Teacher Recognition and Acceptance of Student Methods of Subtraction

Kristi Hanby

University of Michigan

Introduction

Since Shulman (1986) first introduced the notion of *pedagogical content knowledge*, researchers have investigated the components of the domain that make up the overlap between the pedagogical and the content knowledge used in teaching. Particular to mathematics teaching, Ball, Thames, & Phelps (2008) identify particular forms of Mathematical Knowledge for Teaching, one being the knowledge of content and students (KCS), which includes the knowledge required to hear and interpret student mathematical thinking, and another being specialized content knowledge (SCK), which includes knowledge that aids in understanding whether a particular method is mathematically valid or not. Ball et al. (2008) argue that to teach mathematics, teachers need to know both the mathematics of the tasks they ask students to perform and common patterns of student thinking, while also holding the ability to evaluate those students' methods mathematically.

Investigations of student thinking in early subtraction have identified at least two mathematically valid methods used by students in order to make use of a backward count when subtracting (Fuson, 1984). While the two methods are similar in that they both employ a backward count, there is a distinguishable difference. When a teacher recognizes the difference and brings it to the class for discussion, she opens up the instructional space so that students are able to consider the similarities and differences between their methods. Conversely, failing to recognize the difference and proposing that there is just one method for counting backward could narrow the instructional space and lead to possible confusion for students who use the unrecognized method.

In this paper, I study the responses teachers give to students who use the two early subtraction methods to determine if teachers recognize the difference between the two methods and also how they choose to treat them instructionally. I investigate whether teachers are willing to invite the complexity of the instructional space around this mathematical subject, or if they choose to narrow the space and exclude what I hypothesize to be a non-normative method.

This study is meant to determine if teachers recognize, and furthermore are willing to accept, as mathematically valid, students' differing procedures for subtraction when using backward counting to solve a take-away subtraction task. By presenting teachers with representations of instruction in which students share their varied methods, I investigate if a norm exists in instructional situations that involve early subtraction and in what ways the instructional space is open or closed to the interactions of a student sharing his or her preferred method. Appraisal analysis of teacher responses is used to determine if teachers accept students' varied, yet valid, procedures for carrying out a backward counting method and also to investigate how they choose to treat the work of students when they have applied either of the two methods of subtraction under investigation. In doing so, I consider if teachers show evidence of having knowledge of the student methods and in what ways teachers choose to respond to the students' methods, by considering what actions they would choose to carry out, and what reasons they give for these actions.

Norms of Teaching Practice

An instructional situation, as described by Herbst (2006), is the set of common characteristics of instances of classroom instruction in which the participants have customary ways of behaving that permit the teacher to readily exchange the work students do in a type of task for a claim on a specific item of knowledge they are supposed to have or acquire—"for a given course of studies (e.g., algebra I) an instructional situation can be represented by a canonical task (e.g., solving an equation) but the situation itself includes the tacit agreements that determine what needs to be done to do the task, by whom, and when" (Herbst, personal communication, August 20, 2014). The situations and the patterns of activity within them "regulate how teachers and students relate to each other and to the specific subject matter they are learning," (Herbst, Aaron, Dimmel, & Erickson, 2013, p. 2) and by engaging in them the teacher is able to readily exchange the work students do for a claim on the knowledge they are supposed to have or acquire. As teachers and students engage in normative activities in the classroom, each party comes to understand what is expected of them in the particular situation in ways that are held tacitly by participants in the group. However, because they are normative, deviations from these customary activities may call them to the attention of the participants.

To investigate the norms of an instructional situation, researchers have analyzed teacher responses to representations of instances of an instructional situation (Chazan & Herbst, 2012) which are meant to breach (in the sense of Mehan & Wood, 1975) a hypothesized norm. If a norm does exist in a particular instructional situation, then presenting teachers with an instance of a situation that includes a breach of that norm, could produce responses in which it is possible to understand the boundaries of the situation, what stands out as a deviation from the norm and what is offered as a way of describing the differences that are seen in the situation from what may be normative, showing how they may "repair the situation and reduce the magnitude of the breach" (Chazan & Herbst, 2012, p. 20). The presence of a repair of the situation begins to make visible how the norm holds in the situation (Herbst, Kosko, & Dimmel, 2013). Detection of the norm, then, consists of considering the extent or frequency with which teachers, across responses, call attention to the breach. Furthermore, the reasons teachers give for their responses provide evidence to further understand the norm and its boundaries.

While patterns of behavior can be normative, this does not necessarily imply that teachers will always follow normative patterns of behavior. Teachers may find reasons to depart from the norm and when they do, they may justify these departures by using, as a warrant, the professional obligations they have to particular stakeholders (Herbst & Chazan, 2012). In other words, the obligations act as a means by which a teacher could justify a departure from the norm. For example, a teacher may intend to carry out an explanation of an algorithmic procedure, a possible normative activity in which the students might be asked simple, straightforward questions that aid in supporting the teacher's explanation. If an individual child raises their hand to say that the teacher is not doing it the way it was taught to them by another adult, the teacher may deviate from the normative activity and begin to ask questions of the student to explain how they might carry out the algorithm. Of course, the teacher isn't required to follow up on the student's comment, but may choose to do so, justifying this action as an attention to the needs of an individual student. Herbst and Chazan (2012) identify four stakeholders to which the teacher holds professional obligations – the individual students, the interpersonal nature of the classroom, the discipline of mathematics, and the institutions in which teachers carry out their work.

The investigations into instructional situations in mathematics teaching have probed high school geometry instruction in proof (Herbst, Chen, Weiss, & Gonzalez, 2009) and theorems (Herbst, Nachlieli, & Chazan, 2011) and algebra instruction exploring solving equations (Chazan & Lueke, 2009) as well as word problems (Chazan, Sela, & Herbst, 2012). As yet, no studies have been conducted that probe for norms in elementary mathematics classrooms. I propose that studies of norms into elementary classrooms have the potential to be particularly fruitful. Because the cognitive study of children and their thinking in mathematics has been so well documented, studies of norms in elementary classrooms have the potential to probe into teaching practice that intersects with what is known in the literature regarding students' informal methods in mathematics. If students' informal methods are unexpected or unrecognized by a teacher, the student's work could act as a breach of the norm that would originate from the child. The research into student thinking in elementary provides evidence that word problems given to students are likely to elicit various informal methods for solving (Carpenter, Hiebert, & Moser, 1981) which may or may not be expected by the teacher (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). If we take these informal methods as likely to occur in every classroom, then the breach of the norm has the potential to occur with any given problem that is presented by the teacher and can occur spontaneously from the child. Studies that are meant to breach what is normative can be used as a mechanism, then, to present teachers with what is known in the literature regarding student methods, but possibly non-normative for instruction, a phenomenon that would indicate a gap in teaching practice. The current study, then, is meant to provide a means for determining what teachers notice about these methods and whether particular methods for solving are normative. Investigations of this nature can further add to what is known about teaching practice by analyzing responses in order to understand if and why teachers may respond differently to some student methods than others.

Early elementary norms of subtraction. In this study, I investigate the possibility that the informal method of countable subtraction known as the cardinal method (Fuson, 1984) may contradict norms of the early elementary classroom, even though it can be carried out accurately and is mathematically appropriate for a subtractive situation. I posit that teachers view subtraction as a measurement process in which the procedure is carried out by traversing the intervals between numbers on the number line, rather than by counting backward each object as it is removed in a take-away situation. I hypothesize that the cardinal subtraction method breaches this norm and I design an instrument in which teachers are presented with student work, both measurement and cardinal, in response to a subtraction problem. If a norm exists that favors procedures of a particular nature, then it is expected that teachers may call attention to the cardinal method by offering a repair (Herbst & Chazan, 2012). Analyzing instances of repair could provide the means to identify the reasons why teachers may or may not adhere to the norm and would add to an understanding of teacher decision-making.

Responding to Students in Classrooms

While student methods of subtraction could occur in many occasions in which the child is subtracting, teachers are often called upon to respond to student methods when they occur during a class discussion. Teacher responses to students during this time of sharing are important for the learners in the classroom, but are also a productive space in which to investigate the norm since during discussion teachers may consider more than just the child and their method. They may also consider how the other students will react to different methods, what the teacher is conveying about mathematics in general, and whether what the student has said aids in the work they are expected to accomplish. For these reasons, I discuss both why it is important that teachers are able to conduct appropriate interpretations and responses to student methods and describe why these responses are one aspect of the work of teaching that has been shown to be particularly difficult to enact.

As teachers engage students in the discussion of problems, student thinking is made available in the classroom and has the potential to improve student learning, but also requires that a teacher be prepared to respond to students and their ideas. The interactions between students and teacher allow for knowledge to be shared, pondered, and more deeply understood by students in the classroom (Hiebert et al., 1996). The benefits for students that can be realized through the sharing of student thinking are also extended to teachers as they "develop a deeper knowledge about students' thinking" (Carpenter, Fennema, & Franke, 1996, p. 16). When teachers are attentive to student thinking and use it as a basis for response to students, Carpenter and his colleagues argue that teachers open up opportunities for students to learn from each other, but also for teachers to learn from and for their students.

If responses to students are beneficial when there are appropriate interpretations of the students' work, then a norm that favors one method over another would be at least unsupportive of some students, but possibly also detrimental to both the teacher and other students in the classroom. However, one of the particularly difficult tasks of teaching occurs when students in the classroom share knowledge and teachers position themselves to interpret and respond (Ball, 1997). In these interactions, teachers have the opportunity to listen to students and through appropriate interpretation and response, leverage that student thinking in order to make it shared among participants in the classroom. If both students and teachers build knowledge when they engage in discussion around student thinking (Hiebert et al., 1996; Carpenter et al., 1996), conducting appropriate responses to students would be beneficial for both parties and it would be fitting to consider which student methods are more or less difficult for teachers to interpret.

Difficulties in Responding to Students

Ball (1997) describes one role of the teacher as needing to manage two simultaneous realities, one which holds disciplinary knowledge of mathematics and the other which contains an understanding of students and the work that they do. A teacher, being more knowledgeable of the content, is asked to be an ambassador that connects the more formalized disciplinary knowledge with what she observes students doing in their more rudimentary efforts. Being responsive, then, includes making use of knowledge in both of these domains to understand students, make sense of their work, and support students as they develop understandings of the content.

Rather than seeing the teacher as someone who goes between two domains, Wallach and Even (2005) describe that teachers 'hear though' their own personal and social resources, sometimes causing "what the teacher hears . . . to be different from what students are saying or doing" (Wallach & Even, 2005, p. 411). The experiences that teachers accumulate over time may be necessary in order for teachers to connect their more formalized knowledge to that of the learner, but they may also contribute to misinterpretations of student work. This can lead teachers to respond in ways that overlook key aspects of student work or add information to it (Wallach & Even, 2005). If teachers 'hear through' their own experiences and proper interpretations are difficult, then a complete understanding of student thinking may be inhibited by the mathematical experiences the teacher brings with her to instruction. For example, Maher and Davis (1990) found that when the student's representation of the problem situation did not match the teacher's representation, the mismatch caused the teacher to come to the aid of the student by exposing the 'error', having a sense that they are helping, even if the student is not incorrect. Putnam (1987), studying teachers in one-on-one tutoring situations showed that "rather than diagnosis, the teacher's primary goal was to move through [an] organized and sequenced set of skills and activities . . . to correct any student difficulties or misconceptions that might arise" (Putnam, 1987, p. 17). By focusing on the correct response and the mechanics of the algorithms, the teachers spent their time correcting the way the procedures were carried out, rather than determining why a student might think it was appropriate to do the problem in such a way. Each of the above examples could be seen as a case of teaching in which there were misinterpretations of the student's work. What are not described are the complexities of the work of teaching that go on beyond the work that is done between an individual student and the teacher, what the teacher must also manage as these interpretations are occurring.

What some researchers describe as inconsistencies between student thinking and teacher interpretation are also worth considering from the social perspective of the classroom. Not only are there individual actors in the classroom who are learning and teaching, but these actors engage with others in the classroom and consider each other's different ways of working a problem, which can sometimes seem incommensurate with each other. These ideas can become more complex when one tries to reconcile the varied ways that multiple students carry out their work. From this perspective, it is possible that teachers act to maintain some level of consistency within the classroom. This would have the outcome of decreasing the level of complexity that must be managed by the teacher as students share their varied methods and somewhat contain the variability in student work (Cohen, 2011). As Cohen describes,

> "The more constrained and mechanical the teachers' treatment of knowledge, the simpler the intellectual connections that teachers and students make, and the more discrete each connection seems. When teachers and students construe knowledge in this way, they can relatively easily make direct connections . . . In contrast, the more expansive and flexible the treatment of knowledge, the more complex each connection is, and the less discrete it seems. When teachers and learners construe knowledge in this fashion, the intellectual connections they seek are more complex and less direct. . . Mutual understanding is easier in the first case than in the second. Knowledge is a crucial medium for regulating the connections between teaching and learning, but the nature of those connections varies with how knowledge is construed and extended" (Cohen, 2011, p. 35).

If the complexity of the instructional space depends on the treatment of knowledge during instruction, then the teacher plays a role that puts her own comfort at odds with the goals of instruction that attend to student thinking. Namely, the teacher holds the ability to expand or contract the knowledge that is made available in the classroom. In moments when the teacher is called upon to understand the work of the student and connect it to what they know about mathematics themselves, they have their experiences to draw upon and those experiences could determine the breadth of knowledge the teacher is willing to traverse in exploring student ideas. If the teacher's experiences do not include knowledge of what students might do in response to a mathematics task, it might be seen as more comfortable to maintain adherence to what is normative and predictable. If we take this to be plausible, then the teacher's experiences could either compel them to adhere to the norm, or possibly their experiences could allow them to deviate. This would indicate that more than just understanding the formalized mathematics being taught and knowing the students in the classroom, teachers may benefit from other forms of knowledge that aid in the work of listening and responding to student work.

Knowledge for Teaching

Ball and her colleagues (Ball, et al., 2008) argue that the knowledge needed to teach mathematics is broader than knowledge of the subject matter itself. In addition to knowing how to carry out the mathematical procedures that students must learn, teachers must also understand how students come to learn the particular content and in what ways they must be supported in order to understand.

While as early as 1902, Dewey gave attention to the difference between the logical aspect of learning and the ways in which teachers must psychologize the content in the context of school and learning, more recently Shulman (1986) spawned an entire field of study around what he coined *pedagogical content knowledge.* Going beyond knowledge of the subject matter, pedagogical content knowledge includes “an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons” (Shulman, 1986, p. 9). In other words, in addition to understanding the mathematics that will be taught to children, teachers should also understand how students first approach their mathematics and make sense of specific topics, since we can expect that “those learners are unlikely to appear before [teachers] as blank slates” (Shulman, 1986, p. 10).

Mathematical Knowledge for Teaching

In their work to further define pedagogical content knowledge, pertaining to the teaching of mathematics, Ball and her colleagues (2008) identify different facets of teacher knowledge through the close analysis of video representations of teaching practice. They break mathematical knowledge for teaching into content knowledge and pedagogical content knowledge which are further broken down to more clearly describe the varied ways in which teachers must engage in mathematical work that is unique to classrooms, and not likely to occur in other professions. The categories of interest to this study include common content knowledge (CCK) and specialized content knowledge (SCK), both subsets of subject matter knowledge, and knowledge of content and students (KCS), a subset of pedagogical content knowledge. CCK is that knowledge used in settings other than teaching, as well as in teaching, such as canonical procedural knowledge used to carry out procedures to solve a mathematical problem. Once the procedures of mathematics have been learned, this form of knowledge is efficient and predictable and serves as a useful tool to carry out tasks of everyday life. The knowledge of content and students and specialized content knowledge, on the other hand, are considered to be unique to teaching. While someone could use their common content knowledge (CCK) to solve a problem given to an elementary student, teachers must also make use of knowledge of content and students (KCS) to consider the particular ways in which children think about that mathematical concept, to “anticipate what students are likely to think” (Ball et al., 2008, p. 402). Since students do not always solve problems in ways that are common, teachers must be prepared to “figure out what students have done, whether the thinking is mathematically correct for the problem, and whether the approach would work in general” (Ball et al., 2008, p. 397). This entails an analysis of the student’s work in light of what the teacher knows of mathematics, to consider if a student’s method is valid, one aspect of the form of mathematical knowledge for teaching known as specialized content knowledge (SCK). This form of knowledge allows teachers to consider student work and common patterns of error, be flexible in thinking about the variety of methods students use, and consider if they are mathematically valid or efficient to be used effectively. This ability to analyze student work in light of what is true in mathematics would lie between the nature of student's understandings and the nature of mathematics itself, between the mathematical meaning of subtraction and an understanding of a child's work. It provides the resources for teachers during their interactions in the classrooms to listen to students and hear their fledgling ideas and consider those ideas on mathematical grounds.

I posit that knowledge of student methods, such as would be defined by mathematical knowledge for teaching, could provide an allowance for a teacher to deviate from the norm and attend to a particular student's thinking or the understanding of the others in the class. If it can be found that there is a norm that privileges one method over another, then an analysis of the responses in which teachers do deviate from the norm would help us understand under what conditions teachers are willing to sanction a student's non-normative method, possibly leading to benefits for both students and teacher.

Up until this point, I have described the demands of teaching mathematical concepts in general, yet I have not described what could make the teaching of early subtraction difficult, a seemingly simple topic. In the following sections, I describe what is particular to the instructional situation of early subtraction that could create difficulties for teachers as they interpret student work.

What Makes Early Subtraction Complex?

Early subtraction, the subtraction that can be done by counting, could be seen as mathematics that is a relatively simple aspect of what makes up a teacher's common mathematical knowledge, her CCK. While fairly simple for most adults to carry out, for learners, subtraction is not straightforward. While the average adult either knows these basic subtraction facts from memory or does a quick calculation to determine them, those in the process of just learning this concept will repeatedly carry out procedures to determine the difference between two numbers as if it were the first time the problem has been worked, until the facts are known more fluently. For a teacher to be well prepared, they must consider which problems will elicit student work in subtraction, how students are likely to go about that work, and how to respond when students share their varied and sometimes rudimentary methods. One way for teachers to carry this out is to ask that students share the way in which they came to an answer. With varying levels of detail available to them, teachers then assess the information they have gathered from students and use it to guide the instruction in response to the students' work.

Student Methods of Subtraction

To consider a teacher's response to students' informal methods of subtraction, it is important to first consider what is known about these student methods in early addition and subtraction. Interviews with students have shown that children have logical methods for applying the mathematics which they bring with them from their environment, to solve additive and subtractive problem situations, and it is feasible to consider that children will not all use the same strategy to subtract. By introducing students to problem structures which were instances of additive and subtractive situations, Carpenter and his colleagues (1981) showed that students invent strategies even prior to instruction and that when particular problem structures are introduced to students there is a set of expected strategies that children use. Children's earliest attempts at addition and subtraction are done as they count objects in collections to put together, take apart, or compare (Carpenter et al., 1981). Use of a forward or backward count, in conjunction with a concrete or abstract representation, allows children to use counting as a way of manipulating numbers, adding to their growing understanding of mathematical concepts. In order to consider the different counting methods that are of importance to this study, I first describe foundational skills that are necessary in order to enact a counting strategy.

Decomposition of a count. Subtraction that is based on counts can be broken down into three necessary components. The first, the forward or backward number word sequence, is the recitation of number words in sequence, simply 15, 14, 13, 12, for example. Whether starting from the number one or starting somewhere else in the sequence, in order to have an appropriate number word sequence, a child must be able to string the number words in a correct sequential order in a forward or backward manner. This is an early skill for being able to add and subtract and most children acquire at least the ability to recite the forward number sequence in their home. However, this number sequence alone does not constitute a count. In addition to this number sequence "there must be items to count; and, there must be a procedure to make each utterance of a number word to coincide with one of the items that are to be counted" (Steffe, vonGlasersfeld, Cobb, & Richards, 1983, p. 2).

One place in which the counting activity could be recognized is when a child points to an object and counts one, then continues to the next calling it two, and so on. The nature of a child's counting typically progresses from this concrete counting, in which the objects are visible, toward a more sophisticated counting procedure in which the items to which the count corresponds become abstracted. In the concrete instance, the child is able to make use of the visible items by tracking each correspondence between a number word and an object with his eyes or fingers. The items are visible and the correspondence between them and the child's number sequence would be apparent to a watchful observer. In the case of an abstract count, the items have been internalized mentally and the correspondence to the number word sequence becomes more difficult to detect, even for a conscientious observer, since the correspondence is not made apparent to the observer. The correspondence, then, must be inferred.

Subtraction using a forward count. Using generic structures of additive and subtractive situations, Carpenter and his colleagues determined that particular problem structures would elicit particular strategies from students (Carpenter, et al., 1981). The focus of that work was to determine the relationship between certain problem structures and the strategies they are likely to elicit from students. Of interest to this study are those problems that elicit a backward counting procedure, which is true of what Carpenter calls the *separating* problem type. One of seven subtractive situations, the *separating* problem structure is also referred to by teachers as a take-away problem. The generic structure, "Leroy had *a* pieces of candy. He gave *b* pieces to Jenny. How many pieces of candy did he have left?," can be used to produce multiple problems of this form (Carpenter et al., 1981). A common early strategy used to accomplish this task is done with concrete items and carried out by using only a forward count, also called the *separating* strategy (Carpenter et al., 1981). Using the *separating* strategy, the student establishes a collection of concrete objects that match the minuend, then takes away a number of objects that indicate the subtrahend and counts the items that remain. Thus, to solve the problem of how many objects are left from 12 after removing 4, the child would count out 12 objects to make a collection, then count forward as they remove 4 objects from the collection, and finally use a forward count to enumerate the amount remaining in the collection. This strategy is useful for students as they begin to understand the concept of subtraction and requires a relatively low cognitive load given that the items are visible and able to be manipulated and students only need to make use of a forward number word sequence in its execution. In the context of instruction, it might also be easily identifiable since the items are visible and the student's correspondence between the count and the items is apparent. However, while it's not difficult to identify, it's also not possible for the child to use this strategy for any extended period of time since it cannot be accomplished once the concrete materials have been removed and further sophistication of the student's strategy is required in order to work more advanced problems.

Subtraction using a backward count. The *separating* method described above, of course, cannot persist indefinitely since students will necessarily have to subtract without the use of objects at some point in their development. Thus, students will need to advance to the use of a strategy that does not make use of concrete items and instead relies on their abstraction of a sequence of numbers within the range of the subtraction problem. In doing so, the student must be able to make use of a backward number word sequence, since the forward count of the items that remain after separating out the subtrahend would be impossible to enumerate once the items are no longer tangible and using a strategy based on counting up is not likely to be used until later in development (Steffe et al., 1983). Thus, early on and in the absence of concrete objects, one would be most likely to observe the student using a backward number sequence that coincides with an abstract set of items, a backward counting procedure. "This procedure entails a forward count to keep track of the subtrahend . . . [that is] executed while the child counts backward" (Baroody, 1984, p. 205), a noticeably more demanding task. In other words, using an abstract representation, the child must count backward to enact the operation of subtraction while simultaneously tracking the number of counts that have been traversed. Thus, when asked to solve the problem 8 – 5, a student might say, "8 -, 7, 6, 5, 4, 3. The answer is 3" (Carpenter & Moser, 1984, p. 35), while also tracking that they had accomplished 5 counts, a mental correspondence between the subtrahend and the number of backward counts.

If a teacher were to observe the use of this method, it would be possible to distinguish whether the student had an appropriate backward number sequence which corresponds with an appropriate number of counts, matching the subtrahend. However, an inference would need to be made to explain the correspondence to some mental set of items, the nature of which remains unknown. This need to infer what has been counted by the student turns out to be a crucial indicator for understanding the student's method.

Learning Subtraction

In the absence of careful investigation, the correspondence between a student's count and a mental item can only be inferred, a difficult reality connected to the work that students do and the ability of a teacher to interpret that work. The correspondence between a child's number word sequence and the representation is observable when the student is using concrete items, as it can be seen externally, but it becomes relatively unknown to an observer when students transition to an abstract count, since "it is logically impossible. . . to compare a representation with something it is supposed to depict" (vonGlasersfeld, 1995, p. 93). We cannot expect that by listening to a child's count, we can know for sure what they have actually been counting. They could be counting bears or jellybeans, a seemingly unnecessary distinction, but they might also be counting on a numberline, which elicits a different mathematical operation than the jellybeans. Through close observation during cognitive interviews, researchers (Carpenter et al., 1981; Steffe et al., 1983) were first able to construct a plausible abstract representation of a child and I propose that the knowledge of how students count in this way relates to the mathematical knowledge for teaching (Ball et al., 2008).

Discriminating two methods. By giving students access to material objects, there is a natural tendency for a child to use a strategy that requires the least cognitive demand. For instance, in cognitive interviews, when this access was given, each student attempted the forward counting strategy called *separating* (Carpenter et. al, 1981, Carpenter & Moser, 1984), removing from a set of objects equivalent to the minuend, the items indicated by the subtrahend, and counting what remains. In other studies when materials were not available, the same researchers observed a backward counting procedure, which "contains as many counting number words as the given smaller number. The last number uttered in the counting sequence is the answer" (Carpenter et. al, 1981, p. 35). Thus, for the problem, 14 – 3, the student would say "13, 12, 11" which contains three counting number words and lands on the appropriate response of 11. While a teacher could observe whether the student's backward count is properly sequenced and if it involves the correct number of counts, the question of the correspondence to the representation is left to interpretation. The correspondence to the representation is made only slightly more clear in another observation of a child who states the following to solve the problem 5 – 2, "Five, four is one, three is two" (Baroody, 1984, p. 205). In this latter description, the child articulates the backward number sequence as he makes clear the correspondence of his backward count to that of the forward count. And while it is not typical to verbalize the forward number sequence that tracks the subtrahend, it is helpful in beginning to understand the student's use of the strategy, although there is still not a clear indication of what the student is counting. Still, the description of these two students' work is likely acceptable to most adults as an appropriate subtraction strategy for a problem posed as a take-away situation.

Whereas Carpenter and his colleagues (1981) allowed full access to concrete items, Steffe and his colleagues (1983) chose to allow limited accessibility to material objects, preferring instead to observe student methods without the materials, then giving access when this was needed to support their work. Access was given by allowing the objects to be available visually during the delivery of the problem, meant to give the child a sense of the quantities with which he would be working. After having delivered the problem, the objects were then covered with a screen while the child was given time to work through the problem. This initial method is not meant to create an overly demanding task for the child, but rather it is used to encourage the student to use the most sophisticated method they have available to them. If it is observed that the child is unable to complete the problem with the screens in place, they would be removed and the student would be allowed to solve the problem by using the materials to track the count. In the absence of visible materials to support the students in the use of a forward counting strategy, there was a possibility for more abstracted procedures to be elicited, which allowed researchers to recognize more variability in children's backward counting procedures. Compare the child's description of the previous backward counting procedure with a child who was asked to solve 10 – 4 who did not have the objects available to manipulate, "See, I count back. 10 is 1; and 9 is 2; - and 8 is 3; and 7 is 4, so it is six" (Steffe et al., 1983, p. 108). In this method, the child again has an appropriate backward number sequence and articulates the correspondence to a particular representation. However, while the components seem similar to the previous description, a nuanced difference is detectable between the methods employed by this child and those of previous studies. Whereas for one child the correspondence to the representation begins on the number one less than the minuend, the other begins with the correspondence to the minuend itself, a difference that brings the correspondence of the count to the representation into question.

On review, it can be seen that each student has derived a method that when used appropriately consistently arrives at a correct response. Given that students use both methods and teachers must interact with these students in group settings, understanding of the differences would be important teaching knowledge for elementary mathematics.

A closer look. The difference seen in the children's' subtraction behaviors is considered by Fuson (1984) as evidence of two separate methods for counting back to solve a problem that can be described generically as M – S = D. In what Fuson calls the *measurement* method, the student begins to articulate the correspondence to the representation when they state the number one less than the minuend, M – 1. In this case when the student has enacted the appropriate number of backward counts, S, the student will have arrived at the appropriate response. For example, to execute 8 – 3 using the *measurement* method, the student would consider the interval between 8 and 7 as their first count, the interval between 7 and 6 as their second, and the interval between 6 and 5 as their third, giving 5 as their response, as they have tracked the appropriate number of counts. This is shown in Figure 1 using the measurement model of a number line, *a second-order model* (i.e., a model the observer makes of the child's operations; Steffe & Kieren, 1994) that is representative of the work done by the child and results in the same conclusion as the child. The second order model, however, cannot be attributed directly to what a particular child imagines, the *first-order model* (Steffe & Kieren, 1994), as this is unknown to an external observer.

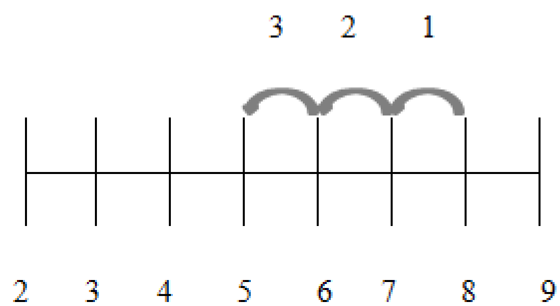

Figure 1: Second order model of the use of the *measurement* method to solve 8 – 3.

In using the other method, called the *cardinal* method, the student begins to articulate the correspondence to the representation on the minuend, M, and after the appropriate number of items have been counted, he will have said the number that is 1 larger than the number that indicates the difference, which necessitates that they say the next number in the sequence in order to give an accurate response. Thus, 8 – 3 sounds like "8 is 1, 7 is 2, 6 is 3 [the count is complete] and so the answer is 5." The correspondence between the number name of the object being removed and the number of counts indicated by the subtrahend is shown in the use of the language that connects 8 with 1, 7 with 2, and 6 with 3. Although children sometimes make use of this strategy, for most adults this seems like the more unusual strategy and may create difficulties in understanding the child. Steinberg explains this ambiguity by translating the student's *cardinal* method, describing "I am taking away the eighth (one finger up), the seventh (two fingers up), and the sixth (three fingers up), so five are left" (Steinberg, 1985, p. 175), as if the child is indicating the number name of the item as it is taken from the collection. As shown in Figure 2, the second-order model of this work represents that the child indicates the number name for each object that is abstractly removed during their backward counting procedure.

Again, the representation that the child is counting can only be inferred, but this representation is consistent with the child's thinking and results in the same conclusion. In this instance, the items that are indicated by the numbers 8, 7, and 6 are removed. When used accurately, the child then indicates that there are 5 objects remaining.

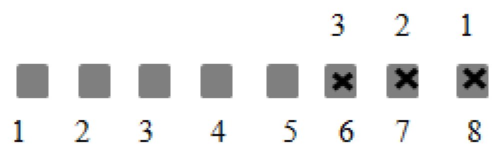

Figure 2: Second order model of the use of the *cardinal* method to solve 8 – 3.

.

Considered together we can see that children who use the *measurement* method, as well as those who use the *cardinal* method enact a backward count of an appropriate number of items, and the number sequence adequately corresponds to a valid mathematical representation. However, the constructed representations differ in that one is a disjointed set of objects and the other indicates the nature of the intervals that are found between each whole number in a sequence. This difference in the representations has a significant impact on the correspondence of the backward number sequence to the representation itself, in particular where the count begins and where the count ends. While students are likely to be observed using their fingers to accomplish either of the two methods, the use of fingers does not help make the distinction between the two methods, since the use of fingers does not make the correspondence to the objects of the intervals clear to an observer. This clarification between *measurement* and *cardinal* subtraction methods is one way in which KCS and SCK play a role in the work that teachers do. Armed or not with knowledge of the two methods (KCS) and disciplinary knowledge that they are both valid methods of subtraction (SCK), a teacher could choose to accept or not, each of the student methods of subtraction. If they possess this knowledge and reject the student's cardinal method, then it seems likely that their reasons for doing so would be based on something other than the mathematical appropriateness of the method. If they do not possess this knowledge, then a rejection of the method might be based on lack of knowledge or some justification based in instruction or both.

Purpose of the Study

The design of the instrument for this study and its analysis are meant to begin to understand the boundaries around the instructional situation of early subtraction, including what teachers notice about student methods, how the teacher responds to students who use informal methods, and how these responses are justified. For this purpose, I represent an early elementary classroom in which the teacher has posed a take-away problem and has prompted a student to share how they came to their solution. Through the use of an online storyboarding tool, I am able to represent multiple plausible scenarios of the situation of early subtraction in which the student uses a particular method to subtract. In analyzing teacher responses to the scenarios, the intent is to determine what teachers notice about the student's method and how that teacher expects she would respond in the classroom to the student who has just used this method, in order to determine if a norm exists in preference to the measurement method as well as what reason teachers may give if it is found that a norm exists. This includes an analysis to determine if teachers recognize the strategies as two unique methods and in which ways they treat the response from the student.

The Study

Instrument Design

General description of an item. One way in which teachers encounter student methods of subtraction is when they engage students in a mathematical task and prompt them to share their reasoning once they have solved the task. For this reason, I scripted context-based scenarios in which students in an early elementary classroom have been given a subtraction problem and been asked to share their thinking. In each scenario the problem is of the *separating* (Carpenter et al., 1981) form. In the work done by Carpenter and his colleagues, the generic example, “Leroy had *a* pieces of candy. He gave *b* pieces to Jenny. How many pieces of candy did he have left?” (Carpenter et al., 1981, p. 31), a backward counting procedure was identified as a typical method for a child to enact. In the scripted scenario, the teacher brings the problem to the attention of the students, giving them time to work through it on their own before asking them to share. The teacher then calls upon a volunteer student to share their response as well as their thinking. By scripting this teaching move, I purposely choose for the student to share their count with the teacher in front of the rest of the students in the classroom because of the way it mirrors the realities of the classroom and positions the teacher to make decisions that are based on what she believes is suitable for classroom instruction, rather than for each individual child. The student responds by giving a solution and then verbally recounts the procedure done to solve the problem, without the use of a manipulative. While manipulatives can often be used in early elementary classrooms, the existence of a manipulative, such as concrete objects, would allow the student to count forward using the separating strategy[1] in order to determine the amount remaining and alter the situation in that students are less likely to count backward when they have discrete manipulatives available to them.

Specific item types. In keeping with the literature that describes student strategies for subtraction (Carpenter et al., 1981; Carpenter et al., 1984; Steffe et al., 1983; & Fuson, 1984), I used the general structure for the scenarios described above to script specific instances in which the shared student methods were *measurement* subtraction, *cardinal* subtraction, and use of derived facts. Using a derived strategy, a student who is asked to solve the problem, 14 – 5, might use the fact that 14 – 4 = 10 and then derive that the result of 14 – 5 would then be 9 (Carpenter et al., 1981). The findings from the use of derived facts are not presented in this paper, but the scenarios were presented in the instrument because of the documented use of derived fact strategies in classrooms. For each scenario in which a student used one of the aforementioned strategies correctly, I also scripted a corresponding scenario in which the student

[1] Using the *separating* strategy, the student establishes a collection of concrete objects that match the minuend, then takes away a number of objects that indicate the subtrahend and counts the items that remain. Thus, to solve the problem of how many objects are left from 12 after removing 4, the child would count out 12 objects to make a collection, then count forward as they remove 4 objects from the collection, and finally use a forward count to enumerate the amount remaining in the collection.

was in error, representative of the work that teachers would encounter when posing problems to a classroom of learners. This structure created conditions for analysis determined by which method the student had used and whether or not the student was correct.

Scripting errors. Since the object of study is the various student informal methods for subtraction and teacher responses to those methods, attempts were made to ensure that the scripted error could not be attributed to an error in the counting itself, since it could distract from the purpose of the instrument. Counting errors can be made in two different ways as a student carries out the procedure of counting backward to subtract. The student could recite their backward number sequence incorrectly or they could carry out fewer or more counts than are indicated by the subtrahend. While it is not uncommon for a student to have an error in their counting, such as missing a numeral in the sequence (e.g. 14, 13, 11, 10), this error was not included in any of the scenarios since a teacher could attribute an error to the sequence of the count and overlook the student's method altogether. Since an appropriate number sequence is a foundational skill that must be in place in order to use any counting method, scripting a missing numeral in the counting sequence would not allow the teacher to focus on a method that involves the use of counting in order to carry it out. For similar reasons, neither was the error scripted to indicate that the student had counted too few or too many counts as would be indicated by the subtrahend. To carry out the method appropriately, then, the child must decide what is being counted and use an appropriate count to correspond appropriately to the indicated problem.

The student's error, then, was scripted to be identifiable based on where the count ended while the method itself could be identified by where the count began. Observations of first grade classrooms and focus groups with elementary teachers indicate that teachers place emphasis on where the count begins when counting backward, but significantly less attention is given to where the count ends. Thus, the methods were scripted in such a way to give emphasis to where the count began, using this as an indicator of each method, with the error occurring when the student chose which number to report as the solution at the end of the count. Documentation of students who use a cardinal subtraction method correctly show that a student would begin their count on the minuend and end their count on the number that is one greater than the difference, stating the number of objects that remain in the set as the solution. I use a count that begins on the number one less than the minuend to indicate the measurement method and cause the error to occur when the student must choose which number to report as the solution, ending the count on the difference but then stating the number that is one less than the difference. Similarly, the expectation is that a student who uses the measurement method correctly would begin the count on the number one less than the minuend and end the count on the difference. Again, this operation for ending the count was applied to a student who had begun the count on the minuend, an indicator of the cardinal method, then ending the count on the number one greater than the difference, stating this as the solution, rather than continuing by quantifying the number of objects that remain. More specifically, to solve the subtraction number model, 14 – 3 = __, a student who solves the problem by using the cardinal subtraction method but makes an error, might say, "I got 12. I counted 14, 13, 12. That makes 12!" This way, if a teacher recognizes the cardinal method as valid, they could recognize that the student began their count on 14, presumably indicating the number name given to the object, but then ended the count on 12 and should have stated what was left. On the other hand, if they do not recognize the cardinal method as valid, then I might expect that they would respond by indicating that the student began their count on the wrong number, hence giving reason that the error occurred. The table below gives examples of how this would be scripted for each method if the problem given to the students could be modeled using the equation, 12 – 4 = __.

Table 1

Scripting of Methods and Correctness in Each Scenario for the Example of Taking 4 From 12

Correctness	Measurement	Cardinal
Correct	"11, 10, 9, 8. I got 8."	"12, 11, 10, 9. I got 8."
Incorrect	"11, 10, 9, 8. I got 7."	"12, 11, 10, 9. I got 9."

Teacher responses to students. Finally, each of the scenarios included a teacher responding to the student that showed if the teacher either positively appraised the student's method by stating that she liked it and asking for someone else to share the same method, or disregarded the method by making a neutral statement and asking for another method to be shared with the class. The scenario was purposefully scripted so that the scripted teacher's attention was focused on how the students carried out the work, rather than on the numerical difference that was found. The scripted teacher's response was included based on experiences in classrooms that indicate the possibility that most teachers would recognize that students use a variety of methods, but also that a teacher's response to the students' methods is something I suspected participants would consider strongly as part of their practice, in that it is the teacher's role to guide the discussion in the classroom to ensure that students come to use appropriate methods in an accurate way. Thus, one could expect that a teacher may react differently to student methods, depending on the inclusion or exclusion of an error, but also depending on whether the scripted teacher was accepting of the method. I hypothesize that a respondent may not be willing to accept a non-normative method from a student and would be more inclined to disagree with a teacher who accepted the hypothesized non-normative method.

To explore that hypothesis, I created twelve scenarios that describe the interactions between a teacher and her classroom focused on the cardinal, measurement, and derived strategies that students have been documented to use, the error a student could make in carrying out those methods, followed by a teacher response. The eight scenarios that included the student using the measurement or cardinal methods, being in error or not, and having a depicted teacher respond in a positive or neutral manner were analyzed for this study.

Medium. The scripted scenarios were depicted using the *Depict* tool in the online environment, Lesson*Sketch* (www.lessonsketch.org). *Depict* is a cartoon storyboarding tool that has been designed to represent a typical classroom environment (Herbst & Chieu, 2011). Depict enables the designer to create classroom scenarios that are comparable to video representations in that they allow the viewer to see the classroom and its participants represented as they carry out classroom interactions and for speech to be associated with particular actors. Chazan and Herbst (2012) have observed that even though a storyboard or animation portrays a specific storyline, teachers are able to look beyond the specific storyboard to project themselves onto it or consider it akin to the teaching they have done. Constructing scripted scenarios in Depict allowed for construction of an online questionnaire that asked questions to respondents in the context of classroom actions, asking teachers to respond to the actual student work of an early elementary student.

In Figure 3, frames one and three of the four-frame depiction are shown. In Frame one the teacher presents the separating problem and in frame three the student shares the count that is meant to describe the method for solving the problem.

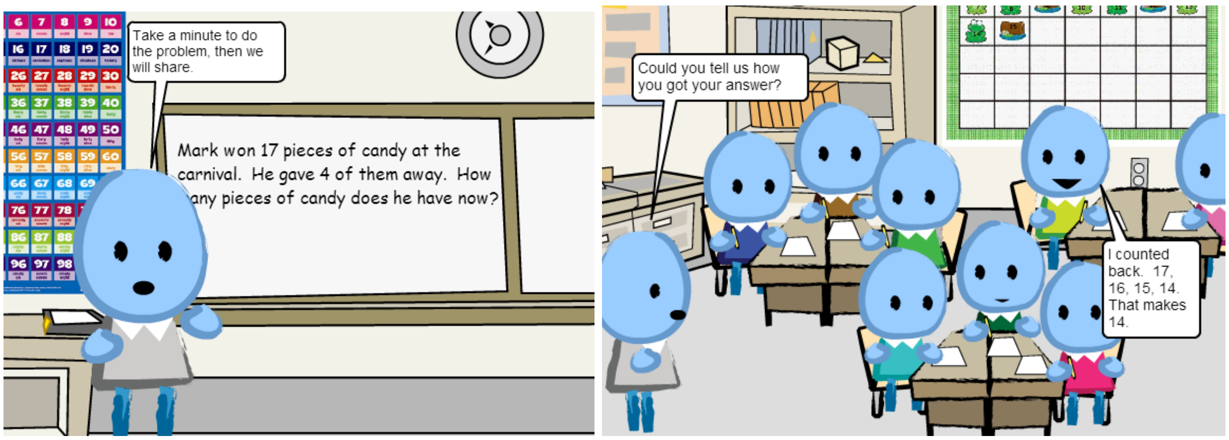

Figure 3: A student's error in applying the cardinal method to a separating problem.

Teacher Sample

The pilot instrument was used with a sample of teachers gathered in the summer, with the revised instrument being used during subsequent fall and winter sessions, drawing teachers from 9 counties around a large Midwestern university. Teachers were asked to respond to the subtraction instrument if they had taught any grade from Kindergarten through fifth grade. A background survey asked teachers to indicate which grade-level bands (K-2 or 3-5) they had taught. While this information was especially useful for the sake of other instruments for a larger study, it served this purpose as well since teachers often change grade levels and are likely to still address subtraction remediation during their instruction for a few years beyond when it is formally taught in school. Of the 39 teachers who participated, 24 had experience teaching in the grade bands, K-2 and 3-5, at some time during their teaching experience. 6 of the participants had experience in the grade band K-2 and 9 teachers had experience in the grade band 3-5. The mean years of teaching experience for the 39 teachers was 13.6 (SD=6.8) years. Teachers came from a range of districts, from urban to suburban. All teachers who participated in the subtraction instrument saw the same depictions of student subtraction methods, but were assigned to one of three forms of the instrument, randomized such that the order in which the items appeared would differ. This was done to ensure that the data was not skewed by having seen a particular method in one of the early responses that could have an impact on respondents' ideas about other scenarios which they would encounter throughout the survey. Responses to the instrument were aggregated by condition.

References

Ball, D. L. (1997). What do students know? Facing challenges of distance, context, and desire in trying to hear children. In *International handbook of teachers and teaching* (pp. 769-818). Springer Netherlands.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching What Makes It Special?. *Journal of Teacher Education*, *59*(5), 389-407.

Baroody, A. J. (1984). Children's difficulties in subtraction: Some causes and questions. *Journal for Research in Mathematics Education*, 203-213.

Carpenter, T. P., Hiebert, J., & Moser, J. M. (1981). Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education*, 27-39.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*,*26*(4), 499-531.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction.*Elementary School Journal*, *97*, 3-20.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, 179-202

Chazan, D., & Herbst, P. (2012). Animations of classroom interaction: Expanding the boundaries of video records of practice. *Teachers College Record*, *114*(3), 1-34.

Chazan, D., & Lueke, H. M. (2009). Exploring tensions between disciplinary knowledge and school mathematics: Implications for reasoning and proof in school mathematics. *Teaching and learning mathematics proof across the grades*, 21-39.

Chazan, D., Sela, H., and Herbst, P. (2012). Is the role of equations in the doing of word problems in school algebra changing? Initial indications from teacher study groups. *Cognition and Instruction*, *30*(1), 1-38.

Cohen, D. K. (2011). *Teaching and its predicaments*. Harvard University Press.

Dewey, J. (1902). The evolutionary method as applied to morality.

Fuson, K. (1984). More complexities in subtraction. *Journal for Research in Mathematics Education*, 214-225.

Halliday, M. A., & Matthiessen, C. M. (2004). An introduction to functional grammar.

Herbst, P. G. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematical tasks. *Journal for Research in Mathematics Education*, 313-347.

Herbst, P., Aaron, W., Dimmel, J., & Erickson, A. (2013). Expanding students' involvement in proof problems: Are geometry teachers willing to depart from the norm?.

Herbst, P., & Chazan, D. (2011). Research on practical rationality: Studying the justification of actions in mathematics teaching. *The Montana Mathematics Enthusiast, 8*(3), 405-462.

Herbst, P., & Chazan, D. (2012). On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM*, *44*(5), 601-612.

Herbst, P., & Chieu, V. M. (2011). Depict: A tool to represent classroom scenarios.

Herbst, P., Kosko, K. W., & Dimmel, J. K. (2013). How are geometric proof problems presented? Conceptualizing and measuring teachers' recognition of the diagrammatic register.

Herbst, P., Nachlieli, T., and Chazan, D. (2011). Studying the practical rationality of mathematics teaching: What goes into "installing" a theorem in geometry? *Cognition and Instruction, 29*(2), 1–38.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., ... & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational researcher, 25*(4), 12-21.

Humphrey, S., Droga, L., & Feez, S. (2012). Grammar and Meaning.

Maher, C. A., & Davis, R. B. (1990). Chapter 6: Building Representations of Children's Meanings. Journal for Research in Mathematics Education. Monograph, 4, 79-210.

Martin, J. R., & White, P. R. R. (2007). *The Language of Evaluation: Appraisal in English*. London: Palgrave Macmillan.

Mehan, H., & Wood, H. (1975). *The reality of ethnomethodology.* Malabar, FL: Krieger.

Putnam, R. T. (1987). Structuring and adjusting content for students: A study of live and simulated tutoring of addition. American Educational Research Journal, 24(1), 13-48.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational researcher, 15(2), 4-14.

Sim, J., & Wright, C. C. (2005). The kappa statistic in reliability studies: use, interpretation, and sample size requirements. Physical therapy, 85(3), 257-268

Steffe, L. P., & Kieren, T. (1994). Radical constructivism and mathematics education. *Journal* *for Research in Mathematics Education*, 711-733.

Steffe, von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy,theory, and application*. New York: Praeger.

Steinberg, R. M. (1985). Instruction on derived facts strategies in addition and subtraction. *Journal for Research in Mathematics Education*, 337-355.

Von Glasersfeld, E. (1995). *Radical Constructivism: A Way of Knowing and Learning. Studies in Mathematics Education Series: 6*. Falmer Press, Taylor & Francis Inc., 1900 Frost Road, Suite 101, Bristol, PA 19007.

Wallach, T., & Even, R. (2005). Hearing students: The complexity of understanding what they are saying, showing, and doing. Journal of Mathematics Teacher Education, 8(5), 393-417.