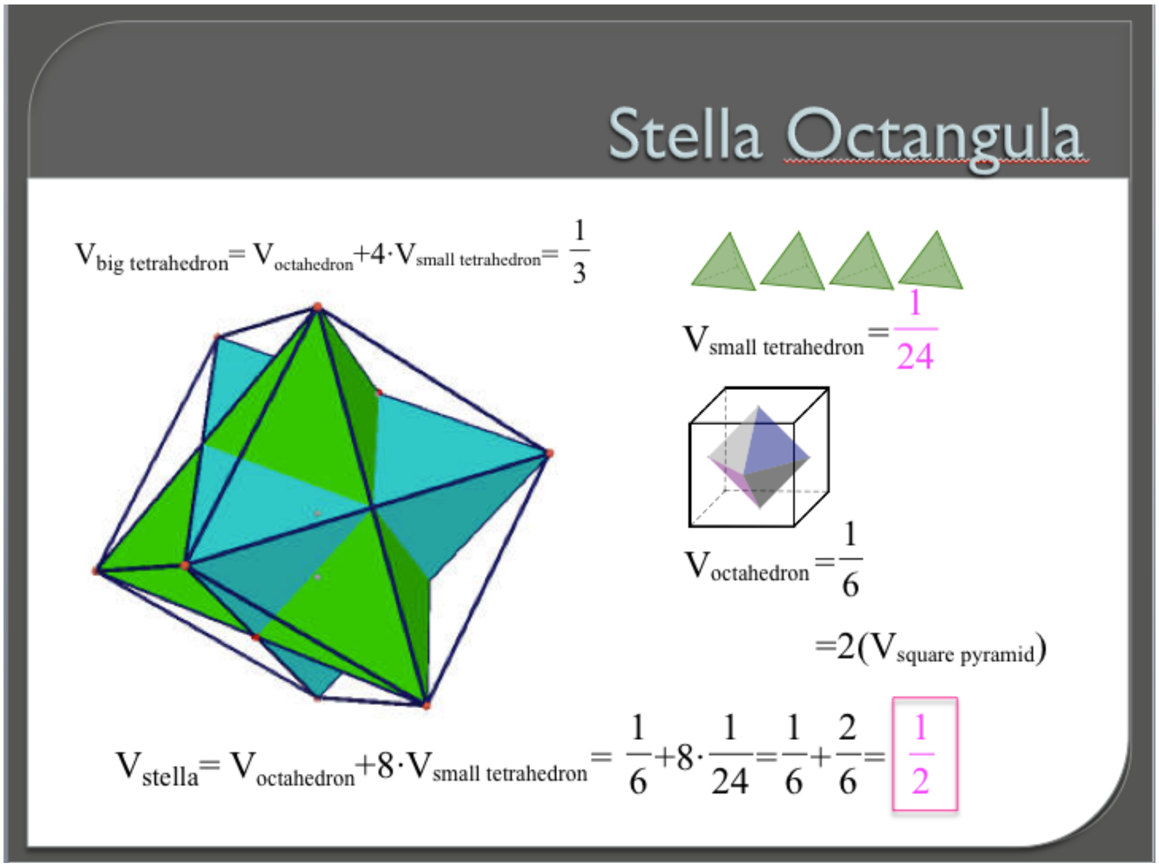
$$\left(1+\sqrt{3}\right)s = 2\sqrt{2}$$

$$s = \frac{2\sqrt{2}}{1+\sqrt{3}} = \frac{2\sqrt{2}}{1+\sqrt{3}} \cdot \frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{2\sqrt{2}-2\sqrt{6}}{-2} = \sqrt{6}-\sqrt{2}$$

Answer:

The largest Equilateral Triangle that will fit in a Unit Square has side length $s = \sqrt{6} - \sqrt{2} = 1.035$ and takes up 46.4% of the area of the Square

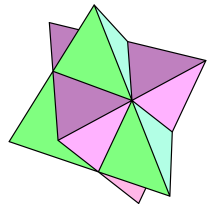
Question:
What is the volume of the Stella Octangula in a Cube of side length, $s = 1$?



Answer:
The Stella Octangula has a side length of $s = \sqrt{2}$.

The volume of the Stella Octangula is exactly $\frac{1}{2}$ the volume of the Cube.

$$V_{stella\,octangula\,,\,s} = \frac{\sqrt{2}}{8}s^3$$



Volume of a Stella Octangula with side length, $s = 1$

$$V_{stella\,octangula\,,\,s=1} = V_{octahedron\,,\,s=1/2} + 8V_{tetrahedron\,,\,s=1/2} \quad = \frac{\sqrt{2}}{24} + 8\left(\frac{\sqrt{2}}{12}\left(\frac{1}{2}\right)^3\right) = \frac{\sqrt{2}}{24} + \frac{2\sqrt{2}}{24} = \frac{\sqrt{2}}{8}$$

Volume of a Stella Octangula with side length, $s = \sqrt{2}$

$$V_{stella\,octangula\,,\,s=\sqrt{2}} = \frac{\sqrt{2}}{8}s^3 = \frac{\sqrt{2}}{8}\sqrt{2}^3 = \frac{1}{2}$$

which is pretty remarkable!

Summary

- The volume of a Stella Octangula is $1/2$ that of the smallest cube into which it can be placed.

- The solid common to both intersecting tetrahedron is a regular octahedron of edge length s.

- The volume of a Stella Octangula composed of two intersecting tetrahedron of edge length $2s$, equals 12 times the volume of the smaller tetrahedron of edge length s

  The Stella Octngula is 3 times the volume of the regular octahedron of edge length s.

- The volume of a regular octahedron of edge length s is 4 times the volume of the smaller tetrahedron of edge length s, or $1/2$ the volume of the tetrahedron of edge length $2s$.

  It is also $1/6$ the volume of the cube in which it is the dual, where it sits with its vertices at the midpoints of each face of the cube.