Case Studies of Computation and Algebraic Reasoning in K – 2 Students[1]

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Abstract

Incorporating algebraic thinking practices into the K-5 mathematics curriculum has been institutionalized through the Common Core State Standards Initiative and National Council of Teachers of Mathematics' published standards. When and how to engage elementary students in algebraic concepts are still under debate and development. This paper contributes to the position that students are ready to use algebraic thinking strategies from the beginning of formal education. In particular, the data presented show how three young students, one each in kindergarten, first grade, and second grade, work through a functional thinking task from a teaching experiment. The analysis of these cases demonstrates that knowledge of the most basic arithmetic operation, addition, can support productive algebraic exploration. In turn, exploration of functions can be a fruitful context for students to build their computational proficiency.

Typically, formal algebra is not evident in mathematics curricula until middle school or high school. Yet algebra, as the science of reasoning about number structures (Devlin, 2011), and arithmetic, as the science of calculating with numbers (Devlin, 2011), share a complementarity that can be (and should be) leveraged throughout K-12 mathematics education. This work is part of a larger teaching experiment, "Exploring K-2 Children's Understanding of Functions" (hereafter called the "K2 Function Study"), in which algebraic reasoning activities were introduced to students in kindergarten, first grade, and second grade. Other analyses provide evidence that students in these grades are capable of generalizing functional relationships (Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2014) and of flexibly expressing those generalizations through natural language, use of examples, and variable notation (Brizuela, Blanton, Gardiner, Newman-Owens, & Sawrey, 2014), thus supporting the call for integrating algebraic reasoning tasks from the beginning of elementary school.

Building from that base, this work takes an intimate look at the interplay between computational and algebraic fluencies in order to address the following research question: *How does students' developing computational fluency interface with their algebraic reasoning*? We analyze three students' work, one student from each of the study's grades, on a linear function task. Specifically, we examine how their computation procedures influenced the representations they produced, and how they described and generalized patterns in their function tables. The analysis of these cases provides evidence that moderate fluency of the most basic arithmetic operation, addition, can support productive algebraic exploration. In turn, exploration of functions can be a fruitful context for students to build their computational proficiency.

Theoretical Perspective

Although formal algebra is not typically present in mathematics classrooms until middle or high school, research in elementary school algebra suggests that there is no educative advantage to delaying algebraic reasoning activities in elementary grades (Blanton, et al., 2015; Carraher & Schliemann, 2007; Russell, Schifter, & Bastable, 2011; Warren & Cooper, 2007). More urgently, such a delay may be a lost opportunity in the education of young children (Carraher, Schliemann, & Brizuela, 2001). Algebra has a complementarity with arithmetic that can and should be leveraged. Along with others (e.g., Blanton, Levi, Crites, Dougherty, 2011; Carpenter, Franke & Levi, 2003), we argue that algebra should be introduced concurrently with the most basic arithmetic from the beginning of formal education.

Contributing to that perspective, this work emerged from a classroom teaching experiment in functional thinking. A functional thinking approach focuses on mathematical relationships between covarying quantities, and can involve finding or applying a function rule to those relationships (Carraher, Schliemann, & Schwartz, 2008; Earnest & Balti, 2008; Moss & McNab, 2011; Schliemann, Carraher, & Brizuela, 2007; Warren, Cooper, & Lamb, 2006). Functional thinking is evident through the explicit representations people make of a functional relationship through natural language and drawings, generalized algebraic equations, tables, or graphs (Carraher, Schliemann, & Schwartz, 2008). It is also evidenced in the ability to reason with those representations to interpret and predict function behavior (Blanton, 2008).

Method

We analyze students' work on a functional thinking task in semi-clinical interviews to address the following research question: *What is the interplay between computational fluency and algebraic reasoning among K-2 students?*

The interviewed students were participants in an 8-week classroom teaching experiment in three classrooms (one each of kindergarten, first grade, and second grade) in an urban, demographically diverse elementary school in the northeast United States. The first four weeks of the intervention focused on functions of the form $y = mx$, and the second four weeks focused on functions of the form $y = x + b$. These two suites of bi-weekly lessons (40 minutes each) were bookended by interviews with individual students (roughly 30 minutes each) at the beginning, middle, and end. Members of the research team led classroom lessons and interviews using protocols developed by the team. Interviews were transcribed verbatim.

Data Sources and Analysis

This work serves as an instrumental case study (Willig, 2013), demonstrating particular ways in which K-2 students use both arithmetic and algebra in exploring a functional relationship. The data set was narrowed to final interviews (post-interviews) with three students[2]: Shay in second-grade, Lea in first-grade, and Del in kindergarten. The three interviews are not intended to be contrasted or compared; as a set they show the different ways in which K-2 students use arithmetic and algebra to explore co-variation.

The final interview task explored the following: if a train picks up two train cars at each stop, what is the relationship between the number of train cars and the number of stops the train has made? This interview was chosen because the task requires more than reiterating the storyline. That is, the story is recursive (two train cars are added at each stop), while the function is a doubling relationship (double the number of stops to calculate the number of train cars). Students can therefore interact with the task from both a recursive and functional perspective. The three particular interviews were chosen because the students' algebraic reasoning and use of

[2] Student names used in this work are pseudonyms.

computation was evident in interviews products and therefore available for analysis. For example, they evaluated the number of train cars at different stop numbers, they articulated their reasons for mathematical choices, and they articulated or used a mathematical rule.

Once the interview data set was selected, a line-by-line review of the three interviews was conducted to flag episodes where students were using an implicit or explicit mathematical rule. Memos, initially simple retellings of the episodes, served to highlight aspects of the story between computation and algebraic reasoning. Through reanalysis of the episodes and in discussion with members of the research team, the memos were refined and elaborated. From here, specific excerpts that we considered faithful representations of the students' thinking and work products were selected to address the research question.

Results

In the post-interviews, students were told this storyline: "There is a train, and as it goes along, it picks up two train cars at each stop. At stop number one, it picks up two train cars. At stop number two, it picks up two more train cars." All three students:

- calculated 2, 4, and 6 train cars at stops 1, 2, and 3 without difficulty;
- generated a function table that captured value pairs, with stop number on the left and number of cars on the right (see Figures 1, 2, and 3); and
- extended the pattern beyond stops one and two to include additional values (like 4 stops) and non-consecutive values (like 10 stops).

Additionally, they each had their own ways of reasoning about the story. As will be described, Shay (2nd grade) used flexible calculation skills to test her mathematical rule. Lea (1st grade) organized the information in a function table and used the table to find a mathematical rule, and Del (kindergarten) tallied how many cars were at each stop without prompting. These students' work provides evidence that:

- functional thinking tasks can be designed for students with basic computation skills such that they engage in algebraic reasoning, and
- functional thinking tasks strengthen computational skills by encouraging alignment between numerical relationships found through computation and contextual relationships embedded in the task.

Shay, a second-grade student

Once the task was introduced, the interviewer asked Shay how many train cars would be on the train at stop four. Shay responded that there would be eight, "Because four plus four equals eight and two times four equals eight. So it goes, two, four, six, eight." Shay's justification covered several arithmetical ways of reaching eight from four: add four twice, multiply four by two, or add two recursively four times. This justification was completely computational, and did not indicate whether she was considering the context of the problem. In fact, it is possible that Shay was not explaining her reasoning in terms of the task context, but that she was instead giving an *ad hoc* list for how eight can be calculated from four and two. Shay's answer does show that she knew more than one way to calculate the output (number of cars) from the input (number of stops), which speaks to her fluency with the computations embedded in the functional relationship.

When prompted to put her values in a table, she drew a function table with two columns which she labeled "# of Days" and "# of cars" and filled in the values row by row (see Figure

1).[3] Through examination of the values in her table, Shay developed a rule, "add the number of days [sic] two times to get a certain number of cars." When the interviewer challenged her to test her rule by figuring out the number of cars at 10 stops, Shay could have simply used her rule to answer the question. Instead, Shay calculated the number of cars two ways to check her rule, as described in the following excerpt.

[Sh1] Interviewer: What if I jumped some numbers and put "10 days" [sic] right here [pointing to the left hand column], what would be the number of cars?

[Sh2] Shay: So. [Gesturing to self with hands for 15 seconds.] Oohhh.

[Sh3] Interviewer: Tell me what you are doing right now…

[Sh4] Shay: I'm counting by twos to get to the right number.

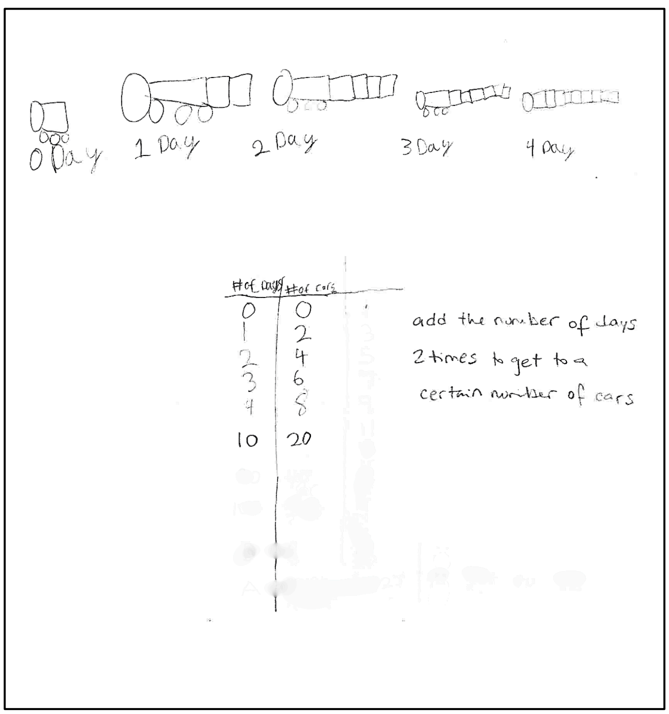

Figure 1: Shay's Function Table for y = 2x Train Function

[3] Shay seemed to interpret the situation as if the train were making one stop each day. The interviewer followed Shay's lead of using "days" instead of "stops."

[Sh5] Interviewer: But then, when you said "ohhh,"... what were you thinking about?

[Sh6] Shay: I was thinking that I should count by twos.

[Sh7] Interviewer: You were thinking that you should count by twos? So you're not using this rule [pointing to the natural language rule in Figure 1].

[Sh8] Shay: No, but I'm adding the number two times to get to the right number.

[Sh9] Interviewer: Ok. So, are you counting by twos or are you adding the number two times?

[Sh10] Shay: No. I'm adding the number two times, and I got to twenty, so I'm counting by two's now to see if I get the same answer.

In finding the number of cars the train would have after 10 stops, Shay calculated 20 train cars using her rule, 10 + 10 = 20, but she also verified that rule using the task context: add two cars at each stop. Thus, the computation procedure "counting by twos" verified to her that her function rule "add the number of days two times to get a certain number of cars" matched the problem context.

Shay's interview demonstrates how a function context built on simple computation (addition and doubling) can promote algebraic thinking in young students. At the most basic level, Shay calculated the number of train cars at a given stop for a range of values. Using those calculations, she generalized a rule for the correspondence relationship between stops and cars. She then verified through computation that her function rule gave the same answer as calculating the number of cars recursively. In each of these moments, Shay was strengthening her computation skills by engaging in the task context while her proficiency with addition gave her the opportunity to explore the functional task in a variety of ways.

Lea, a first-grade student

The interviewer introduced the train context, and Lea quickly surmised, "So every stop, it picks up two cars." The interviewer concurred, then asked what Lea was going to do, and Lea suggested a T-chart (two-column function tables), and filled in values for the number of stops and number of cars with apparent ease (see Figure 2). Looking for insight into Lea's work process, the interviewer asked, "How do you get from this number [pointing to "1" in the left column] to this number [pointing to its pair, "2," in the right column]? What's your rule to get from one number to the next?" Lea's answer was that you "Skip zero." The interviewer asked for further clarification:

[L1] Interviewer: You skip zero, you said? … How do you get from 2 to 4? Or from 3 to 6?

[L2] Lea: You skip one.

[L3] Interviewer: Always skip one?

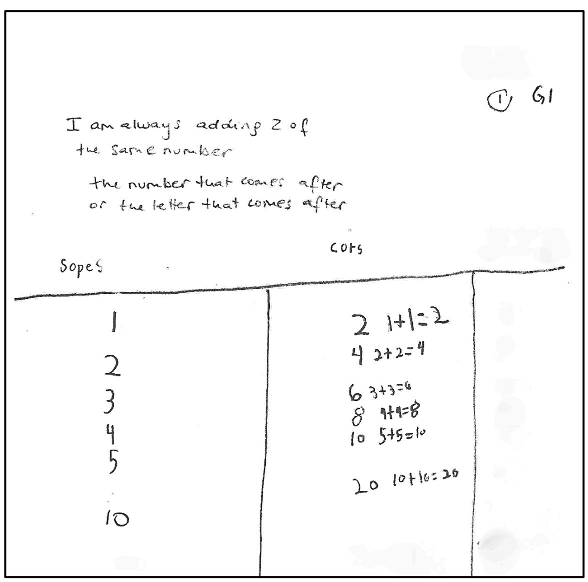

Figure 2: Lea's Function Table for y = 2x Function

[L4] Lea: No. [Pause.] If- with five you skip: one, two, three, four. [Touches the number line taped to the desk at the numbers 6, 7, 8, and 9, to get from five to ten.] You skip four.

The skipping that Lea referred to (i.e., line L3 and L4) is a way of using the number line as an aid in addition. For example, if we consider adding five and five, the starting number is five, and a student puts their finger on the number line at the number five. To complete the addition, they then move their finger through five jumps (or skips) to reach 10. Students sometimes count how many numbers they skipped over, rather than how many skips they made. When Lea said that to get from one to two you "skip zero," she was noticing that two is right next to one on the number line (evidenced in next excerpt, see line L8). Similarly, to get from two to four, one number is skipped (the number three), and to get from five to ten, four numbers are skipped (six, seven, eight, and nine). Carraher, Schliemann, Brizuela, and Earnest (2006) described this as a "fencepost" problem (p. 107). Both counting the intervals and counting the skipped numbers have logic, and it is easy for students to confound what needs to be counted.

Luckily for Lea, the interviewer sought common ground for the computations underlying the values in Lea's table. The excerpt below directly followed statement L4 above:

[L5] Interviewer: Well, let's think about that. Let's think about how many you're skipping. So when you're at one, to get to two you skip one. Right?

[L6] Lea: [Shakes her head, "No."] It doesn't always work.

[L7] Interviewer: Well, let's figure it out.

[L8] Lea: You skip zero because after one comes two.

[L9] Interviewer: Yea, so you...oh, ok. The next one. And when you're at two… Well, actually, what I'm saying is...you .. you have to do plus one. That's what I meant. Plus one. Right? Now when you're at two to get to four, what do you do? [Her index fingers are on the number line at two and four.]

[L10] Lea: You add two!

[L11] Interviewer: Oh, OK.

[L12] Lea: I get it!

First, the interviewer shared her understanding of how skipping works. When Lea disagreed, the interviewer reframed the task from skipping to addition by suggesting "What I'm saying is you have to do plus 1" as she showed the movement on the number line. She then encouraged Lea to supply the operation to get from two to four. Lea accepted the reframing, and happily declared, "You add two!" Together, Lea and the interviewer produced the addition expressions for stops one through five (see Figure 2). Lea's sense that the same mathematical rule applied to any number of stops, her nascent "sense of functions" (Eisenberg, 1992), created a context for her to explore and resolve her understandings about addition. Later in the interview, Lea articulated the general addition pattern as a mathematical rule, linking the number of stops to the number of cars, saying, "I'm always adding two; two of the same number!"

Del, a kindergarten student

Del computed the number of train cars after the train made one, two, and three stops as the interviewer shared the train story. He then put those values in a function table (see Figure 3) at the request of the interviewer. When asked about stop four, he wrote "4" on the left-hand side of the chart and "8" on the right-hand side of the chart. His reasoning for how he was finding these additional values is "by plusses":

[D1] Interviewer: By plusses? What do you mean "by plusses"?

[D2] Del: Like, one plus one: two; two plus two: four; three plus three: six; four plus four: eight. [Del touches each number in the function table as he says this.]

[D3] Interviewer: Wow. What if I tell you ten? [Writes "10" in left column.] What's gonna go on this side? [Pointing to right hand column.]

[D4] Del: Twenty. [Writes "20" in the right column.]

[D5] Interviewer: Okay. So tell me the plus that you would do to get from here [taps left (number of stops)] to here [taps right (number of cars)].

[D6] Del: Ten plus ten equals twenty.

As Del heard the storyline, he recursively added two cars to his total count. When he saw the values in the function table, he recognized a "plusses" correspondence relationship

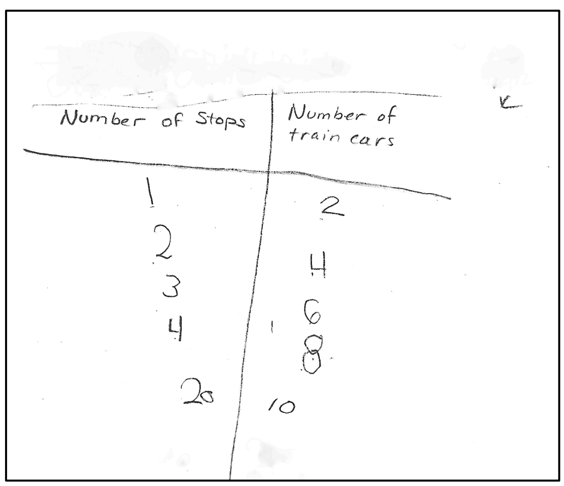

Figure 3: Del's Function Table for y = 2x Train Function

between the columns. Although he never articulated a mathematical rule, Del was later able to adapt his implicit rule when the context of the situation changed. Near the end of the interview, the interviewer suggested considering the engine as part of the train car count, which would change the function from $y = 2x$ to $y = 2x + 1$. Del acted on his "plusses" function directly as mathematical object (Sfard, 1991).

[D7] Interviewer: Alright, so now, let me give you a hard one. Let's say, um, -- okay, so when I said ten stops, you told me twenty train cars. What if I say ten stops with this one [referring to the function table which includes the engine in the total number of cars (see Figure 4)]? How will the answer be different?

[D8] Del: Uh... wait a second. Twenty-one, right?

[D9] Interviewer: How'd you get that?

[D10] Del: I don't know… When I put eleven—[Writes "11" in left-hand (stops) column, see Figure 4]. That would be twenty-two, right? Yeah, twenty-two.

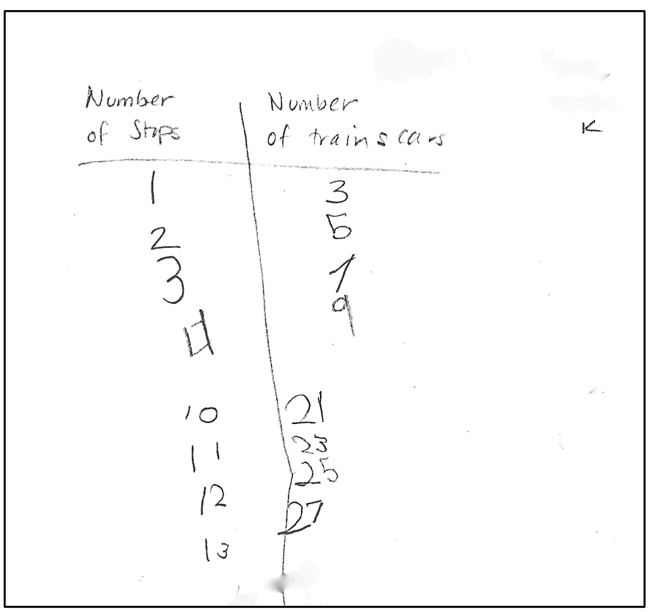

Figure 4: Del's Function Table for y = 2x + 1 Train Function

[D13] Interviewer: Okay. Does that --

[D14] Del: Yeah, twenty-three! [Writes "23" in right-hand (cars) column.]

[D15] Interviewer: Ahh. Nice. Okay.

[D16] Del: And-- [Writes (12, 25), (13, 27) in the function table, Figure 4.]

[D17] Interviewer: So if you had to explain to somebody how to get from ten cars to-- I mean, ten stops to twenty-one cars, is there a way to explain it?

[D18] Del: No.

The interviewer created an opportunity for Del to simply add one more to his previous work when she directed is attention to his first table (line D7). However, Del's work on 11 stops (lines D10 – D14) indicate that he instead changed the algorithm to accommodate the inclusion of the train engine. Although his "function" was never made explicit through either natural language or a mathematical equation, he was able to calculate accurate solutions for the new situation (much like Vergnaud's theorems-in-action, 1996). That is, Del's interaction with the train problem function was embedded in computational cases, but he was applying a general calculation procedure to those cases. His ability to reason with a general procedure (in this case, adjust it to accommodate counting the engine) is an important characteristic of algebraic reasoning (Blanton et al., 2014).

Significance of the Results

A fundamental tenet in mathematics education is that computational proficiency develops over time with practice in varied situations (Russell, 2010; Tall, 2013). The functional thinking task gave Shay, Lea, and Del opportunity for purposeful, connected and varied computation.

Shay's interview shows the development of a function rule from exploration of the train task. The fluidity and flexibility with which Shay handled the computations around the function gave her many options to consider: adding two recursively, adding a number twice, and/or doubling a number. Noticing the doubling relationship in rows of the table, she then applied that rule to find the number of cars after the train made 10 stops and verified her answer by the recursive procedure in the task description.

Lea needed more computational assistance than Shay, yet she too developed a general rule for finding the number of train cars given the number of stops. For Lea, exploration of the algebraic task required her to resolve some of her understandings about addition. The coordination of the value pairs with unexecuted expressions gave Lea purposeful practice with doubling addition facts. Furthermore, the work Lea and the interviewer did to explore those patterns in the table addressed the ambiguity of "skipping" as a procedure for addition. In this way, the algebraic exploration was an avenue for developing more robust computational practices.

Del generated his function table following the description of the story. He was able to identify the functional relationship between the columns of his function table because he recognized that each row was one a doubles addition facts. Del did not develop an explicit mathematical rule for the problem context, yet he developed an implicit rule that he was able to modify to account for counting the train engine. Although evidence of Del's algebraic thinking is more subtle, he was able to generalize an implicit mathematical rule, building experience in both arithmetic and algebra. Moreover, Del's computational work on specific cases indicated he was reasoning about a generalization that remained implicit.

These cases present evidence that basic knowledge of addition was enough for students to engage in algebraic thinking practices of generalizing, representing, and justifying. The functional aspect of the problem context, in turn, was excellent opportunity for practicing and generalizing number operations. The cases further highlight how arithmetic and algebra are mutually supporting ventures in the mathematical education of young students.

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