Interpreting Student Work:

What Secondary Mathematics Teacher Candidates Bring to Preparation

Erin E. Baldinger

Arizona State University

Author Note

Dr. Erin Baldinger is an Assistant Professor of Mathematics and Teacher Preparation at Mary Lou Fulton Teachers College, Arizona State University. The research for this paper was conducted at as part of a dissertation study at Stanford University, and was supported in part by a Stanford Graduate School of Education Dissertation Support Grant.

Abstract

Being able to make sense of students' written work is a key practice that relies on having robust subject-matter knowledge and pedagogical content knowledge (Ball, Thames, & Phelps, 2008). It also is connected to a teacher's ability to notice students' mathematical thinking (e.g., Sherin, Jacobs, & Philipp, 2011). In this paper, I explore the strategies pre-service secondary math teachers use to analyze student written work based on in-depth, task-based interviews with eight participants. The participants engaged primarily in mathematical analysis of the written work, as well as analysis by comparison to their own solutions to the math tasks. One participant engaged in pedagogical analysis. The proficiency with which participants made assertions about the student work was related to their proficiency in solving the task as well as whether they were making high-inference or low-inference assertions. The results have implications for supporting teachers to implement the practices described in *Principles to Actions*.

Interpreting Student Work:

What Secondary Mathematics Teacher Candidates Bring to Preparation

NCTM's *Principles to Actions* (2014) describes critical mathematics teaching practices that build on a teacher's ability to make sense of student thinking. Interpreting students' written work is a key component of making sense of student thinking and implementing productive and equitable assessment practices. To inform teacher education efforts, in this study I investigate how secondary pre-service teachers reason about student written work at the beginning of their teacher preparation programs.

Literature Overview

Interpreting student thinking is a critical component of high quality instruction and assessment (NCTM, 2014; Teaching Works, 2012). However, this practice can be difficult for novice teachers to implement (Sleep & Boerst, 2012; Shaugnessy, Boerst, & Ball, 2014). To make sense of student work, teachers must draw on both knowledge of mathematics and knowledge about students, different aspects of mathematical knowledge for teaching (MKT).

Ball and colleagues (2008) argue that knowledge of content and students is a component of pedagogical content knowledge. In the context of interpreting student work, drawing on knowledge of content and students might take the form of recognizing a common student error (e.g., knowing students might believe $(a + b)^2 = a^2 + b^2$ when learning about the distributive property). Teachers might also draw on subject-matter knowledge when interpreting student work. For example, teachers rely on their own mathematical knowledge when making sense of a novel approach to a particular problem. Research at the elementary and secondary levels has shown that MKT is correlated with student achievement (Baumert et al., 2010; Hill, Rowan, & Ball, 2005). Work is on going at the elementary level to integrate thinking about high-leverage practices and MKT (Shaugnessy et al., 2014). However, research at the secondary level has yet to fully unpack the relationship between MKT and the high-leverage practices emphasized in *Principles to Actions*.

Another key factor in interpreting student work is the ability to notice key aspects of student mathematical thinking (e.g., Jacobs, Lamb, & Philipp, 2010; Sherin, Jacobs, & Philipp, 2011; van Es & Sherin, 2002). Jacobs and colleauges (2010) note that pre-service and novice elementary teachers struggle to identify and interpret students' mathematical strategies. These results motivate a consideration of what types of reasoning pre-service teachers do engage in when asked to analyze student work. Since expertise around noticing can be learned (Jacobs et al., 2010), it is important to tease out what tools pre-service teachers have to work with as they enter teacher preparation.

Drawing on Ball et al.'s (2008) conceptualization of MKT, I investigate the relationships between pre-service secondary teachers' interpretations of student work and MKT. Building on the literature around teacher noticing, I use in-depth interviews and a multi-case approach (Miles & Huberman, 1994) to address the following questions:

(1) What strategies do pre-service secondary teachers use to analyze written work?

(2) How are these strategies related to teachers' own solutions to mathematical tasks?

Methods

Data Collection

Data are drawn from a larger project investigating pre-service secondary teachers' development of MKT. This analysis considers participants at two preparation programs. Both are small, selective, one-year programs culminating in a secondary teaching credential and master's degree. East University[1] emphasizes strong mathematical preparation. West University has a strong focus on social justice and the integration of research into coursework. The preparation programs were purposively sampled to highlight the role of mathematics content courses in teacher preparation (East requires content courses for teachers; West does not). Within each program, four participants were purposively sampled with two main criteria: (1) participants had a range of mathematical knowledge for teaching; and (2) participants were matched across sites (to the extent possible) based on their mathematical knowledge for teaching and prior experiences in education (see Baldinger (2014) for more details on sampling logic). Though these eight participants are not a representative sample of all future secondary math teachers, their mathematical preparation is consistent with that of typical pre-service secondary math teachers (Graham, Li, & Buck, 2000). Table 1 shows participants' backgrounds.

Table 1: Participants' math and teaching backgrounds

School	Name	Math Background	Teaching Background
East	Daniel	Engineering major	Tutoring
East	Laura	Math major	Paraprofessional
East	Sam	Math major; associates degree in engineering	Substitute teacher
East	Tim	Math and physics major	A few teacher education courses
West	Dylan	Math and engineering major	Tutoring
West	Kendra	Math major	Summer small group teaching, After school teaching, Tutoring
West	Lisa	Math major	Tutoring
West	Nate	Engineering major	Peace Corps teaching (not math)

Participants completed in-depth, task based interviews during which they thought aloud while solving two high school math tasks (Ericsson & Simon, 1980; Ginsburg, 1981; Goldin, 1997). The problems were chosen to be accessible to participants with a range of mathematical proficiency, to be challenging, to provide multiple possible solution strategies, and to take

[1] All school and participant names are pseudonyms.

approximately 15 minutes. The problems were non-familiar; that is, even though they dealt with secondary level content, they were not problems participants were likely to have seen or completed prior to the interview. Participants were not told whether or not they had solved the problems correctly.

After solving, participants analyzed a sample of student written work on the same tasks. They were asked: "What does the student understand, and what doesn't the student understand?" Each student work sample was designed to reflect common student errors. Participants first completed an algebra task (see Figure 1), where the student work sample illustrates a student not attending to all of the conditions in the problem statement. Second, participants completed a geometry problem (see Figure 2), where the student work sample suggests a student reasoning based on what the diagram looks like, rather than attending to the relevant geometric properties of the shape. The interviews were audio and video recorded to capture participant writing and were transcribed for analysis.

Prove the following statement:

If the graphs of linear functions *f(x) = ax + b* and *g(x) = cx + d* intersect at a point P on the *x*-axis, the graph of their sum function *(f+g)(x)* must also go through P.

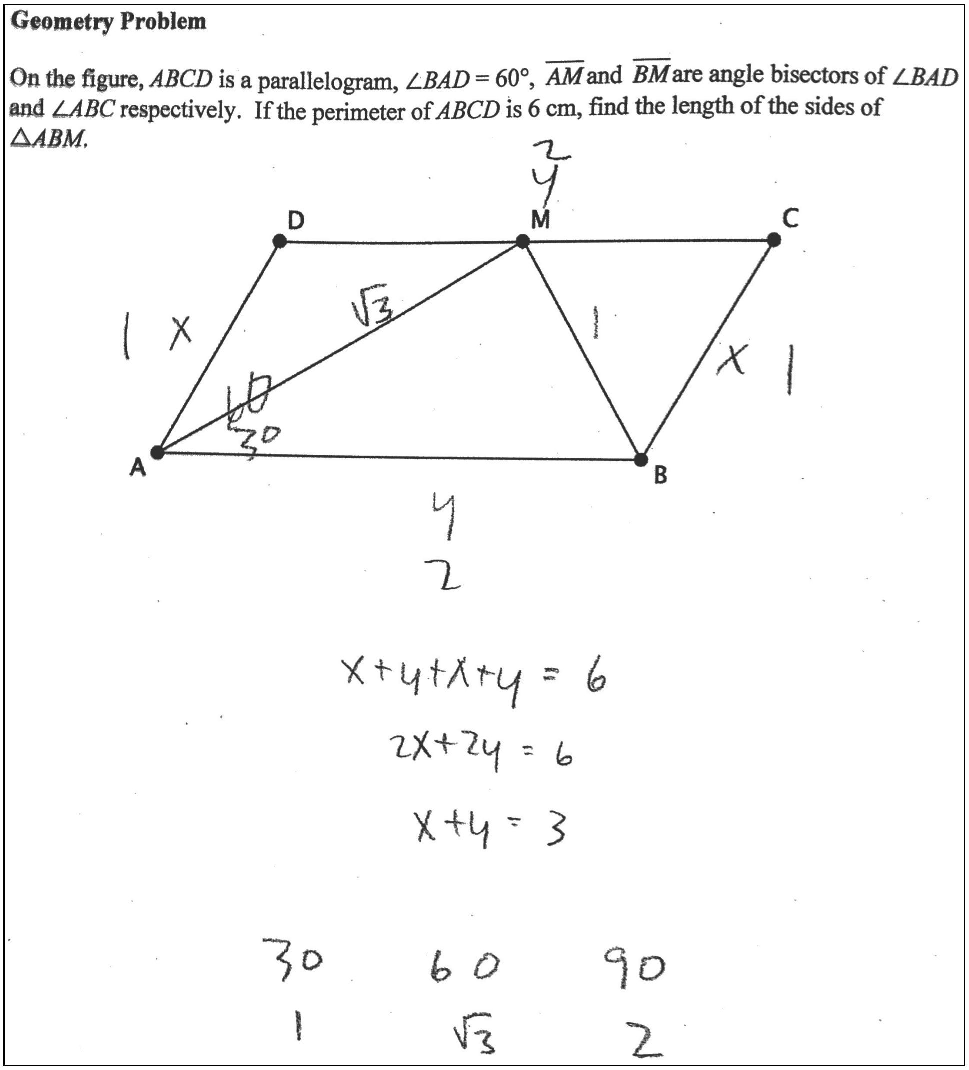
Counter example:

$y = x + 2$ $\quad$ $y = -x + 2$

$(0, 2)$

$y = 4$

Figure 1: Algebra problem and student work



Geometry Problem

On the figure, $ABCD$ is a parallelogram, $\angle BAD = 60°$, $\overline{AM}$ and $\overline{BM}$ are angle bisectors of $\angle BAD$ and $\angle ABC$ respectively. If the perimeter of $ABCD$ is 6 cm, find the length of the sides of $\triangle ABM$.

Figure 2: Geometry problem and student work

Data Analysis

To address the question of what strategies participants used, I divided transcripts of the student work analysis into assertions. Assertions occurred when the participant made a statement about what the student did or did not yet understand. I analyzed these assertions for two possible types of discrepancies (Sleep & Boerst, 2012): discrepancies between the assertion made and the evidence used to justify it, and discrepancies between the evidence used and the evidence actually available in the written work. Next I coded the type of reasoning used to make the assertion (see Table 2).

Table 2: Types of reasoning used in student work analysis

Type of Reasoning	Explanation
Mathematical	Mathematical critique of the student work
Comparison to participant's solution strategy	Participant compares the student work to the participant's own solution to the problem
Pedagogical	Participant draws on common student errors or teaching practices to support analysis

I developed cases for each task analyzed, noting the number of assertions made, types of discrepancies, and types of reasoning used. I looked across participants for patterns in the types of reasoning and discrepancies. Finally, I compared results across content areas.

Preliminary Findings

Strategies for Reasoning about Student Work

Mathematical analysis. Participants utilized several different approaches when analyzing student work. All participants engaged in at least some mathematical analysis; for one participant (Dylan) this was his only approach, for several others it was their primary approach. When utilizing this approach, the participants were engaged in the mathematical practice of critiquing the reasoning of others, as described in the Common Core standards (2010). For example, while analyzing the student work on the algebra task, Dylan reasoned,

> Well, they understand that these are two examples of linear equations, that have different slopes, and so they're going to intersect. And when they draw these arrows down and say the point (0, 2), they intersect at that point, and then $y = 4$. Okay, and then when you do the sum function, the x's cancel and you get 4. […] Where there is a little gap is they're not quite sure that the x-axis means that the y-coordinate has to be zero. And so because of that, this counterexample doesn't work.

This shows Dylan carefully examining each step that the student made and seeing what about it made sense, while at the same time acknowledging the student error.

In some instances, this type of mathematical analysis resulted in interpretations that were too broad given the available evidence. For example, in looking at the student work for the geometry problem, Tim reasoned,

> However, based on this math right here [*the set of equations written below the diagram*], because x plus y is 3, the way, the way that is, I see that the student is taking into account that this [*segment CB*] may be slightly less than uh no slightly more than half of this, or slightly less than half of this [*segment AB*]. So I feel like the generality is there [*indicates the set of equations*].

Tim is arguing that because the student work includes generalized equations that the student recognizes the possibility of non-integer lengths for segments *AB* and *BC*. However, the available evidence does not fully support this conclusion.

Participants may also have been impeded from utilizing mathematical reasoning to analyze student work when they were struggling with the mathematics themselves. The clearest example of this occurred in Nate's analysis of the algebra student work. As a result of his own mathematical error, Nate was conflicted about whether the sum of the equations $y = x + 2$ and $y = -x + 2$ should be $y = 4$ or $2y = 4$. After thinking through this issue, Nate concluded, "I actually can't say whether or not they understand the concept of adding two linear functions together." He recognized that a lack of mathematical understanding was getting in the way of his ability to make sense of student thinking.

Analysis by comparison. Six of the participants compared the student work and their own solutions. For example, Daniel twice made connections to his own work when analyzing the algebra student work. Early in his analysis, he made a supportable assertion that the student understands function notation. He said, "So this student has said that y equals x plus 2, so presumably they're so familiar with function notation that they've done, made the same interchange I did, that y is standing in for $f(x)$." Later, Daniel made an assertion about the student work that was too broad, saying, "Their wrong answer resulted from […] not having understood the picture […] I approached this as a picture myself first, so I'm leaning that way…" Daniel relied on his own approach to the problem, where he drew a picture to gain insight into the conditions of the problem, to explain why the student might have chosen the erroneous example equations. However, there is no evidence to indicate whether or not the student understood "the picture".

Just as mathematical analysis could be impeded by a lack of mathematical understanding, analysis by comparison could also be impeded when a participant had errors within their own solutions. For example, on the algebra task, Lisa made essentially the same error as the student work sample. She pointed out that in the student work sample,

> There's no proof. On my little paper over here, I did draw a proof that came to this, and I'm not sure I would expect that of a high school student to draw proof like that [*indicates her own algebraic work*]. But I did prove that it would equal $2y$ of whatever that P point was, or 2 of the y-value of that point P would be the y-value at $f(x) + g(x)$'s x_1 point, I guess. So there was no proof and no why it didn't work.

However, she also concluded that the student "did give me a counterexample that works, basically. Their counterexample works. It shows, you know, sum of two functions does not always go through P. That's basically what I meant by that. They recognized that that is not true." Comparison to her own work gave her the expectation of seeing a counterexample, but that the student did not write as much as she had, so the student's proof by counterexample was insufficient.

Laura's analysis by comparison was impeded in a different way. On the geometry task, Laura did not arrive at a solution, but did make two conjectures about some possible relationships in the geometric figure. She hypothesized that if M was the midpoint of segment CD, then angle AMB was 90 degrees. Then in her analysis of the student work, she asserted, "I don't know personally I feel like they shouldn't assume that this [*point M*] is the midpoint, angle bisector doesn't mean that it's the midpoint of the line. And so I think they're taking that for granted." However, there is not actually evidence in the student work related to angle bisectors and midpoints.

Pedagogical analysis. Only one participant engaged in pedagogical analysis of the student work. This entailed explaining or making sense of observations by connecting them to knowledge of common student errors, for example, or knowledge of common teaching strategies. In analyzing the algebra task student work, Kendra made three assertions that relied on pedagogical analysis. First, she asserted,

> So it looks like they understand [that] in some situations you set *x* equal to zero and you see what happens to a function, which is – most times that's presented in class for a procedure for finding the *y*-intercept to kind of start graphing. So they understand that that is a procedure sometimes used with linear functions.

For this assertion, Kendra drew on her knowledge of the school curriculum, and in particular how linear functions are commonly taught in school. Kendra also asserted that one possibility for explaining the student error would be that the student did not double-check the work. She reasoned that students commonly "[rush] through with the first answer they got without double checking it." In this case, Kendra was relying on knowledge of common student errors. Finally, in analyzing the student's proof, Kendra asserts,

> I think depending on what's been presented or what's the expectation of proofs, the student might not understand how proofs should be explained. Because for some classes, this would be valid, for others, this would just be kind of your scratch paper notes and this wouldn't be accepted as a formal proof of this statement. So I think it depends on what expectations have been set or what the student's mathematical background is.

This shows an analysis based on knowledge of typical teaching practices. Interestingly, Kendra did not engage in any pedagogical analysis on the geometry student work.

Overall, participants engaged primarily in mathematical analysis of student work, and many of the participants also utilized their own solutions as a tool for comparative analysis. Only one participant engaged in any pedagogical analysis, and she did not do so frequently.

Each type of analysis has the potential to be a productive way to interpret student work, but can also lead to potential discrepancies in analysis. In the next section, I explore the relationships between participants' own success solving the tasks and their ability to reason about the related student work.

Proficiency with Interpreting Student Work

Participants' abilities to make reasoned interpretations of the student work seemed to vary according to their own success with a task. Dylan, Daniel, and Tim all produced completely correct or nearly correct responses to both problems, and their analyses of the student work were similarly careful and detailed. They attended closely to the evidence and mostly (but not entirely) made supportable assertions about student understanding. Dylan engaged entirely in mathematical analysis, and made no references to his own work. Daniel and Tim primarily utilized mathematical analysis, but did draw some connections to their own solutions. Interestingly, in Daniel's case, the few instances where there were discrepancies between his assertions and the available evidence occurred when he was making comparisons to his own work (though he also was able to reason by comparison without discrepancies). Tim's work does not share this pattern.

In contrast, participants who struggled more with the mathematics exhibited some difficulties in interpreting the student work. Sam, for instance, was confident that his incorrect solution to the algebra task was correct, and so looked for evidence of his approach in the student work. Not seeing it, he asserted that the student showed little understanding. Additionally, Sam tried to make sense of the student work by analyzing the student's equations without accounting for the fact they were chosen as a counterexample. Laura and Lisa both made similar errors on the algebra task to the student work. Lisa was confident that her approach was correct, and so interpreted the student work accordingly, as shown above. Laura was much more uncertain about her approach was correct, and so felt unable to make many assertions about student understanding.

Laura and Kendra both spent very little time on their solutions for both tasks. For Laura, this was a result of mathematical uncertainty, which limited the number of assertions about student work she was willing to make. Her lack of comfort with the mathematics made it more difficult for her to engage in mathematical analysis. She also had a limited amount of work available for comparison. Kendra illustrates a different pattern. She attributed her lack of work on the mathematics tasks to feeling ill, and expressed more confidence in her ability to analyze the student work mathematically. Like Laura, she had a limited amount of work available for comparison. One possibility is that Kendra may have compensated for this by engaging in pedagogical reasoning on the algebra task.

Participant proficiency in making assertions about the student work samples also showed variation based on the nature of the assertions, particularly on the geometry task. When participants made low-inference assertions, such as that the student knows the definition or formula for perimeter, there were fewer discrepancies. Participants tended to have more discrepancies when making higher-inference or broader assertions. For example, Tim's high-inference assertion that the student admits the possibility of non-integer side lengths and maintains generality through the solution is too broad for the available evidence. Another potential pattern is that the participants tended to make more high-inference assertions when discussing what the student did not understand and they made more low-inference assertions when discussing what the student did understand.

Overall, participant success in engaging in a critique of student work seemed to be related to their success with engaging in the mathematics themselves. There is also a potential relationship between the types of analysis strategies utilized and participants' success with solving the problem or understanding the relevant mathematics. Finally, the quality of the assertions also seemed related to the level of inference required to make the assertions.

Implications

These findings suggest several implications for teacher preparation. First, mathematically reasoning about another's solution is part of the standards for mathematical practice described in the Common Core (2010), and requires sufficient knowledge of the relevant mathematics. Thus the use of mathematical reasoning in analyzing student work suggests that strong subject-matter knowledge and the ability to engage in mathematical practices are key tools in enacting this high-leverage practice. Further, misunderstandings of the relevant mathematics can lead to incorrect interpretations of student work. So supporting pre-service teachers to develop subject-matter knowledge and engage in mathematical practices not only helps their own knowledge development, it also supports them in more successfully engaging in this critical high-leverage practice of interpreting student work.

A second implication from this work is that it emphasizes the value and importance of teachers engaging in the same tasks they assign their students. Comparisons between student work and correct personal solutions supported valuable interpretations about student understanding. Since pre-service teachers frequently utilized analysis by comparison, doing the task provides a key resource in interpreting student thinking. Alternately, one could argue that because participants had been required to complete the task first, they had access to a type of analysis they otherwise might not have engaged in. Future work might consider whether or not participants would choose to solve a task in order to analyze a piece of student work. Participants engaged in reasoning by comparison whether or not they produced correct solutions themselves, which points to the importance of supporting novice teachers in finding the necessary resources to determine whether or not they have correctly solved a particular problem.

Finally, the fact that only one participant reasoned pedagogically suggests the possibility that pre-service teachers at the beginning of their preparation are potentially unfamiliar with common student approaches and errors. This echoes the findings from the teacher noticing literature (e.g., Jacobs et al., 2010). It suggests the value of supporting novices in learning explicitly about common errors and teaching techniques. Developing awareness of common student errors will likely support novices in engaging in pedagogical reasoning, and will give them an additional resource to draw on when interpreting student work. Having access to pedagogical analysis may also support pre-service teachers in determining an appropriate course of action for the student, whereas mathematical analysis and analysis by comparison do not lend themselves as easily to assisting with that choice.

The more resources teachers have to interpret student work, the better able they will be to enact the practices detailed in *Principles to Actions* (2014). Understanding what skills novice teachers bring into teacher preparation will help teacher educators strategically design learning opportunities to target key areas for growth. Future work will consider how teacher preparation programs can support novices in developing these skills.

References

Baldinger, E. (2014). *Studying abstract algebra to teach high school algebra: Investigating future teachers' development of mathematical knowledge for teaching* (Unpublished doctoral dissertation). Stanford University, Stanford, CA.

Ball, D. L., Thames, M. H., & Phelps, G. C. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 398–407.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., … Tsai, Y.-M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, *47*(1), 133–180. doi:10.3102/0002831209345157

Ericsson, K. A., & Simon, H. A. (1980). Verbal reports as data. *Psychological Review*, *87*(3).

Ginsburg, H. (1981). The clinical interview in psychology research on mathematical thinking: Aims, rationales, techniques. *For the Learning of Mathematics*, *1*(3), 4–11.

Goldin, G. A. (1997). Chapter 4: Observing mathematical problem solving through task-based interviews. *Journal for Research in Mathematics Education. Monograph.*, *9*(1997), 40–62.

Graham, K. J., Li, Y., & Buck, J. C. (2000). Characteristics of mathematics teacher preparation programs in the United States: An exploratory study. *Mathematics Educator*, *5*(1/2), 5–31.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*(2), 371–406. doi:10.3102/00028312042002371

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook* (2nd ed.). Thousand Oaks, CA: Sage Publications.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices, Council of Chief State School Officers.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.). (2011). *Mathematics teacher noticing: Seeing through teachers' eyes*. New York: Routledge.

Sleep, L., & Boerst, T. A. (2012). Preparing beginning teachers to elicit and interpret students' mathematical thinking. *Teaching and Teacher Education*, *28*(7), 1038–1048. doi:10.1016/j.tate.2012.04.005

Teaching Works. (2012). High-Leverage Practices. Retrieved from http://www.teachingworks.org/work-of-teaching/high-leverage-practices

Van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education*, *10*(4), 571–596.