Extending the Empirical Basis of a Hypothetical Learning Trajectory for Length Measurement

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Overview and Research Questions

Researchers have highlighted the importance of hypothetical learning trajectories (LTs) in mathematics education, and have called for studies that provide an empirical basis for this work beyond the elementary grades (Daro, Mosher, Corcoran, 2011). This session reports data from NSF-supported research aimed at extending an LT for length measurement (Clements et al., in press) into the middle and secondary grades. A written length LT-based assessment (Clements et al., in press) was designed and administered to 82 students from Grades 4, 6, 8, and 10. By investigating the levels of thinking exhibited by students across a wide span of development, this study addressed questions about LTs that could not be addressed in prior studies focused mainly on elementary children's thinking (Battista, 2006; Clements et al., in press):

1. What levels of an LT for length measurement do elementary, middle, and secondary students exhibit in a typical US educational context?
2. How does an LT for length measurement extend into the middle and secondary grades?

Theoretical Framework

We view the LT for length measurement (Clements et al., in press) through the lens of hierarchic interactionalism (HI), which is a cognitive theory of development that synthesizes aspects of nativism, empiricism, and interactionalism (Clements & Sarama, 2007). From this perspective, development is an "interactive interplay among specific components of knowledge and processes" (p. 464). The LT for length measurement originates from an assumption of HI, which postulates that children progress through domain-specific levels of understanding in ways that can be characterized by concepts and processes that build hierarchically on previous levels.

From the theory of HI, each level develops gradually out of preceding levels (Sarama, Clements, Barrett, Van Dine, & McDonel, 2011; Sarama & Clements, 2009). That is, children often can be considered partially at one level while *reaching up* to show characteristics of the next. Thinking or operating characteristics of lower levels are not abandoned; children *fall back* to exhibit a lower level of thinking if a task is complex, or *reach back* to use a lower level of thinking if a task is simple. Therefore, a claim that a child is "at" a particular level is a probabilistic statement about a child's likely response in a context in which the level is relevant.

LT for Length Measurement

The LT for length measurement "describes an important sequence of knowledge about quantity, based on a ratio between a unit and the measured object, and other measured lengths as ratios" (Barrett, et al, 2012, p. 51). In the following sections we summarize the concepts and processes that characterize this hierarchic development, which begins with the recognition of length as an attribute (Clements et al., in press). Across the first two levels, children use continuous mental processes as they evaluate continuous extents. At the earliest level, *Length Quantity Recognizer* (LQR), children identify length (the extent of an object from end-to-end) and distance (the amount of space between two points) as attributes; however, they do not yet understand length as a comparative. The second level, *Length Comparer* (LC), involves two sub-levels, *Length Direct Comparer* (LDC) *and Indirect Length Comparer* (ILC). At the LDC sublevel, children physically align a pair of objects for the purpose of determining which is longer, and children at the ILC sublevel use a third object to compare the lengths of two objects.

The transition into the third level, the *End-to-End* (EE) level, marks a significant conceptual advance over the first two levels because it marks the development of the implicit concept that lengths can be composed of repetitions of shorter lengths. Students at this level understand that the number of repetitions of shorter lengths (or units) that fit along an object describe its length, and they typically lay units end-to-end to measure the length of an object. By the next level, the *Length Unit Relater and Repeater* (LURR) level, children measure by repeating, or iterating, a unit. They also understand that more shorter units or fewer larger units are needed to measure the same object and can add two lengths to determine the length of a whole. Because the goal of this study was to extend the empirical basis of the length LT into the middle and secondary grades, we targeted levels beyond the development of the concept of unit iteration. That is, we sought to target LT levels above LURR.

By the next level beyond LURR, *Consistent Length Measurer* (CLM), children simultaneously imagine and conceive of an object's length as a total extent and a comparison of units. At this level, children see length as a ratio comparison between the unit and the object measured. They begin to estimate reasonably, measure straight paths consistently, use equal-length units, understand the zero point on the ruler, and can partition units to make use of units and subunits for the purpose of increasing precision. However, when determining the length of a bent path, children at this level may make rounding errors when measuring each segment and may not equate the sum of the parts of the bent path to the length of the whole. In addition, they may not be perturbed with geometric inconsistencies when coping with perimeter items. For example, when asked to draw a rectangle with a specified perimeter, a child at the CLM level may draw a rectangle with opposite sides that are not congruent.

By the *Conceptual Ruler Measurer* (CRM) level, children have an "internal" measurement tool. That is, they employ *explicit* strategies to estimate lengths, such as mentally iterating internal units of length or partitioning a length into equal parts. Children at the CRM level project or translate given lengths to determine missing lengths. When asked to draw a rectangle with a specified perimeter, children at this level notice or are perturbed by geometric inconsistencies; they no longer accept rectangles with opposite sides that are not congruent.

The *Integrated Conceptual Path Measurer* (ICPM) level marks a transition point in the LT from an emphasis on the development of the concept of unit iteration toward an emphasis on linear quantity, geometric/spatial characteristics, and reasoning toward proof (see Barrett et al., 2006; Clements et al., in press). This level, and the subsequent level, both characterize the hierarchic development of increasingly sophisticated logical thought and justification in geometric situations. At the ICPM level, children integrate and compare sets of units along each section of a bent path. When reflecting on the measure of a bent path or the perimeter of a polygon, they regard a group of units as a flexible object, a "string" of units wrapped around the entire perimeter or along the entire path. Therefore, in the context of a fixed perimeter or fixed path length item, children at the ICPM level compensate for changes made to one side of a figure by adjusting other sides to maintain the fixed overall length. Although, they can find several related cases of polygons with the same perimeter, they may not yet be able to organize and synthesize a set of related polygons based on perimeter to formulate and justify a valid argument.

The highest level of the current length LT is the *Abstract Length Measurer* (ALM) level. At this level, children have developed a continuous sense of length, and engage dynamic imagery to coordinate and operate internally on collections of units of units as well as collections of complex paths. Within the context of a fixed perimeter or path length item, they can synthesize sets of figures based on perimeter to formulate and justify a valid argument. Children at this level can coordinate multiplicative and additive reasoning in fluent ways and can engage in proportional reasoning about coordinated cases of paths for the purpose of reflecting on patterns among cases. Table 1 below summarizes the length LT levels that are most relevant to this study.

Table 1

Length LT Levels Relevant to the Present Study

Level	Summary*
Consistent Length Measurer (CLM)	Measures straight paths consistently, uses equal-length units, understands the zero point on the ruler, partitions units
Conceptual Ruler Measurer (CRM)	Mentally partitions lengths by projecting a mental unit, a ruler, or a sequence of units onto an unpartitioned object; projects or translates given lengths to determine missing lengths
Integrated Conceptual Path Measurer (ICPM)	Integrates and compares a set of units along each section of a bent path; regards a group of units as a flexible object, a "string" of units wrapped around the entire perimeter or along the entire path; copes with sub- and superordinate units
Abstract Length Measurer (ALM)	Develops a continuous sense of length; engages dynamic imagery to coordinate and operate internally on collections of units of units and collections of complex paths

Methodology

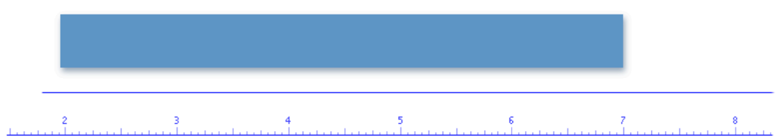
Participants and Data Sources

We designed and administered an open-response length LT-based assessment to 71 students in the Midwest from Grades 4, 6, 8, and 10 (26 in Grade 4, 18 in Grade 6, 20 in Grade 8, and 7 in Grade 10). We recruited participants from two different private schools in the Midwest, one for pre-K – 8 students and another for pre-K – 12. At the pre-K – 8 school, we selected all consenting students from two classes each in Grades 4, 6, and 8. We selected the seven consenting Grade 10 participants from the pre-K – 12 school, where there were a total of 22 Grade 10 students enrolled in Algebra I, Algebra II, and advanced math. Data sources are students' written responses to six assessment items.

Instrument Design

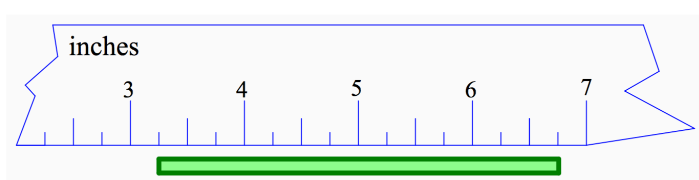
We designed Items to elicit observable strategies that indicate particular mental actions and objects that differentiate the levels of the length LT (Clements et al., in press). Some Items were designed to reveal thinking at a variety of LT levels. For the purpose of designing the instrument, we mapped Items to the highest length LT level they were shown to elicit in prior research (Clements et al., in press). To provide confidence in the level placement assigned by this instrument, we included two Items for each of the CLM and CRM levels. Because we experienced difficulties with designing Items that differentiate students at the highest two levels of the length LT in our prior work (Clements et al., in press), we included two Items to probe students' thinking at the ICPM and ALM levels (see Figures 5 and 6). The following sections describe the design and the purpose of including each item, as well as the methods or procedures that our research team used to analyze students' written responses.

CLM level items. Assessment Items 1 and 2 shown in Figures 1 and 2 below have been shown to elicit thinking at the EE, LURR, or CLM levels of the length LT (Barrett et al., 2012). Therefore, in the present study, we regarded them as CLM-level items.



Using the drawing of a part of a ruler as a guide, measure the strip of paper shown above it. How many inches long is the strip?

Figure 1. Written LT-based assessment CLM level item, Item 1.

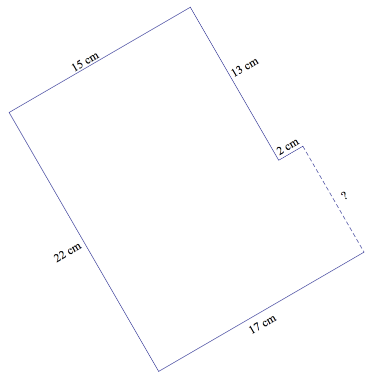


This is a picture of a rod just below a broken section of a ruler. Use this picture to measure the length of the rod. How long is the rod?

Figure 2. Written LT-based assessment CLM level item, Item 2.

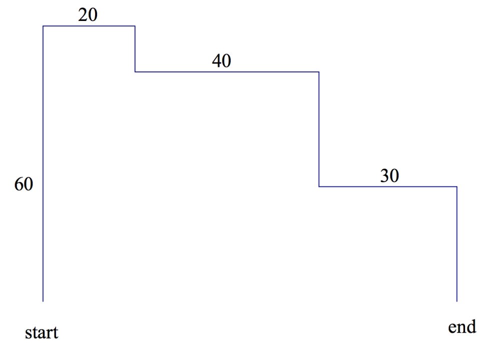
The CLM level items shown in Figures 1 and 2 were designed to investigate students' ability to integrate intervals and endpoints of those intervals (Barrett et al., 2012; Cullen, 2009).

CRM level items. Assessment Items 3 and 4 (Figures 3 and 4) have been shown to indicate whether students are operating as high as the CRM level (Clements et al., in press).



Find the measure of the missing side length.

Figure 3. Written LT-based assessment CRM level item, Item 3.



Find the length of the total path, from start to end, shown above.

Figure 4. Written LT-based assessment CRM level item, Item 4.

Specifically, items 3 and 4 (Figures 3 and 4) were designed to distinguish students' ability to apply the CRM-level strategy of projecting or translating given lengths to determine missing lengths (Clements et al., in press) from a CLM-level (less sophisticated) strategy of estimating the missing lengths.

ICPM and ALM level items. Assessment Items 5 and 6 (Figures 5 and 6) have been shown to be accessible to children at the CLM, CRM, ICPM, and ALM levels of the length LT (DRL 0732217; DRL 1222944).

Imagine making an L-shaped path from a string that is 10 cm long.

a. How many different L-shaped paths would you be able to form in all?

b. Use the space below to explain how you got your answer and why you think your answer is correct.

Figure 5. Written LT-based assessment ICPM and ALM level item, Item 5.

a. Use the space below to sketch **two different** rectangles, each having a perimeter of 2 inches. For each of your rectangles, label the lengths of all four sides.

b. How many **more** rectangles have a perimeter of 2 inches? _____________

Figure 6. Written LT-based assessment ICPM and ALM level item, Item 6.

Items 5 and 6 were designed to explore students' ability to find several related cases of bent paths or polygons with the same path length or perimeter and to relate those cases to one another by logical comparison, which is ICPM-level thinking (Clements et al., in press). Item 6 also reveals students' abilities for coping efficiently and precisely with subordinate units in the context of finding related cases of polygons with the same perimeter. Part b for both Items 5 and 6 also have the potential to reveal whether students are aware that subdividing a unit into subunits is a process that is potentially unlimited, which is ALM-level thinking.

Analysis

Analysis of students' responses to the six items in the assessment proceeded according to three phases. In the first phase, our research team collaborated to group students' responses to each of the six Items into emergent categories, which we generated through a constant comparative method of analysis (Merriam, 2009). In doing so, we considered only students' final responses. For this first phase, through an initial cycle of open coding (Corbin & Strauss, 2008), we made comparisons among participants and developed categories to identify qualitatively different observable written responses that reoccurred with regularity. This yielded six response categories for Item 1, 20 categories for Item 2, 13 categories for Item 3, 13 categories for Item 4, 17 for Item 5, and 23 categories for Item 6.

For the second phase, we used the LT for length measurement to extract thematic categories that represented the LT levels from the set of response categories that were generated in the first phase. For this second phase of the analysis, the research team met and discussed the mapping of each response category for each item to one of the levels of the length LT until consensus was reached. Our team coded students' written responses that were not consistent with any of the levels of the length LT as "No Claim." This phase of analysis yielded an LT level claim for each of the six assessment items for each of the 71 participants. Based on the levels observed on the six individual assessment items, the research team also collaboratively identified the predominant overall LT level exhibited by each participant.

After mapping students’ responses (to individual items) to the length LT via the process of creating response categories, and then collaboratively assigning an aggregate LT level claim for each participant, we then subjected this coded data to a third phase of analysis, a frequency analysis. In this third phase, we tracked the frequency of the appearance of each LT level per participant and per item, as well as predominant level claims, using spreadsheet software. Next, the research team examined developmental patterns in the variability exhibited within and across participants, grade-by-grade. Findings from this frequency analysis constitute the extension of the empirical basis for the LT for length measurement for the four levels that we addressed in this study: the CLM, CRM, ICPM, and ALM levels.

Results and Discussion

In the sections below, we first demonstrate the frequency of appearance of each LT level within and across grades for the group of CLM level items (Items 1 and 2), CRM level items (Items 3 and 4), and ICPM and ALM level items (Items 5 and 6). Next, we illustrate the distribution of predominant levels exhibited by participants on the collection of six assessment items. Finally, we examine individual differences exhibited within and across each grade level.

Grouped item-by-item analysis. The figure below (Figure 7) illustrates the predominant level within each grade for the CLM-level Items (Items 1 and 2). The most predominant level within each grade for each item is indicated by the darkest shade of blue.

Item 1

	Grade 4	Grade 6	Grade 8	Grade 10
CLM	5	9	15	6
LURR	14	4	4	0
EE	7	5	1	1
No Claim	0	0	0	0

Item 2

	Grade 4	Grade 6	Grade 8	Grade 10
CLM	8	9	15	7
LURR	17	5	4	0
EE	1	4	1	0
No Claim	0	0	0	0

Figure 7. Grouped item-by-item results for CLM-level Items (Items 1 and 2).

For the CLM-level items (Items 1 and 2 in Figure 7 above), the LURR-level strategy of inefficiently coordinating tick marks and intervals peaked at Grade 4 and decreased across Grades 6, 8, and 10. By Grade 6, half of the participants exhibited the CLM-level concepts of identifying zero as the starting point for motion along a ruler, and seeing the numbers as reports of completed motion. By Grade 8, 75% of participants exhibited these CLM-level concepts. This increased to approximately 100% by Grade 10 (with only 1 participant exhibiting EE-level thinking on one of the CLM-level items).

The following figure (Figure 8) shows the predominant level within each grade for the CRM-level Items (Items 3 and 4).

Item 3

	Grade 4	Grade 6	Grade 8	Grade 10
CRM	2	6	15	7
CLM	19	11	4	0
not yet CLM	5	1	1	0
No Claim	0	0	0	0

Item 4

	Grade 4	Grade 6	Grade 8	Grade 10
CRM	0	3	10	7
CLM	12	5	5	0
LURR	13	10	5	0
No Claim	1	0	0	0

Figure 8. Grouped item-by-item results for CRM-level Items (Items 3 and 4).

For the CRM-level Items, (Items 3 and 4 in Figure 8 above) indicates that most of the Grade 4 and 6 students exhibited the CLM–level approach of estimating to determine missing lengths on Item 3. By Grade 8, 75% of students used the CRM-level strategy of projecting or translating missing lengths to determine a missing length. This was 100% by Grade 10. For Item 4, approximately half of Grade 4 and 6 students ignored the missing lengths, and added the labeled segments to determine path length (which we deemed as consistent with the LURR-level). Interestingly, none of the Grade 4 students used the CRM-level project and translate approach for Item 4. This first appeared at Grade 6, and the proportion of participants who used this approach consistently increased across Grades 6, 8, and 10. This suggests that Item 4, which was counched in a context of path length, may be more difficult than Item 3.

The figure below (Figure 8) depicts the predominant level within each grade for the ICPM/ALM-level Items (Items 5 and 6).

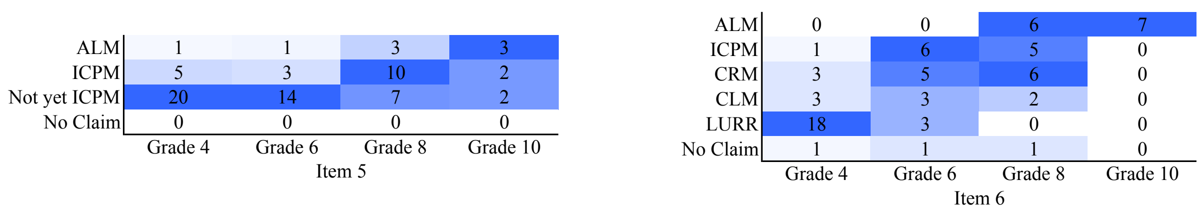


Figure 9. Grouped item-by-item results for ICPM/ALM-level Items (Items 5 and 6).

For the ICPM/ALM-level Items (Items 5 and 6 in Figure 9 above), students in Grades 4 and 6 predominantly exhibited strategies indicating that they were not yet at the ICPM level for Item 5, with just over 20% in each of Grades 4 and 6 exhibiting ICPM or ALM-level thinking. Half of the Grade 8 participants exhibited ICPM-level thinking, with 15% and 35% showing evidence of ALM-level thinking and “not yet ICPM” level thinking, respectively. For Item 5, participants’ responses were rather evenly distributed across the “not yet ICPM,” ICPM, and ALM-level categories. For Item 6, the majority of Grade 4 students exhibited LURR-level thinking, with only 12% (3 participants) showing evidence of CLM or CRM and 4% (1 participant) reaching up as high as the ICPM level. In Grade 6, just over 60% of the participants operated at the CRM-level or above on Item 6. By Grade 8, this number grew to 85%. Finally, by Grade 10, all participants exhibited ALM-level thinking for Item 6.

Predominant LT Levels

Based on responses to the set of six items, our research team reached consensus about the predominant length LT level exhibited by each of the 71 participants. We tracked the distribution of these claims within and across Grades 4, 6, 8, and 10. This distribution is depicted in Figure 10 below.

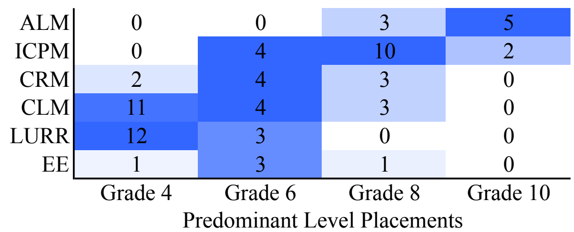


Figure 10. Distribution of predominant length LT levels within each grade.

In Grade 4, most of the participants predominantly exhibited LURR and CLM-level thinking across the six items on the assessment, with only one and two instances of predominant EE and CRM-level placements, respectively. We observed the most variability across participants in Grade 6, with nearly the same number of participants predominantly exhibiting every level from EE to ICPM. By Grade 8, this variability diminished, with half of the participants exhibiting predominantly ICPM-level thinking and the other half of the class mainly either still showing CLM and CRM-level thinking or showing growth into the ALM level. By Grade 10, most of the participants exhibited predominantly ALM-level concepts and processes across the six items of the written assessment.

These results indicate that, even when considering the aggregate level claim, or predominant level exhibited by each participant, individual differences in length measurement knowledge exist across the elementary, middle, and secondary grades. These individual differences are most pronounced in the middle grades, and appear to dissipate at the secondary level.

Overall Performance by Grade

After mapping coding categories to the LT for length measurement, the research team reached consensus on an aggregate level claim for each participant. Next, we examined the variability exhibited within and across participants for grade level. In each of the following figures, participants are represented by columns and levels are represented by rows. Each participant's predominant level is indicated with blue, and other levels reflected in the participant's responses to the set of items are indicated with grey.

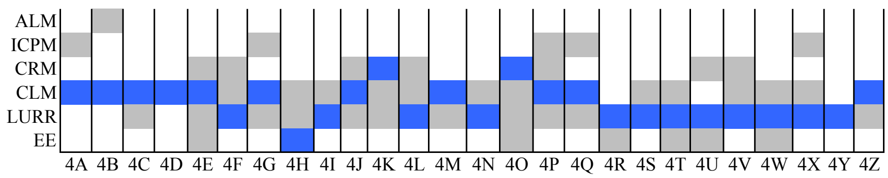


Figure 11. Length LT levels exhibited by Grade 4 participants.

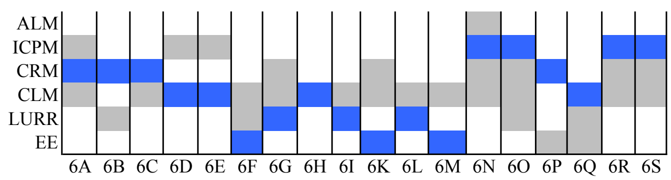


Figure 12. Length LT levels exhibited by Grade 6 participants.

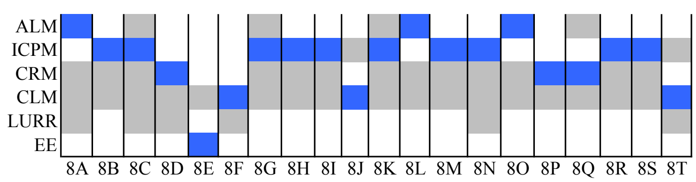


Figure 13. Length LT levels exhibited by Grade 8 participants.

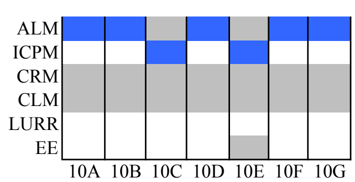


Figure 14. Length LT levels exhibited by Grade 10 participants.

Figures 11 - 14 demonstrate how LT levels develop gradually out of preceding levels by illustrating the *fall back*, *reach back*, and *reach up* that we predict, based on the hierarchic interactionalist perspective with which we view LTs (Clements et al., in press). For example, only three of the participants (4D, 4Y, and 6H) exhibited just their predominant level placement across all six items, with the remaining 69 participants showing some evidence of thinking at other levels.

Forty-eight participants exhibited levels below their predominant level. Here, it is important to note that none of the six items were deemed as LURR or EE-level items. Therefore, the 25 of these participants who showed evidence of EE or LURR below their predominant levels are instances of *fall back* to a prior level (For example, the research team determined that participant 4C exhibited predominantly CLM level thinking overall on the assessment, however, he or she showed evidence of LURR thinking on at least one of the items.) This is in contrast to instances of *reach back* exhibited by participants as they used strategies from LT levels lower than their predominant levels to meet the demands of the item. (For example, all of the Grade 10 participants were operating predominantly at the ICPM and ALM levels and exhibited instances of CRM and CLM. These instances of CRM and CLM occurred on Items 1 through 4, and they are not a *fall back* because CRM and CLM are the highest levels that these items can measure.)

Thirty-seven participants exhibited instances of reaching higher up into LT levels that were above their predominant level placements. For example, the research team determined that participant 4E was operating predominantly at the CLM level, but showed evidence of the subsequent level, CRM, on at least one of the assessment items. Fourteen of these instances involved a participant exhibiting an LT level that was both higher than and not adjacent to their predominant level. Four instances (4U, 6K, 6M, and 8E) involved a predominantly EE or LURR-level participant exhibiting the nonadjacent CLM or CRM levels. Next, ten of these instances (4A, 4B, 4G, 4Q, 4X, 6D, 6E, 8J, and 8T) involved LURR- or CLM-level students showing evidence of the nonadjacent ICPM or ALM levels. Finally, one instance (8Q) involved a CRM-level participant showing evidence of the non-adjacent ALM level on at least one of the items. These results indicate that the increasingly sophisticated logical thought and justification in geometric situations that is articulated in the highest two levels of the length LT (the ICPM and ALM levels) may begin to develop while the concept of unit iteration is still developing (at the EE, LURR, CLM, and CRM levels).

Conclusions and Implications

The present study addressed a significant gap in the literature with respect to the length LT levels exhibited by a cross-section of elementary, middle, and secondary level students. Prior to this study, researchers had described elementary children's thinking and learning for length measurement, as measured by the length LT (Clements et al., in press). Therefore, this study provides an extension of the empirical basis of length LT beyond its foundation established by the body of literature on elementary children's concepts and strategies for length measurement.

Researchers previously reported observing LURR and CLM level thinking predominantly in Grades 2 and 3 (Clements et al., in press); however, results reported here suggest that these levels are relevant for students in Grades 4, 6, 8, and 10. Taken together, these findings suggest that children continue to progress through the levels of the length LT beyond their elementary years into middle and secondary school in a typical educational context in the Midwestern United States. Furthermore, the highest level of the length LT did not appear consistently before the secondary level. This suggests that the scope of the current length LT is sufficient for describing most elementary, middle, and secondary students' thinking; however, more levels may be needed to describe and differentiate students' thinking beyond Grade 10.

In addition, results reported here indicate that the increasingly sophisticated logical thought and justification in geometric situations that is articulated in the highest two levels of the length LT (the ICPM and ALM levels) may begin to develop while the concept of unit iteration is still developing (at the EE, LURR, CLM, and CRM levels). This finding suggests that students operating predominantly at the EE, LURR, and CLM levels could potentially benefit from instructional activities designed to elicit and develop ICPM and ALM-level thinking (see Clements, et al., in press). More research is needed to investigate how children transition into the ICPM and ALM levels and the nature of the instruction that can support this growth.

Furthermore, the findings discussed in the sections above highlight individual differences in length measurement knowledge, which are present in elementary school, prominent in the middle grades, and nominal at the secondary level. This may be attributed to an intensive exposure to a year of study of geometry, which typically coincides with the transition into the secondary level. Alternatively, the decreased variability in length measurement knowledge observed in Grade 10 might be attributed to a ceiling effect of the LT. That is, there may be levels above the currently articulated highest level, the ALM level, which could bring individual differences in thinking for length measurement into focus. Future research is needed to explore these competing hypotheses.

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