


Each State shall appoint, in such Manner as the Legislature thereof may direct, a Number of Electors, equal to the whole Number of Senators and Representatives to which the State may be entitled in the Congress.

Up until the 1960 presidential election, there were 535 electors ( 435 from the House of Representatives and 100 from the Senate) who selected the president. The Twenty-third Amendment to the Constitution, ratified in 1961, granted the District of Columbia three electoral votes. Since 1964, there have been 538 electors; a simple majority consists of 270 electoral votes.

Bush carried thirty states and netted 271 electoral votes to Gore's twenty states and 266 votes (one Gore elector from the District of Columbia abstained from voting). This was the fourth U.S. presidential election in which the winner did not receive the plurality of the popular votes.

The 2004 presidential election was another close contest. President Bush carried thirty-one states and received $50.7 \%$ of the popular vote to beat Senator John Kerry. In the 2008 presidential election, then-Senator Barak Obama won a landslide victory. Carrying twenty-eight states and netting 365 electoral votes, he received about ten million more popular votes than Senator John McCain.

## THE WEIGHTED VOTING SYSTEM

Students understand intuitively that California, the most populous state with $37,253,956$ people (according to the 2010 U.S. Census), should have more say in electing a U.S. president than Wyoming, the least populous state with 563,626 people. Reflecting this population difference, California had 55 electoral votes to Wyoming's 3 in the 2012 presidential election. (To survey each state's electoral vote apportionment for the 2012 presidential election, see the website http://electoral-vote.com.)

During the 2008 election, Obama's 365 electoral votes came largely from winning the more populous states, such as California ( 55 electoral votes), New York (31), Florida (27), Illinois (21), Pennsylvania (21), and Ohio (20). In short, the electoral voting system is an example of weighted voting.

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\(\{A, B\},\{A, C\},\{B, C\}\),
\(\{A, B, C\},\{A, B, D\},\{A, B, E\},\{A, B, F\},\{A, C, D\},\{A, C, E\}\),
    \(\{A, C, F\},\{B, C, D\},\{B, C, E\},\{B, C, F\}\),
\(\{A, B, C, D\},\{A, B, C, E\},\{A, B, C, F\},\{A, B, D, E\},\{A, B, D, F\}\),
    \(\{A, B, E, F\},\{A, C, D, E\},\{A, C, D, F\},\{A, C, E, F\},\{B, C, D, E\}\),
    \(\{B, C, D, F\},\{B, C, E, F\}\),
\(\{A, B, C, D, E\},\{A, B, C, D, F\},\{A, B, C, E, F\},\{A, B, D, E, F\}\),
    \(\{A, C, D, E, F\},\{B, C, D, E, F\}\),
\(\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}\)
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Fig. 1 Thirty-two winning coalitions are listed here.

Let's consider a simple example to understand this concept. First, we need to define three important notions. In a weighted voting system, the power of a voter is his or her ability to influence a decision; a coalition is a set of voters who support a measure that is being voted on; and a quota is the minimum number of votes needed to form a winning coalition. We will use the following notation to represent the quota and each voter's weight: [quota: weight , weight $_{2}$, weight $_{3}$, weight $\left._{4}, \ldots\right]$. For example, $[16: 9,9,7,3,1$, 1] indicates that there are six voters with varying weights. To form a winning coalition, at least 16 votes are needed. Hence, $\{9,9\}$ represents a winning coalition that consists of the two voters with the most weight; $\{9,7\}$ represents another winning coalition.

A careful inspection reveals that the three voters with weights 3,1 , and 1 will never make a difference. In other words, they have no voting power. For instance, $\{9,7,3\}$ represents a winning coalition because the sum of their weights (19) is at least the quota (16). However, the voters with weights 9 and 7 could have formed a winning coalition by themselves. In fact, these two voters can pass any measure. A voter who has no voting power is called a dummy. A situation such as [16: $9,9,6,2,2,2]$ has no dummies. In [51: 60, 40], the voter with weight 60 is called the dictator. Surprisingly, in [51: 49, 48, 3], all three voters are equally powerful. Hence, to determine a voter's power in weighted voting, we need to examine not only how much weight the voter has but also how often he or she can form winning coalitions.

## THE BANZHAF POWER INDEX

In 1965 John F. Banzhaf III, a noted attorney, advanced the perspective that a voter is "a critical voter if the outcome would be different if that voter, and no other voter, changed his or her vote" (Banzhaf 1965; Taylor, Conrad, and Brams 2013, p. 389).

Recall the earlier example [16: 9, 9, 7, 3, 1, 1]. For Banzhaf, this example had real-life implications. He studied the Nassau County (NY) Board of Supervisors' voting system that apportioned the total of 30 votes to the county's municipalities in the following manner:

| Hempstead 1: 9 | (A) |
| :--- | ---: |
| Hempstead 2: 9 | (B) |
| North Hempstead: 7 | (C) |
| Oyster Bay: 3 | (D) |
| Glen Cove: 1 | (E) |
| Long Beach: 1 | (F) |

For convenience, we have assigned the letters A through F to represent the municipalities.

A simple majority of 16 votes will form a winning coalition. Examining the combinations, Banzhaf discovered 32 winning coalitions (see fig. 1).

In $\{A, B, D\}$, without either $A$ or $B$ there is no winning coalition. In other words, both $A$ and $B$ are critical voters-in this case, critical municipali-ties-in this winning coalition. Hence, the following underlined notation $\{\underline{A}, \underline{B}, D\}$ represents a winning coalition in which both A and B are critical municipalities. The municipality D is a dummy. However, in $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$, none of the three municipalities is critical to form this winning coalition. Banzhaf omitted this winning coalition and others that did not contain any critical municipalities. According to this criterion, he winnowed the winning coalitions to 24 (see fig. 2).

Banzhaf then tallied each municipality's critical votes:

| Hempstead 1:16 | (A) |
| :--- | :--- |
| Hempstead 2:16 | (B) |
| North Hempstead: 16 | (C) |
| Oyster Bay: 0 | (D) |
| Glen Cove: 0 | (E) |
| Long Beach: 0 | (F) |

Therefore, the Banzhaf Power Index for the Nassau Board of Supervisors is as follows:

Hempstead 1: 16 out of 48 critical votes, or $1 / 3$ (about $33 \%$ )
Hempstead 2: 1/3 (about 33\%)
North Hempstead: 1/3 (about 33\%)
Oyster Bay: 0
Glen Cove: 0
Long Beach: 0

Banzhaf observed that the Hempstead 1, Hempstead 2, and North Hempstead municipalities were equally powerful in passing measures. He also argued that the voting arrangement shown above gave no power to the Oyster Bay, Glen Cove, and Long Beach municipalities (one-sixth of Nassau County residents); he later sued the Board of Supervisors on their behalf.

In an attempt to equalize the representation of residents in municipalities of different sizes, some county supervisorial boards in New York State are constituted according to a form of Banzhaf's index (Dubey and Shapley 1979; Imrie 1973; Lucas 1976). Today, the Banzhaf Power Index is an acknowledged method of measuring voting power.

## THE SHAPLEY-SHUBIK POWER INDEX

In 1954, a mathematician, Lloyd S. Shapley, and an economist, Martin Shubik, articulated the ShapleyShubik Power Index to assess power in voting systems. They noticed that in many situations, winning coalitions are built one vote at a time. The most important voter (the pivotal voter) is the one who turns a losing coalition into a winning coalition.
$\{\underline{A}, \underline{B}\},\{\underline{A}, \underline{C}\},\{\underline{B}, \underline{C}\}$,
$\{\underline{A}, \underline{B}, D\},\{\underline{A}, \underline{B}, E\},\{\underline{A}, \underline{B}, F\},\{\underline{A}, \underline{C}, D\},\{\underline{A}, \underline{C}, E\},\{\underline{A}, \underline{C}, F\}$, $\{\underline{B}, \underline{C}, D\},\{\underline{B}, \underline{C}, E\},\{\underline{B}, \underline{C}, F\}$,
$\{\underline{A}, \underline{B}, D, E\},\{\underline{A}, \underline{B}, D, F\},\{\underline{A}, \underline{B}, E, F\},\{\underline{A}, \underline{C}, D, E\},\{\underline{A}, \underline{C}, D, F\}$, $\{\underline{A}, \underline{C}, E, F\},\{\underline{B}, \underline{C}, D, E\},\{\underline{B}, \underline{C}, D, F\},\{\underline{B}, \underline{C}, E, F\}$, $\{\underline{A}, \underline{B}, D, E, F\},\{\underline{A}, \underline{C}, D, E, F\},\{\underline{B}, \underline{C}, D, E, F\}$

Fig. 2 Only 24 coalitions contain a critical municipality.

## Table 1 Determining Pivotal Voters

| Permutations | Accumulative <br> Weights | Pivotal Voters |
| :---: | :---: | :---: |
| ABC | 234 | B |
| ACB | 234 | C |
| BAC | 134 | A |
| BCA | 124 | A |
| CAB | 134 | A |
| CBA | 124 | A |

Table 2 Power Indices for [3: 2, 1, 1]

| Voters | Nominal | Banzhaf | Shapley-Shubik |
| :---: | :---: | :---: | :---: |
| A | $1 / 2(50 \%)$ | $3 / 5(60 \%)$ | $2 / 3($ about $67 \%)$ |
| B | $1 / 4(25 \%)$ | $1 / 5(20 \%)$ | $1 / 6($ about $17 \%)$ |
| C | $1 / 4(25 \%)$ | $1 / 5(20 \%)$ | $1 / 6$ (about $17 \%)$ |

Hence, the power index comprises an examination of permutations.

In $[3: 2,1,1]$, we represent the three votes using the letters A through C. How do we determine the pivotal voter if the votes are cast in the order $A, B$, C? In this permutation, voter A votes first. Voter A's weight is 2, but this value does not attain the needed quota of 3 . Because voter $B$ votes next, we combine voter A's weight (2) and voter B's weight (1) to reach 3. Hence, voter B is the pivotal voter.

Table 1 denotes the six permutations, given three voters. The second column lists the corresponding accumulative weights, and the bold values indicate the quota of 3 . The pivotal voters correspond to the bold values.

On the basis of the pivotal voters' tally, we can compute the Shapley-Shubik Power Index:

$$
\begin{array}{ll}
4 / 6 \text { (about } 67 \% \text { ) } & \text { (A) } \\
1 / 6 \text { (about } 17 \% \text { ) } & \text { (B) } \\
1 / 6 \text { (about } 17 \% \text { ) } & \text { (C) }
\end{array}
$$

Finally, let's define the voter's nominal power as the ratio of its weight to the sum of all voters' weights. On the basis of [3:2,1,1], we can compute, compare, and contrast the three power indices (see table 2).


Fig. 3 Would the green states turn red or blue?

## Table 3 Winning Coalitions for Swing States

| Number of <br> States | Number of <br> Winning <br> Coalitions | Number of <br> Winning Coalitions <br> with Critical States |
| :---: | :---: | :---: |
| 1 | 0 | 0 |
| 2 | 1 | 1 |
| 3 | 10 | 10 |
| 4 | 14 | 14 |
| 5 | 6 | 2 |
| 6 | 1 | 0 |
|  | 32 Winning <br> Coalitions | 27 Winning Coalitions with <br> Critical States |

As an exercise, students can verify the Banzhaf Power Index values shown in table 2. In this particular situation, the Banzhaf and Shapley-Shubik Power Indices designate more power ( $60 \%$ and $67 \%$, respectively) to voter A than indicated by the Nominal Power Index value (50\%).

## THE POWER INDICES AND THE 2000 ELECTION

On November 6, 2000, just one day before the election, Rajghatta (2000) reported, "The latest Reuters count gave Bush a slight lead in the Electoral College with 209 votes solid or leaning strongly toward him. Gore had 196 votes and 133 were too close to call." The swing states (also known as battleground states) were Florida ( 25 electoral votes), Pennsylvania (23), Missouri (11), Tennessee (11), Washington (11), Wisconsin (11), Iowa (7), Oregon (7), Arkansas (6), New Mexico (5), West Virginia (5), Maine (4), Nevada (4), and Delaware (3).

It is quite plausible that the candidates would devote more resources, energy, and time in the six swing states with the greater electoral votes (Florida, Pennsylvania, Missouri, Tennessee, Washington, and Wisconsin). (See fig. 3 for a map of these
six states and their respective electoral votes.) On November 5, 2000, Gore was campaigning in Pennsylvania, and Bush was campaigning in Florida. If we assume that the electoral votes for all but the six most populous swing states split 223-223, we are left with candidates requiring a majority of 47 of the 92 electoral votes available in those swing states: $[47: 25,23,11,11,11,11]$ (see fig. 3). These six states could determine the election.

Naturally, these states received much more attention from the candidates, the media, and the voters. The question was, How much power did the identified swing states wield?

To determine the Banzhaf Power Index, students need to identify 27 winning coalitions containing these critical states. With the teacher's guidance, students should explore more efficient methods of finding these coalitions. For instance, students could organize their work according to the number of states in each winning coalition (see table 3).

To determine the Shapley-Shubik Power Index, students need not find all 6 ! permutations. Because four states have the same weight, only 30 permutations need to be considered.

Table 4 represents the Nominal Power Index, the Banzhaf Power Index, and the Shapley-Shubik Power Index for the six swing states.

Both the Banzhaf and the Shapley-Shubik Power Indices clearly indicate that Florida has much more power ( $39 \%$ and $40 \%$, respectively) than its Nominal Power Index (27\%). Moreover, Dubey and Shapley concluded: "The actual numerical values that issue from the Banzhaf and Shapley-Shubik models are quite often similar, and the two can be regarded as equivalent for many practical purposes if we grant that law and politics are far from being exact sciences" (1979, p. 100). The comparable power values for the other five states seem to support this assertion. (For a comprehensive treatment of the power indices, see Taylor, Conrad, and Brams [2013].)

## IMPLICATIONS FOR PRESIDENTIAL ELECTIONS

In recent years, Republican presidential candidates have ceded very liberal states such as New York, and Democratic presidential candidates have ceded very conservative states such as Texas. One website (www.270towin.com) projects that for the 2012 presidential election there will be ten swing states: Florida (29 electoral votes), Pennsylvania (20), Ohio (18), North Carolina (15), Virginia (13), Wisconsin (10), Colorado (9), Iowa (6), Nevada (6), and New Hampshire (4).

During the previous presidential elections, various articles (see, e.g., Fournier 2004; Sidoti 2004; Elliott 2008) provided glimpses into campaigns' decision making, setting of priorities, and allocation of

Table 4 Power Indices for Swing States

| States | Nominal Power Index | Banzhaf Power Index | Shapley-Shubik Power Index |
| :---: | :---: | :---: | :---: |
| FL | $\begin{gathered} 25 / 92 \\ \text { (about } 27 \% \text { ) } \end{gathered}$ | $\begin{gathered} 22 / 56 \\ \text { (about 39\%) } \end{gathered}$ | $\begin{aligned} & 12 / 30 \\ & (40 \%) \end{aligned}$ |
| PA | $\begin{aligned} & 23 / 92 \\ & (25 \%) \end{aligned}$ | $\begin{gathered} 10 / 56 \\ \text { (about 18\%) } \end{gathered}$ | $\begin{gathered} 6 / 30 \\ (20 \%) \end{gathered}$ |
| MO | $\begin{gathered} 11 / 92 \\ \text { (about } 12 \% \text { ) } \end{gathered}$ | $\begin{gathered} 6 / 56 \\ \text { (about 11\%) } \end{gathered}$ | $\begin{gathered} 3 / 30 \\ (10 \%) \end{gathered}$ |
| TN | $\begin{gathered} 11 / 92 \\ \text { (about } 12 \% \text { ) } \end{gathered}$ | $\begin{gathered} 6 / 56 \\ \text { (about 11\%) } \end{gathered}$ | $\begin{gathered} 3 / 30 \\ (10 \%) \end{gathered}$ |
| WA | $\begin{gathered} 11 / 92 \\ \text { (about 12\%) } \end{gathered}$ | $\begin{gathered} 6 / 56 \\ \text { (about 11\%) } \end{gathered}$ | $\begin{gathered} 3 / 30 \\ (10 \%) \end{gathered}$ |
| WI | $\begin{gathered} 11 / 92 \\ \text { (about 12\%) } \end{gathered}$ | $\begin{gathered} 6 / 56 \\ \text { (about 11\%) } \end{gathered}$ | $\begin{gathered} 3 / 30 \\ (10 \%) \end{gathered}$ |

resources. President Obama's campaign is projected to raise $\$ 1$ billion toward his reelection. Particularly in the swing states, the candidates and their respective campaign teams will decide how much time the candidates will spend there and how many advertisements to purchase. The Banzhaf and the ShapleyShubik Power Indices provide objective, mathematical decision-making tools for these deliberations.

## IMPLICATIONS FOR

## MATHEMATICS EDUCATION

The Banzhaf and the Shapley-Shubik Power Indices are based on probabilistic models. An understanding of combinations and permutations as counting techniques is a prerequisite. NCTM's Principles and Standards for School Mathematics (2000) has advocated that students learn these concepts in grades $9-12$. Recently, the Common Core State Standards for Mathematics has received much attention from the mathematics education community. For high school students, this document states: "Use permutations and combinations to compute probabilities of compound events and solve problems" (CCSSI 2010, p. 82). Surprisingly, this is the only standard related to the concepts of combinations and permutations. Further, this particular standard is described as "additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics" (CCSSI 2010, p. 57). High school mathematics courses such as algebra, precalculus, or discrete mathematics will typically cover the concepts.

## THE 2012 PRESIDENTIAL ELECTION: A LESSON PLAN

With regard to students' understanding of probability, Shaughnessy (2003) recommends: "Adopt
a problem-solving approach to probability. Give students opportunities to investigate probability problems or chance situations on their own and to conduct their own stochastic projects" (p. 224). Specifically, a lesson on the power indices should entail the following:

- The teacher chooses a lesson theme and assigns tasks to students. For example, students may volunteer to serve as campaign advisers for one of two candidates, President Barack Obama or Governor Mitt Romney.
- A class discussion on the importance of the swing states should lead to this research question: How much money should each candidate allocate to the designated swing states? Computing the power indices for the ten swing states from the 2012 U.S. presidential election will overwhelm students. Teachers may want to limit the number of swing states to three or four and provide students an option for selecting their preferred swing states.
- Students should be grouped by twos or threes and given opportunities to conjecture, investigate, and share ideas (Copeland 1984). Most likely, students will distribute the money according to the states' Nominal Power Index values. However, the class should take time to attend to other opinions and the respective rationales.
- The teacher introduces the Banzhaf and the Shapley-Shubik Power Indices.
- Students determine the Nominal Power Index and the Banzhaf and Shapley-Shubik Power Indices values for their swing states. To provide students ample time, the teacher should assign this part as homework.
- A class discussion on all the groups' findings will conclude the lesson. The premise in using the Shapley-Shubik Power Index is that the permutations of the swing states are equally likely. However, the permutations of knowing the winners of the swing states are not equally likely. In particular, the teacher may choose to point out that polling places on the East Coast close before those on the West Coast. Shortly after the polls' closing, the media, quite accurately, announce the projected winner for each state. Knowing the winner of Colorado, a western swing state, before the winner of New Hampshire, an eastern swing state, is unlikely.

The 2012 U.S. presidential election is predicted to be another close race. This setting provides an authentic context to build students' mathematical knowledge. Having learned the Banzhaf and the Shapley-Shubik Power Indices within the social sciences context, students will appreciate the utility and relevance of mathematics. Specifically, defining the precise notions, establishing certain assumptions, determining combinations and permutations of identified sets, and calculating the power indices will develop students' mathematical reasoning.

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