# Responsive Teaching with Fractions

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**Victoria R. Jacobs**

*University of North Carolina at Greensboro*
vrjacobs@uncg.edu

**Susan B. Empson**
empson@austin.utexas.edu

**Gladys H. Krause**
gladyshrueda@utexas.edu

**D'Anna Pynes**
dannapynes@utexas.edu

*The University of Texas at Austin*

*Abstract:* A model for teaching that is responsive to children's thinking about fractions is introduced and, as part of this model, a framework for characterizing teaching moves that support and extend children's thinking in one-on-one interactions in interview and classroom settings is presented and illustrated. Data for the framework draw on three cases of expertise in teaching in grades 4 – 5 that is responsive to children's fraction thinking. Five major categories of supporting and extending moves include: (a) ensuring the child is making sense of the story context, (b) exploring details of a child's existing strategy, (c) encouraging the child to consider other strategies, (d) inviting the child to generate symbolic notation, and (e) adjusting the problem to match the child's understandings. The category, exploring the details of the child's existing strategy, is further decomposed into five subcategories, because of the necessity of eliciting children's thinking in order to respond to it.

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## Responsive Teaching With Fractions

In this paper we introduce a model for teaching that is responsive to children's thinking about fractions and focus on describing a part of this model: a framework for characterizing teaching moves that build on children's thinking during instruction. We conceptualize *responsive teaching* as a type of teaching in which teachers' instructional decisions about what to pursue and how to pursue it are continually adjusted during instruction, rather than determined in advance, in response to children's content-specific thinking. We draw on three cases of teachers of grades 4–5 who are skilled at responsive teaching with fractions to illustrate our emerging categories of teaching moves involved in responsive teaching.

### *Theoretical Framework*

Our focus on responsive teaching echoes recent agenda-setting documents that highlight the enormous potential of instruction in which teachers draw on knowledge of children's mathematical thinking. For example, in the recently published *Principles to Actions: Ensuring Mathematical Success for All,* "elicit and use evidence of student thinking" was highlighted as one of the eight high-leverage practices that should be included in teachers' daily work (National Council of Teachers of Mathematics, 2014, p. 10). This interactive view of instruction is challenging to enact, however, because teachers must make "on the spot decisions concerning what mathematics to pursue and how to pursue it" (Sherin, 2002, p. 122), and it requires knowledge and instructional practices that may be unfamiliar to many teachers.

Consistent with current efforts to decompose teaching into core activities that can be discussed and rehearsed (Grossman et al., 2009), we highlight what we consider to be core practices in responsive teaching and the knowledge that supports these practices. Specifically, we decompose responsive teaching into knowledge of children's mathematical thinking and a set of generative instructional practices: noticing, and supporting and extending children's mathematical thinking. These practices are domain-specific in the sense that they call upon specific knowledge of the target domain—in this case, fractions—and learning in the domain.

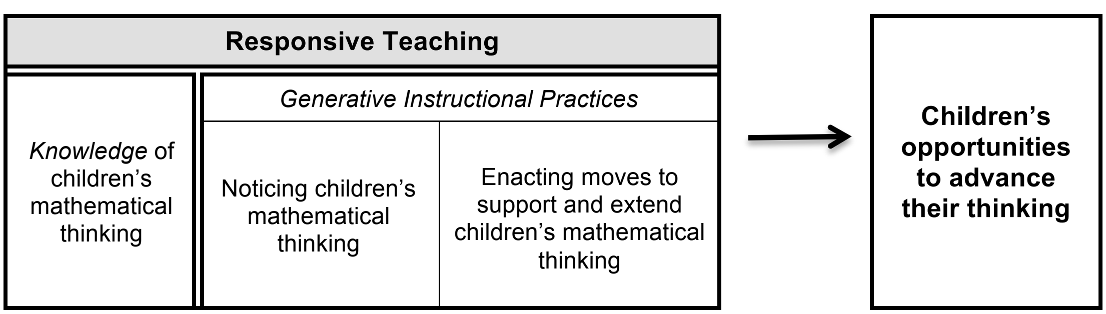


*Figure 1.* Model of Teaching that is Responsive to Children's Mathematical Thinking

***Knowledge of children's mathematical thinking.*** In responsive teaching, teachers draw on a knowledge base that we conceptualize as integrated and linked with practice (Hiebert, Gallimore, & Stigler, 2002). We highlight knowledge of children's mathematical thinking that is *usable* by teachers while they engage in instruction, and we draw from a framework for fractions that focuses on how children's informal ideas of partitioning quantities and their early understanding of the fundamental properties of operations and equality support the development of understanding of fractions (Empson & Levi, 2011). The content addressed is consistent with the vision for mathematics content and practices set forth in the *Common Core State Standards in Mathematics* (CCSSO/NGA, 2010) and current efforts to map students' learning progressions in various mathematical domains (Daro, Mosher, & Corcoran, 2011).

***Noticing children's mathematical thinking.*** One of the main challenges of responsive teaching is the complexity of classrooms in which teachers cannot respond to every event and must instead make choices. We argue that prior to responding to an event, teachers must first *notice* that event and this hidden practice of noticing is, by itself, worthy of study. Mathematics education researchers have shown renewed interest in teacher noticing—how teachers attend to and make sense of events in classrooms—and this work builds on the expertise literature (Bransford, Brown, & Cocking, 2000) and on research that highlights the distinct patterns of noticing particular to professions (Goodwin, 1994; Mason, 2002; Stevens & Hall, 1998). See (Sherin, Jacobs, and Philipp, 2011) for a compilation of this work in mathematics education.

In responsive teaching, we are interested in a particular type of noticing—*professional noticing of children's mathematical thinking*—that highlights how and the extent to which teachers notice children's mathematical thinking (Jacobs, Lamb, & Philipp, 2010). This practice is conceptualized in terms of a set of three interrelated skills that are temporally and conceptually linked in teachers' daily work. First, teachers must *attend* to the details in children's strategies. Second, they must *interpret* children's understandings as reflected in those strategies. Third, they must *decide how to respond* on the basis of those understandings. Note that this third skill does not include the execution of the response, but rather the reasoning that occurs before a teacher acts (i.e., *intended*— as separate from *actual*—responding).

***Supporting and extending children's mathematical thinking.*** Once teachers have noticed children's mathematical thinking and considered their intended responses, they act. We focus on teaching moves that take into account evidence of children's thinking and purposefully build on that thinking rather than redirect or ignore it. These moves involve responding to what children actually say or do—rather than responding on the basis of an instructional script or routine, as is prevalent in many classrooms (Fraivillig, Murphy, & Fuson, 1999; Kawanka & Stigler, 1999; Minstrell, Anderson, & Li, 2011). These teaching moves support and extend children's thinking (Jacobs & Ambrose, 2008; Jacobs, Ambrose, Philipp, & Martin, 2011).

## *Methods and Data Sources*

Through a multiple case-study approach, we investigated the teaching moves of three teachers of grades 4–5 who are skilled in responsive teaching with fractions. The teachers include Ms. K, Mr. S, and Ms. B. These teachers were purposefully selected from three diverse geographical areas and based on recommendations of local professional developers who were familiar with the teachers' knowledge and classroom instruction. They come from a larger set of seven cases of expert teaching in which we analyzed a snapshot of the practice of teachers who were already skilled at responding to children's fractional thinking.

Our analysis draws from two types of data from each of these teachers' classrooms: (a) video of 2 days of classroom instruction in mathematics; and (b) video of one-on-one problem-solving interviews conducted by the teacher with each of 5–7 students. Problems discussed in this paper were posed during lessons and the one-on-one problem solving interviews and these problems were written or adapted by teachers (Table 1).

Table 1
*Story Problems Given During the Lessons and Interviews*

| Event | Problem |
| --- | --- |
| Ms. K Lesson 1 | Kenzie loves to go on long hikes. She often hikes with her college friends. She knows it is important to drink water when she hikes. She drinks 2/3 cup of water for every mile she hikes. Her water bottle holds 4 cups of water. How many miles can she hike before her water runs out? |
| Ms. K Lesson 2 | 8 people want to share 13 sandwiches so that they all have the same amount of sandwich. How much sandwich should each person get?<br><br>8 children are sharing 5 hamburgers equally. How much hamburger does one child get? |
| Ms. K Interviews with Individual Children | 8 children want to share 14 cookies so that each child gets the same amount. How much can each child get?<br><br>Meleri is making some cookies. It takes 1 1/2 cups of brown sugar to make each batch. If Meleri has 10 cups of brown sugar, how many batches of cookies can she make?<br><br>Ms. Hughes is making lemonade for the class party. She has 8 cups of sugar. She needs 2/3 cup of sugar to make a pitcher of lemonade. How many pitchers of lemonade can she make? |
| Mr. S Lesson | Mary made ______ pounds of fudge. She and her friends ate some of the fudge and now there are _____ pounds of fudge left. How many pounds of fudge did Mary and her friends eat?<br><br>(Children chose [3, 1 3/4], [5 1/2, 2 9/10] to replace the blanks.) |
| Mr. S Interviews with Individual Children | Junior wants to prepare candy bags to pass out during his birthday party. Junior has 6 2/3 pounds of jelly beans to make party favors. It takes 2/3 pounds of jelly beans to fill 1 bag. How many bags jelly beans can Junior fill with his jelly beans? (alternate number set: 20, 5/8)<br><br>6 children want to equally share 20 peanut butter sandwiches, with no leftovers. How much can each child have?<br>(alternate number set: [8, 5 2/4]) |
| Ms. B Lesson 1 | Ms. Veronica is making cookies. Each batch of cookies takes 2/3 of a stick of butter. How many sticks of butter will Ms. Veronica need if she wants to make 9 batches of cookies? |
| Ms. B Lesson 2 | Maddy is helping her dad feed cows. They have 10 bags of feed. If there are 6 cows that all get the same amount, how much feed can each cow get? |

Analysis of teachers' supporting and extending moves began with an existing framework for children's whole-number thinking (Jacobs & Ambrose, 2008) and, through iterative coding, is being refined and customized for the domain of fractions (Jacobs & Empson, accepted). Our overall analysis strategy follows Yin's (2003) recommendations for high-quality analyses: attending to the use of all the evidence, identifying and investigating rival interpretations, and bringing to bear our own expertise in both theory and practice.

## *Findings: Categories of Supporting and Extending Moves*

Previous research on teachers' supporting and extending moves focused mainly on teacher-student interactions in one-on-one settings (Jacobs & Ambrose, 2008), but we have found evidence of the importance of supporting and extending children's mathematical thinking in both one-on-one settings *and* classroom settings, particularly as teachers interact with students during the circulating phase of a lesson, after a problem has been posed and students are working on solving (Jacobs & Empson, accepted). In the classrooms of the case study teachers, students are encouraged to solve problems by making sense of the story context and using what they know about quantities and relationships to generate solutions.

Teachers' supporting and extending moves as they interacted with children around story problems fell into 5 major categories (a) ensuring the child is making sense of the problem, (b) exploring details of a child's existing strategy, (c) encouraging the child to consider other strategies, (d) inviting the child to generate symbolic notation, and (e) adjusting the problem to match the child's understandings (Figure 2). Category (b), exploring the details of the child's existing strategy, is further decomposed into 5 subcategories. Expert noticing of children's mathematical thinking underlies all of these supporting and extending moves but is not a focus of this paper. In the following sections, we share selected examples to illustrate these categories of supporting and extending moves.

### Ensuring Making Sense of Story Problem

One of the biggest challenges for children when they are solving story problems is to understand what is happening in the story and what question is being posed. Thus, the first major category of teaching moves involves ensuring that the child understands the problem. These moves often, but not always, occurred early in the interaction.

The teachers regularly checked in with children's understanding of the story, highlighting the specific question they were trying to answer. Sometimes, the discussion focused on understanding the entire problem such as in Jane's[1] interview with Ms. K when she was unsure how to begin solving the problem about 8 children sharing 14 cookies. Ms. K asked her to make sense of the problem: "What are you thinking when you read this problem—what is the picture in your head? Could you retell me the problem, like in your own words? What is it saying to you?"

Other times, teachers focused on specific parts of the story, such as vocabulary, that were problematic for a particular child. In Ms. B's lesson, after she posed the problem about 6 cows sharing 10 feedbags, a student said in reference to feedbags, "I can't get that in my head or anything." Ms. B responded, "Well, do you have dogs? Think about your dog food, how you'd feed your dogs dog food. And if a bag of food bothers you, you could in your mind change it to

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[1] All children's names are pseudonyms.

something else. It could be a bale of hay or whatever you want it to be to make it make sense." After Ms. B clarified the vocabulary and encouraged the student to relate the quantities to something more familiar to her, the student better understood the story and was able to solve the problem.

<table>
<tr><th colspan="2">Categories of<br>Supporting and Extending</th><th>Sample Teaching Moves</th></tr>
<tr><td colspan="2">Ensuring the child is making sense of the problem</td><td>• What are you thinking when you read this problem—what is the picture in your head? Could you retell me the problem, like in your own words? What is it saying to you?</td></tr>
<tr><td rowspan="5">Exploring details of the child's existing strategy</td><td>Posing starter questions to the child</td><td>• Tell me what you did.<br>• What was the first idea that came to your head?</td></tr>
<tr><td>Pressing the child for a detailed explanation of his or her problem-solving process</td><td>• So I noticed you cut this [cookie] into eighths, why did you choose to cut that one into eighths?<br>• Talk to me about how you got 1 and 4 sixths … What if someone says I do not understand how you added that? What would you tell them?</td></tr>
<tr><td>Probing the child's representation to highlight connections to the problem context</td><td>• Let me ask you a question. What do these numbers (2, 4, 6) represent? These whole numbers, in the problem?<br>• It says it takes 1 1/2 cups of brown sugar to make each batch. Where on your grid here does it show 1 1/2 cups of brown sugar?</td></tr>
<tr><td>Questioning the child about quantities and their relationships in the strategy</td><td>• Is 2/3 more or less than 1 whole cup? (Child says 'less.') So do you think she'll be able to go more than 4 miles then if she's drinking less water?<br>• Is this true? (writes) 1 3/4 + 1/4 + 1 = 3</td></tr>
<tr><td>Inviting the child to think ahead before executing a problem-solving step</td><td>• (posed prior to solving) Would the children all get one hamburger?<br>• (for 10 × 5/8) Before you figure it out, can you kind of guess what that might be?</td></tr>
<tr><td colspan="2">Encouraging the child to consider other strategies</td><td>• Do you think there is a second way you could check … so you're like Okay this makes sense to me what I'm doing.<br>• Is that the only way you can share those last two sandwiches?</td></tr>
<tr><td colspan="2">Inviting the child to generate symbolic notation</td><td>• Do you want to write that [relationship] down so you are recording it as you are working?<br>• Can you represent that [share for each person] without shading it in? How would you represent it?</td></tr>
<tr><td colspan="2">Adjusting the problem to match the child's understanding</td><td>• (After child had difficulty solving a separating problem with the change quantity unknown) So let me ask you a question. I'm gonna change the question a little bit, okay? So if Mary bought 5 1/2 pounds of fudge and then her mom gave her 2 9/10 as well, how much fudge does she have in total?</td></tr>
</table>

*Figure 2.* Framework of supporting and extending moves used in one-on-one interactions, in classroom lessons and during problem-solving interviews

**Exploring Details of Child's Existing Strategy**

This second major category of teaching moves captures the essence of being responsive to children's mathematical thinking because teachers need to understand children's existing thinking before they can build on it. This category contains five distinct subcategories of teaching moves that are distinguished on the basis of the types of details, or aspects of children's thinking, that were explored: (a) posing starter questions, (b) pressing for detailed explanations of the problem-solving process, (c) probing representations for their connections to the story context, (d) questioning quantities and their relationships, and (e) inviting the child to think ahead.

***Posing starter question.*** To begin conversations about children's ideas, teachers need to elicit what children have already thought about and done, even if their work is incomplete or only partially correct. These questions tended to be general in nature, because teachers did not have knowledge of children's thinking about a specific problem at the beginning of an interaction. For example, Ms. B circulated while students were solving the batches of cookies problem and, as she approached several students, began with, "Tell me what you did" to initiate a conversation about students' thinking.

We most often saw these starter questions after children reached an answer or were at a stopping point in the interviews, or when a teacher first arrived at children's seats during the class lessons, although teachers sometimes used them when a child was at a decision point. Gloria was solving the problem in which 8 children equally shared 5 2/4 sandwiches. She drew the sandwiches (Figure 3) and then paused for 12 seconds, at which point Mr. S asked her, "What was the first idea that came to your head?" Gloria responded, "I might cut them into eighths," and resumed drawing out her strategy.

*Figure 3.* Gloria's start for equally sharing 5 2/4 sandwiches among 8 children.

Although these starter questions effectively initiated conversations and tended to be easy to enact because they were less situation-specific than other categories, they were generally insufficient by themselves for supporting and extending children's thinking.

***Pressing for detailed explanation of process.*** Children's explanations of their strategies and ideas are often incomplete or procedural in nature (i.e. focused on the steps they performed rather than why they performed them), and teachers' questions can help children make these explanations richer and more complete—a feature of instruction that has been positively linked to student achievement (Webb et al., 2014). Teachers' questioning in this category tended to focus on mathematically important steps in the problem solution (e.g., how they partitioned or how they combined fractional parts to arrive at a final answer) as well as decision points where children exhibited uncertainty.

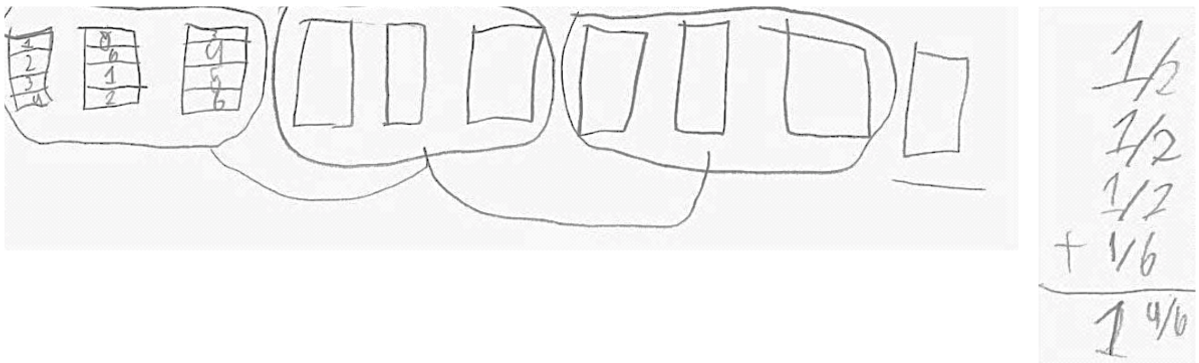


*Figure 4.* Emory's written work for 6 cows sharing 10 feedbags.

For example, when Ms. B first asked Emory during her lesson how he had gotten 1 4/6 for the problem about 6 cows sharing 10 feedbags, he gave an incomplete explanation. First he described how he had given each cow 1/2 feedbag from each of the three groups of feedbags (after experimenting with splitting the first three feedbags into fourths) and then split the last feedbag into sixths (Figure 4). He went on to say that he added 1/2, 1/2, 1/2, and 1/6 and got 1 4/6. Ms. B pressed Emory for details about how he added the fractional quantities:

T: And so, talk to me about how you got 1 4/6. You told me completely, your one half came from here (pointing to first group of 3 feedbags), your one half came from here (pointing to second group of 3 feedbags), your one half from here (pointing to third group of 3 feedbags), and your one sixth came from here (pointing to last feedbag). I got that. Talk to me about this answer (1 4/6).
S: It's just adding up all of what was in there. Because I just used mathematics.
T: You sure did. What if someone says, 'I do not understand how you added that.' What would you tell them?
S: I used mathematics!
T: You did. Can you explain it to me? What kind of math did you use?
S: I just used like 1 whole, 1 half, like, things like that.
T: Tell me how you knew that this was going to be one. Where did your one come from?
S: These two halves.
T: Gotcha. And what about the 4/6?
S: Half of that (struggles to find words). Well, 3 sixths came from there (the third group of feedbags, which provided 1/2 feedbag per cow), and the fourth one came from this one (the last feedbag, which provided 1/6 feedbag per cow).
T: Nice. Good thinking, Emory. I like it.

At first Emory's explanation was vague. However Ms. B pressed him for details about how he combined the fractional quantities, first by inviting him to imagine explaining his thinking to someone who did not understand what he did, and then by asking him a question about a specific quantity in his answer, one, and where it came from in his strategy. This question prompted him to explain that he combined 1/2 and 1/2 to make one whole, and then 3/6 (from the third 1/2) and 1/6 to make 4/6.

Teachers often pressed for several different details within one interaction. In her interview with Ms. K, Alice solved the problem about 8 children sharing 14 cookies (Figure 5) by drawing 8 rectangles (children) and giving each child one cookie (marking "1" for each cookie in the rectangles). She tried to give them each another cookie (again, marking "1" for each cookie in the rectangles), but at the sixth child stopped and crossed out the 6 cookies she had tried to distribute in the second round (the second "1" in the first 6 rectangles). She then drew 6 new rectangles (cookies) and partitioned the first rectangle into 8 parts, passed out 1/8 to each child by writing "1/8" in each rectangle, looked at her remaining 5 cookies (but did not partition them), and answered that each child would receive 1 6/8 cookies.

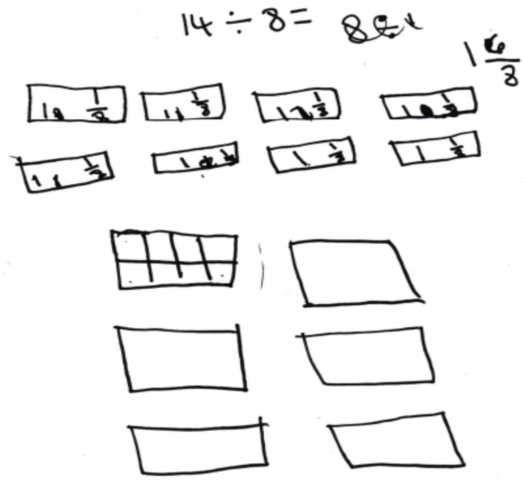


*Figure 5.* Alice's work for 8 children sharing 14 cookies.

Ms. K pressed Alice for three mathematically important details of her problem-solving process: (a) why she stopped passing out whole cookies at the sixth child ("I see you went on and then you stopped yourself here [points to the $6^{th}$ child], why?"); (b) why she partitioned the first cookie into eighths ("So I noticed you cut this one into eighths [points to the partitioned cookie], why did you choose to cut that one into eighths?"); and (c) why she did not need to partition the final five cookies ("Here you made this model [points to the first partitioned cookie], how come you didn't go ahead and make the lines on there?" [points to the 5 cookies that were not partitioned]).

Unlike the starter questions in the previous category, questions that press for details about the child's problem solving process can not be planned in advance because they focus on strategy details that a teacher can only know about by paying close attention to students' thinking and explanations.

***Probing representation for connections to story context.*** A child's representation of the problem situation—whether it involves pictures, tables, manipulatives, or written numbers—is an important component of problem solving and articulating how representations connect to quantities and actions in a story problem enhances children's understanding. Teachers' questioning prompted children to make these connections explicit in two ways.

In the first way, teachers asked about how children's representations connected to the story context. For example, in her problem-solving interview with Mr. S, Liliana was solving the problem about how many bags of jelly beans could be made from 6 2/3 pounds of jelly beans, if 2/3 pound went into each bag (Figure 6). She was using skip counting to keep track of the amount in each bag and when she reached 6 pounds, she reported that 9 bags of party favors could be made, temporarily ignoring the additional 2/3 pound of jelly beans in the total. Mr. S drew her attention to the total by asking how her representation was connected to the story problem: “Let me ask you a question. What do these numbers (2, 4, 6) represent? These whole numbers, in the problem?” Liliana replied, “They represent (pause) the pounds.” After further discussion, Liliana realized that there was an additional 2/3 pound of jelly beans to put into bags.

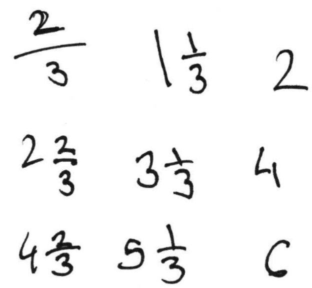


*Figure 6.* Liliana’s written work for figuring out how many bags of 2/3 pound of jelly beans could be made from 6 2/3 pounds of jelly beans

In the second way, teachers started with the story context and asked children how it was reflected in their representations. In her interview with Ms. K, Alice used a table to figure out how many batches of cookies Meleri could make with 10 cups of brown sugar if it took 1 1/2 cups of brown sugar to make each batch (Figure 7). Ms. K asked her to locate parts of the story in her representation [a table or “grid”] with questions such as “Where are your batches?”; “Where’s your brown sugar?”; “It says it takes 1 1/2 cups of brown sugar to make each batch. Where on your grid here does it show 1 1/2 cups of brown sugar?”; and “Where on your grid here does it show 10 cups of brown sugar?”

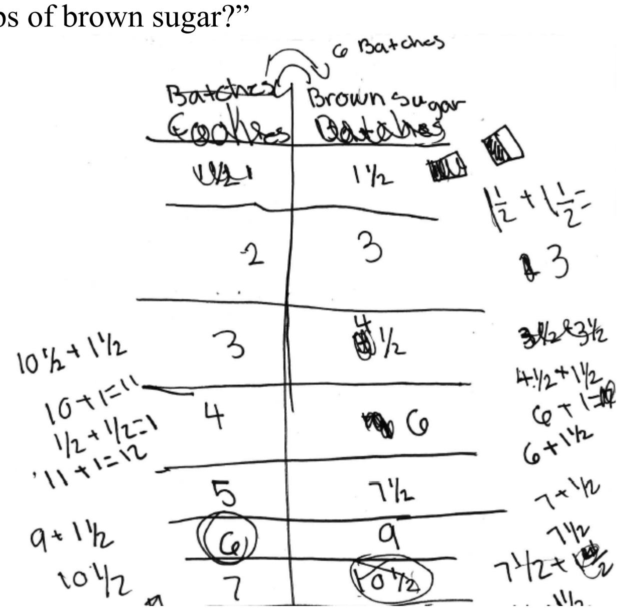


*Figure 7.* Alice’s table for determining the number of batches from 10 cups of brown sugar if she uses 1 1/2 cups each batch.

***Questioning quantities and relationships.*** The ability to reason about relationships among quantities is a hallmark of understanding arithmetic. Children’s problem-solving strategies generally involve multiple quantities and thus strategy discussions provide fertile ground for exploring fractional quantities and their relationships.

Ethan was solving the problem in Ms. K's class about how far Kenzie could hike with 4 cups of water if she drank 2/3 cup every mile. When he gave his initial answer as 4 miles, Ms. K engaged him in an exploration of the relationship between the amount of water and the number of miles by asking, "If she drank *one* cup of water for every mile, how many miles would she go?" This question emphasized the relationship between cups and miles and built on Ethan's answer of 4 miles by highlighting a situation where the answer would be 4 miles. After determining that Kenzie could go 4 miles if she drank one cup of water for every mile, Ms. K asked, "Is 2/3 more or less than 1 whole cup?" When Ethan indicated that he knew 2/3 was less than 1, Ms. K followed up with "So do you think she'll be able to go more than 4 miles then if she's drinking less water?"

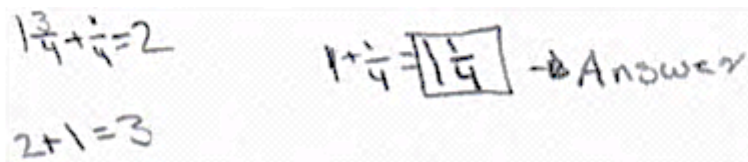


*Figure 8.* Esperanza's written work for the problem in which fudge was given away, with 3 pounds as the start quantity and 1 3/4 pounds as the end quantity, showing how she added on

Esperanza had solved the fudge problem in Mr. S's class using an incremental strategy in which she added on and kept track of the amount she added on for her answer (Figure 8). Mr. S. saw her written work and asked her to explain her strategy. Esperanza explained that she first added 1/4 to 1 3/4 to get 2, then added 1 onto 2 to get up to 3. She then added 1/4 and 1 to get her final answer. To further explore the relationships she used in her strategy, Mr. S wrote the equation 1 3/4 + 1/4 + 1 = 3 and asked Esperanza if it was true or false:

> T: Is this true? … 1 3/4 + 1/4 + 1 = 3
> S: (Thinks.) True.
> T: Why?
> S: Because if you add 1 whole and 1 whole that equals to 2 wholes, and I know that 3 fourths and 1 fourth equals to 1 whole and that 1 whole plus 1 whole plus 1 whole equals to 3 wholes.
> T: So I noticed in here (in her strategy) you were also trying to make it to the whole numbers. Can you explain why?
> E: Cause it's more easier to get the whole numbers than the fractions.
> T: And does this (points to the equation he wrote) match what you did?
> E: Yeah.
> T: Yeah, can you write that on your paper?

***Inviting to think ahead.*** Children sometimes go through a problem-solving process not taking advantage of what they already know or tending to stick with strategies that are familiar. In these cases, teachers' questioning can encourage children to use what they know to solve a problem in a more efficient manner. To address these kinds of issues, teachers in our study asked children at strategic points during their problem solving to anticipate the outcome of certain steps before executing them or to reflect on inefficient strategies, if they seemed ready to move on.

For example, in Ms. K's second class lesson, Jarvis had successfully completed the first problem about 8 children sharing 13 sandwiches, and was ready to move on to the second problem about 8 children sharing 5 hamburgers. After asking what the problem was about, Ms. K asked him to predict whether they would all get 1 hamburger (similar to how all children received one whole sandwich in the previous problem). Jarvis was able to think ahead and explained that they would not because with only 5 hamburgers and 8 children "you've gotta split them up."

Ricardo was working with Mr. S in an interview solving the problem how many bags of 5/8 pound of jelly beans could be made from a total of 20 pounds of jelly beans. He had figured that 5 bags with 5/8 pound of jelly beans would use 3 1/8 pounds of jelly beans and was repeating this information to use up the 20 pounds in increments of 3 1/8 pounds (Figure 9). As he began to write a third line of 5/8 × 5, Mr. S. stopped him:

T: I want to challenge you a little bit, before you write the next number. So you said you wanted to make big leaps, right? Big jumps? So far you've made a big jump of 5 and you know that that was 3 wholes and 1/8. Is there another big jump that you can probably take?
S: 10
T: 10. Before you figure it out, can you kind of guess what that might be?
S: (Thinking.)
T: More than 3, less than 3?
S: More. I think it's going to be 6 wholes and 2/8.
T: And you figured that out by?
S: Because I know that 5/8 times 5 equals 3 wholes and 1/8. And plus, again, will get me to 6 wholes and 2/8. Because it's 10, when you double the 5 it's 10.

In this interaction, Mr. S invited Ricardo to think ahead in two ways, first by choosing a more efficient number of bags – taking a "big leap" – and second by estimating the product of his big leap. Ricardo chose to work with 10 bags, which used up the jelly beans faster (although it did not necessarily make his strategy more efficient, because he was still working with mix-number quantities). These kinds of invitations challenged Ricardo to mentally anticipate his problem solving steps and work toward more efficient, generalizable strategies.

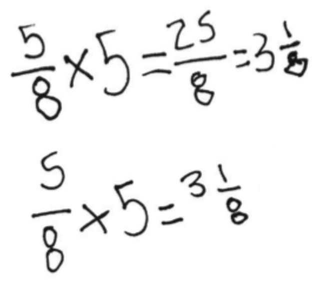


*Figure 9.* The beginning of Ricardo's strategy for finding how many bags of 5/8 pound of jelly beans could be made from 20 pounds

**Encouraging Consideration of Other Strategies**

Flexibility in strategy use is an important characteristic of successful problem solving because not only do children learn alternate paths for when they reach an impasse but comparisons of strategies can also be useful in highlighting important mathematical relationships (Heinze, Star, & Verschaffel, 2009). However, children are often hesitant to consider alternative strategies, and thus the third major category of teaching moves involves encouraging children to consider other strategies, within their own problem solving and by comparing their strategies with those of other children.

The teachers sometimes used the suggestion of other strategies to help children complete strategies successfully and with confidence. In Ms. K's first class lesson, Amir was trying to figure out how many miles Kenzie could hike with 4 cups of water if she drank 2/3 cup of water every mile. When Ms. K arrived at his seat, he had used several relationships that he knew ($3 \times 2/3 = 2$ and $2 \times 2 = 4$) to conclude that Kenzie could hike 12 miles because $3 + 3 + 3 + 3 = 12$. In sharing his strategy, he revised his answer three times, first to 18, then 9, and finally to the correct answer of 6, but he was still not confident. Ms. K encouraged him to solve it another way so that it made sense *to him*:

> Do you think there is a second way you could check because you've thought about 12, you've thought about 18, and then you thought 9 and now you're thinking, wait a minute. Could you make a grid or some other way that you could think about solving it? I just want you to double check so you're like, 'Okay this makes sense *to me* what I'm doing.'

In his interview with Mr. S, Juan solved the problem about 6 children sharing 20 peanut butter sandwiches by passing out 3 sandwiches to each child and then splitting the two remaining sandwiches each into sixths (Figure 10). He got 3 2/6 sandwiches for his answer. He had only recently begun to solve problems such as this one by linking the number of parts in a partition of a whole unit to the number of sharers in the problem and so Mr. S asked him if that was the only way he could share the last two sandwiches. Juan thought for a moment and responded that he could also split each sandwich into thirds; he re-drew the two remaining sandwiches and split them both into thirds.

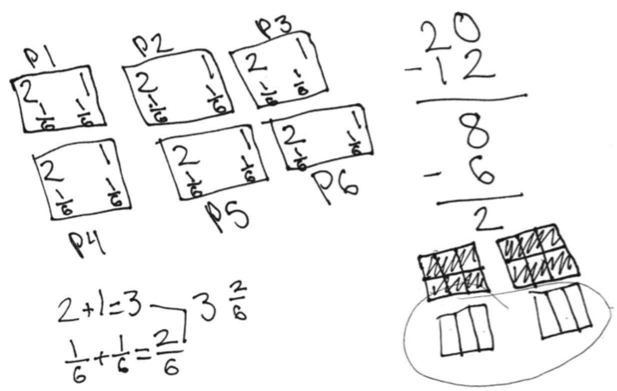


*Figure 10.* Juan's written work for sharing 20 sandwiches among 6 children

**Inviting Symbolic Notation**

The fourth major category of teaching moves highlights the important role of symbolic notation and building meaning for such notation in mathematical understanding. In contrast to previously discussed categories in which teachers questioned children about their existing representations, here teachers invited children to *generate* numbers, expressions, or equations that connected to their thinking.

Teachers made some notation requests during the child's problem solving, such as in Ethan's interview when he began figuring out how many pitchers of lemonade Ms. Hughes could make from 8 cups of sugar when it takes 2/3 cup of sugar for each pitcher. He stated a relationship he knew ("two thirds times three is two"), and as a way to support his reasoning, Ms. K suggested that he record that relationship: "Do you want to write that [relationship] down so you are recording it as you are working?" Encouraging children to record their ideas as they are working not only helps them view notation as a thinking tool (vs. only as a way of communicating a solution), but the resulting representation can also serve as a focal point for further discussion of the details of the child's thinking.

Other times, teachers made notation requests to help children link their informal strategies to more formal notation to forge connections among multiple representations. For example, Gloria was solving the problem about 8 children sharing 5 2/4 peanut butter sandwiches in her interview with Mr. S (Figure 11). She had split each whole sandwich into eighths wanted to represent each share by shading pieces. She was about to start shading when Mr. S invited her to do something different:

T: Can you represent it without shading it in?
S: Yeah
T: How would you represent that?
S: With numbers
T: Try it

Gloria proceeded to build up each child's share using fraction notation, saying as she went, "I would just write them up. I am going to do the same thing for all of them." As she wrote "1/8" for each sharer, Mr. S reminded her that she had already counted the number of pieces each and determined that each person would get 5 eighths, and so she wrote "5/8" for this part of the problem and then moved on to share the last 2/4 sandwich among 8 children.

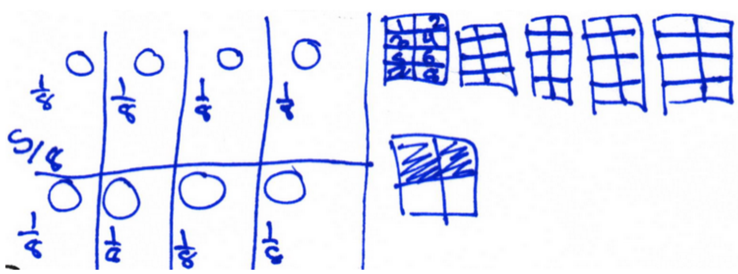

*Figure 11.* Gloria's partial written work for sharing 5 2/4 sandwiches among 8 children

**Adjusting the Problem**

To build understanding, children need to engage in solving problems that present a level of challenge that is within the zone of their understanding. Children's struggle to solve a problem may indicate that they are not able to work from what they understand to make sense of the problem. An adjustment in this case would bring the problem within their zone of understanding. Alternatively, children who answer problems with automaticity or recall may need to be challenged with a more difficult problem that requires more intellectual effort.

Mateo was working on the fudge problem in Mr. S's class, in which he had to figure out how much fudge was given away if Mary started with 5 1/2 pounds and ended up with 2 9/10 pounds. He initially added the whole-number amounts to get 7 wholes and his drawing shows that he represented each quantity separately, suggesting he did not understand the problem structure (Figure 12). Mr. S responded: "So let me ask you a question. I'm gonna change the question a little bit, okay? So if Mary bought 5 1/2 pounds of fudge and then her mom gave her 2 9/10 as well, how much fudge does she have in total?" This problem represented an easier structure yet still required Mateo to reason with fractional quantities. After Mateo added the whole-number amounts but ignored the fractional amounts, Mr. S decided to adjust the fractional quantities in the problem to 5 1/2 pounds and 2 1/2 pounds, often a more accessible combination for children. Again, Mateo did not combine the fractional amounts, perhaps because he did not understand a half as an amount that could be combined with another fractional amount. Thus, although Mr. S's teaching moves were responsive to Mateo's mathematical thinking, in this case, the child did not complete a successful strategy and Mr. S did not insist that he use a strategy he did not understand.

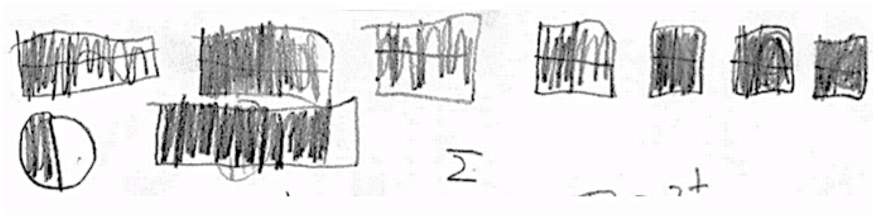

*Figure 12.* Mateo's initial attempt to solve the problem in which fudge was given away, with 5 1/2 as the start quantity and 2 9/10 as the end quantity, represented separately

## ***Educational Implications***

The categories of teaching moves that we have shared in this paper comprise a framework for instructional interactions that build on children's thinking in fractions. The framework was derived on the basis of studying teachers who had recognized expertise in responding on the basis of children's thinking about fractions, during individual problem-solving interviews and also during one-on-one interactions in class and it extends earlier research on teachers' work with children in problem-solving interviews with whole-numbers.

This work addresses the growing need for research to help professional developers support teachers in developing expertise in responsive teaching—teaching in which children's mathematical thinking is central and teaching moves are adaptive. Despite widespread enthusiasm for this approach to instruction, expertise has proven challenging to develop. In particular, teachers often struggle with how to be active but not too directive in advancing children's learning. A common recommendation is that teachers should "not tell" students how to solve problems, but this recommendation is insufficient because it is about what *not* to do instead of what to do (Chazan & Ball, 1999). Our framework for teaching that is responsive to children's mathematical thinking addresses these issues by identifying and sharing examples of the types of teaching moves skilled teachers actually *do* in pursuit of advancing children's thinking.

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