

Presentation 151

Shapes in Shapes

Building Number Sense and Formula Sense

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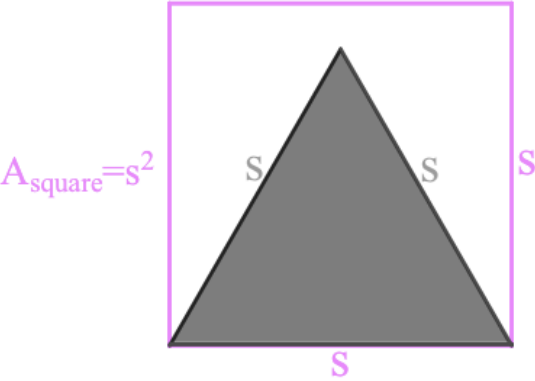
- Starting with **Triangles**, **Squares**, and **Circles**, fit shapes inside other shapes and compare relative **Side Lengths** and **Areas**
- Build **Unit Tetrahedron** out of envelopes
- Each table will collaborate to build a **Stella Octangula**
- Discuss further properties of these basic 3D shapes and how they fit within each other using the readily available **Stella Octangula** model

Question:

What part of the area of a Square does an Equilateral Triangle cover?

Shapes in Shapes

Draw a square.
Draw an equilateral triangle with the same side length inside of the square sitting along the bottom.

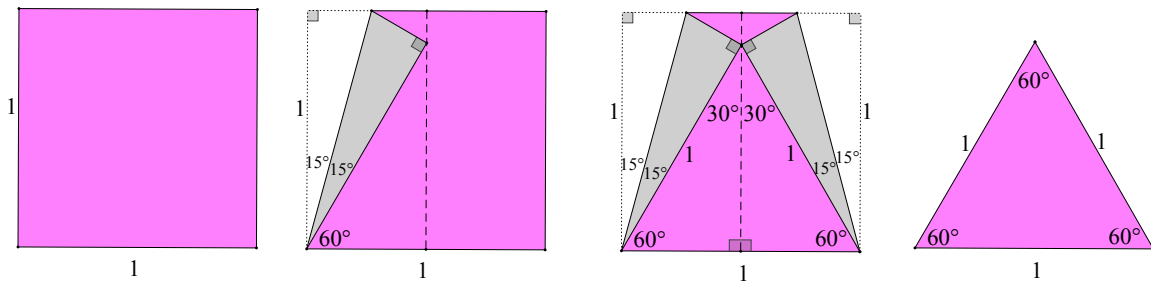


$A_{\text{square}} = s^2$

$A_{\text{equil tri}} = \frac{\sqrt{3}}{4} s^2$
 $\approx 0.433 \cdot s^2$

What part of the area of the square does the triangle cover?

To fold an equilateral triangle from a square (patty paper works great):



Answer:

The ratio of Areas of an Equilateral Triangle to a Square is $\frac{\sqrt{3}}{4} : 1$.

A Unit Equilateral Triangle takes up 43.3% of the area of a Square.

Question:

What part of the area of a Square does a Circle cover?

Shapes in Shapes

Draw a square.
Draw an inscribed circle.

$A_{\text{square}} = s^2$

What part of the area of the square does the circle cover?

$$A_{\text{circle}} = \pi \left(\frac{s}{2}\right)^2 = \pi \frac{s^2}{4} = \frac{\pi}{4} s^2$$

$\approx 0.785 \cdot s^2$

Answer:

The ratio of Areas of a Circle to a Square is $\frac{\pi}{4} : 1$.

A Circle takes up 78.5% of the area of a Square.

Question:

What is the largest Equilateral Triangle that will fit in a Unit Square with $s = 1$?

Largest Equilateral Triangle that will fit in a Unit Square

Scale the cube side to a unit length
 \overline{EB} is 3.5% longer than $s=1$
 not so much longer...

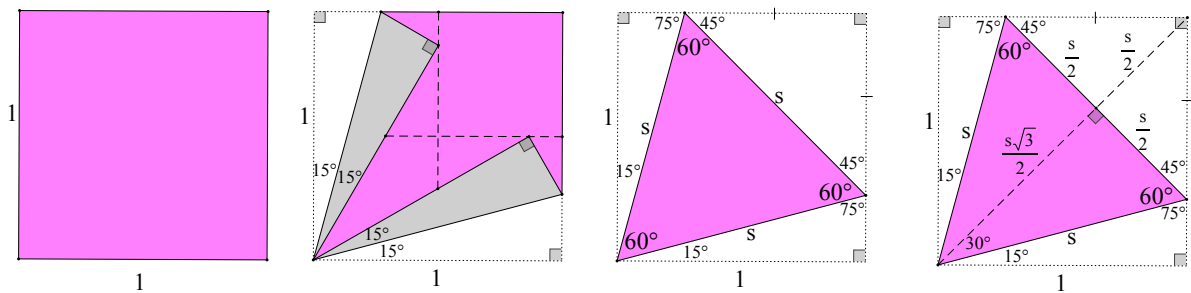
But what about the ratio of Areas?

$$\frac{A_{\text{equil tri}}}{A_{\text{square}}} = \frac{2\sqrt{3}-3}{1} \approx 0.464$$

a bit bigger!

$$\begin{aligned} A_{\text{equil tri}} &= \frac{\sqrt{3}}{4} \cdot (\sqrt{6}-\sqrt{2})^2 \\ &= \frac{\sqrt{3}}{4} \cdot (8-4\sqrt{3}) \\ &= 2\sqrt{3}-3 \end{aligned}$$

To fold the largest equilateral triangle in a square (patty paper works great):



$$d = \frac{s\sqrt{3}}{2} + \frac{s}{2} = \sqrt{2}$$

$$(1 + \sqrt{3})s = 2\sqrt{2}$$

$$s = \frac{2\sqrt{2}}{1 + \sqrt{3}} = \frac{2\sqrt{2}}{1 + \sqrt{3}} \cdot \frac{1 - \sqrt{3}}{1 - \sqrt{3}} = \frac{2\sqrt{2} - 2\sqrt{6}}{-2} = \sqrt{6} - \sqrt{2}$$

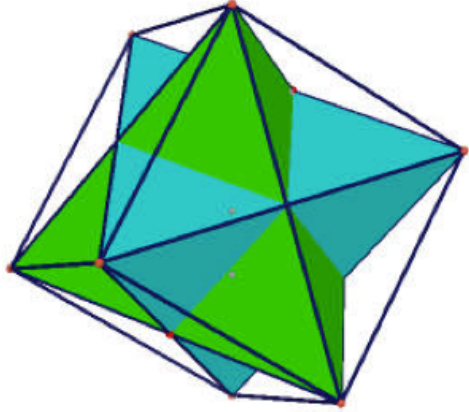
Answer:


The largest Equilateral Triangle that will fit in a Unit Square has side length $s = \sqrt{6} - \sqrt{2} = 1.035$ and takes up 46.4% of the area of the Square


Question:

What is the volume of the Stella Octangula in a Cube of side length, $s = 1$?

Stella Octangula

$$V_{\text{big tetrahedron}} = V_{\text{octahedron}} + 4 \cdot V_{\text{small tetrahedron}} = \frac{1}{3}$$




$$V_{\text{small tetrahedron}} = \frac{1}{24}$$


$$V_{\text{octahedron}} = \frac{1}{6}$$

$= 2(V_{\text{square pyramid}})$

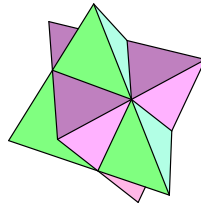
$$V_{\text{stella}} = V_{\text{octahedron}} + 8 \cdot V_{\text{small tetrahedron}} = \frac{1}{6} + 8 \cdot \frac{1}{24} = \frac{1}{6} + \frac{2}{6} = \boxed{\frac{1}{2}}$$

Answer:

The Stella Octangula has a side length of $s = \sqrt{2}$.

The volume of the Stella Octangula is exactly $\frac{1}{2}$ the volume of the Cube.

$$V_{\text{stella octangula}, s} = \frac{\sqrt{2}}{8} s^3$$



Volume of a Stella Octangula with side length, $s = 1$

$$V_{\text{stella octangula}, s=1} = V_{\text{octahedron}, s=\frac{1}{2}} + 8V_{\text{tetrahedron}, s=\frac{1}{2}} = \frac{\sqrt{2}}{24} + 8 \left(\frac{\sqrt{2}}{12} \left(\frac{1}{2} \right)^3 \right) = \frac{\sqrt{2}}{24} + \frac{2\sqrt{2}}{24} = \frac{\sqrt{2}}{8}$$

Volume of a Stella Octangula with side length, $s = \sqrt{2}$

$$V_{\text{stella octangula}, s=\sqrt{2}} = \frac{\sqrt{2}}{8} s^3 = \frac{\sqrt{2}}{8} \sqrt{2}^3 = \frac{1}{2}$$

which is pretty remarkable!

Summary

- The volume of a Stella Octangula is $\frac{1}{2}$ that of the smallest cube into which it can be placed.
- The solid common to both intersecting tetrahedron is a regular octahedron of edge length s .
- The volume of a Stella Octangula composed of two intersecting tetrahedron of edge length $2s$, equals 12 times the volume of the smaller tetrahedron of edge length s .

The Stella Octngula is 3 times the volume of the regular octahedron of edge length s .

- The volume of a regular octahedron of edge length s is 4 times the volume of the smaller tetrahedron of edge length s , or $\frac{1}{2}$ the volume of the tetrahedron of edge length $2s$.

It is also $\frac{1}{6}$ the volume of the cube in which it is the dual, where it sits with its vertices at the midpoints of each face of the cube.