The Impact of Multiple Representations First versus Algorithms First On Student Abilities

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Abstract

This study examined the impact of the ordering of teaching approaches on students' abilities to solve percent problems. Group 1 received multiple representations first followed by traditional algorithms. Group 2 received these in reverse order. Participants included 43 seventh grade students from an urban Midwestern US middle school. Results indicated ability gains in both groups however no significant differences. Comparisons of effect size however indicated larger growths in abilities for students in Group 1.

Introduction

Research suggests that the ability to solve complex problems and transfer skills to new situations is related to how well students' procedural and conceptual knowledge have developed (Hiebert & Carpenter, 1992). Procedural knowledge in mathematics refers to the ability to execute algorithms and encompasses knowledge of procedures, symbols, and domain conventions (Hiebert & Lefevre, 1986). Conceptual knowledge, on the other hand, is more networked, connected, and rich in relationships between the concepts of a domain. For mathematics and other domains both kinds of knowledge have been hypothesized to contribute to *procedural flexibility*, or the ability to solve a range of problems flexibly and efficiently (Blöte, van der Burg, & Klein, 2001; NRC, 2001; Star & Seifert, 2006). Howe (1999) proclaimed that there is no serious conflict between procedural knowledge and conceptual knowledge. In fact, many leaders in mathematics education today support the idea that students must have a balance of both conceptual understanding and procedural fluency in all areas of mathematics (Capraro & Joffrion, 2006; NCTM, 2000). In addition, the writers of Common Core State Standards for Mathematics (CCSSO & NGA, 2010) share the belief that both conceptual understanding and procedural fluency are essential for student mathematics learning, and Hiebert and Lefevre (1986) add that there are many benefits when conceptual and procedural knowledge are linked.

The teaching of many mathematical topics in the U.S. however, often relies heavily on algorithmic approaches which emphasize procedural skills (National Research Council [NRC], 2001; Lee, Brown, & Orrill, 2011; Ma, 1999, 2010; Van de Walle & Louvin, 2006). For example, instruction on rational numbers (e.g., fractions, decimals, and percents) and their manipulations is traditionally algorithmic, rule-based, and relies on sets of procedures aimed at making students quick and accurate when solving problems (National Research Council [NRC], 2001). Traditional algorithmic instruction, a form of direct instruction, often begins with teachers stating an algorithm (e.g., “to divide by a fraction, invert and multiply”), teacher-led demonstrations of how the algorithm works by presenting several examples, and then student practice, independently or in groups, on similar exercises. While algorithmic approaches have been found to be efficient methods for teaching students how to solve problems (Newton & Sands, 2012), major issues arise when, as a result of these approaches, students begin to view mathematics as sets of rules and give up their own mathematical sense-making while carrying out the steps of an algorithm (Fosnot & Dolk, 2002). The National Research Council (NRC) finds that the "rules for manipulating symbols are being memorized but students are not connecting those rules to their conceptual understanding nor are they reasoning about the rules" (National Research Council [NRC], 2001, p. 234).

In efforts to promote deeper understanding of mathematical topics various methods have been used and recommended. One successful teaching approach for helping students make better sense of mathematics and develop deeper conceptual understanding is the use of multiple representations (Fosnot & Dolk, 2002; Van den Heuvel-Panhuizen, 2003; Ng & Lee, 2009). Research suggests that engaging students in mathematics through multiple representations (MRs)—such as diagrams, graphical displays, and symbolic expressions— helps them better visualize, simplify, and make sense of abstract mathematical topics, and using representations flexibly is a key characteristic of skilled problem solvers (NRC, 2001; Lamon, 2001; NCTM, 2000; Dreyfus & Eisenberg, 1996). Representations refer not only to the product, or student created model, but to the process and act of capturing mathematical concepts or relationships using student created models (NCTM, 2000). Van den Heuvel-Panhuizen (2003) proclaims, “it is not the models in themselves that make the growth in mathematical understanding possible, but the students’ modeling activities” (p. 29). Therefore, when using MRs, it is important for teachers to place emphasis beyond the model and more on students’ focus on sense making while using the models, their justification, and the use of multiple methods to find solutions. Lamon (2001) found that by using different representations of rational numbers students gained a deeper understanding of them and were better able to transfer their knowledge from one model to another. Furthermore, students have been shown to gain confidence in their abilities by exploring problems through multiple approaches and justifying their choices (Newton & Sands, 2012). Hence, representations are recommended as an essential component of mathematical activities and means for capturing mathematical concepts (e.g., Goldin & Shteingold, 2001).

While traditional algorithmic instruction and multiple representation instruction are both useful for helping students achieve a balance of conceptual understanding and procedural fluency, questions still remain as to how these approaches should be integrated to best meet the learning needs of students. For example, should teachers begin with teaching approaches that help students develop conceptual understanding first (e.g. using multiple representations), or teaching approaches that help students develop procedural fluency first (e.g. using traditional algorithms). The research questions that this study investigate are:

1). Does the order of teaching approaches (MR first versus TA first) impact students’ abilities to solve math problems that involve fractions, percents, and decimals?

2). Does the order of teaching approaches (MR first versus TA first) impact students' confidence to solve math problems that involve fractions, percents, and decimals?

3). Does the teaching approach (MR versus TA) impact students' on task behaviors?

Theoretical Perspectives

For many decades, researchers have attempted to examine how conceptual and procedural knowledge influence and impact each other (Byrnes & Wasik, 1991; Canobi, Reeve, & Pattison, 1998; Dixon & Moore, 1996; Gelman & Gallistel, 1978; Gelman & Meck, 1983; Greeno, Riley, & Gelman, 1984; Hiebert, 1986; Resnick & Ford, 1981; Rittle-Johnson, Siegler, & Alibali, 2001; Sophian, 1997). From this body of research, various theoretical perspectives have emerged (see Rittle-Johnson, Siegler, & Alibali, 2001; Schneider & Stern, 2010). *Concepts-first* theory conjecture that students initially gain conceptual knowledge and then derive and develop procedural knowledge from it through repeated practice with solving problems (Gelman & Williams, 1998; Halford, 1993). Empirical evidence supporting the concepts first perspective has been found for the teaching of various mathematics concepts including simple arithmetic and proportional reasoning (Byrnes, 1992; Cowan & Renton, 1996; Dixon & Moore, 1996; Hiebert & Wearne, 1996; Siegler & Crowley, 1994; Wynn, 1992; see Rittle-Johnson, Siegler, & Alibali, 2001). *Procedures-first* theory, on the other hand, state the opposite and suggest that students first learn procedures and from practice with those procedures, gradually develop conceptual knowledge (Karmiloff-Smith, 1992; Siegler & Stern, 1998). Similarly, empirical evidence have also been found in support of the procedures first approach for teaching various mathematical concepts such as counting and fraction multiplication (Briars & Siegler, 1984; Byrnes & Wasik, 1991; Frye, Braisby, Love, Maroudas, & Nicholls, 1989; Fuson, 1988; Hiebert & Wearne, 1996).

While these two perspectives continue to be debated, the importance of these perspectives are their implications for how teaching approaches should be sequenced to best meet the needs of students. In support of the concepts first theory, the researchers of this study hypothesize larger gains in student learning outcomes for those presented with MR approaches before TA approaches.

Purpose of the Study

The purpose of this study was to examine and compare the impact of the order of two teaching approaches (e.g. multiple representation (MR) instruction and traditional algorithmic (TA)) on students' abilities to solve math problems that involve fractions, decimals, and percents, and confidence in their abilities to solve these types of problems. Additionally, this study sought to examine whether the teaching approaches (e.g. MR versus TA) impacted on task behaviors while learning.

Method

To be successful in algebra, students should be fluent with rational numbers, their operations, and the ability to convert between equivalent forms (i.e. decimals, fractions, and percents) (Bottoms, 2003; Stacey & MacGregor, 1997). Due to the importance of these mathematical skills, the population of interest for this study was middle school aged pre-algebra students and the mathematical topic of interest was percentage problems that involved decimals and fractions.

Sample

The participants for this study included forty-three 7th graders enrolled in two pre-algebra sections in an urban middle school in Midwestern U.S. Over half were males (56%), and all students were 12-13 years old. Students came from very diverse ethnic backgrounds including 19% African American, 9% were Asian American, 5% were Hispanic, 37% were White, 26% were Mixed Race, or 5% indicated Other.

Content

The content for this study included a 7^{th} grade pre-algebra unit on percentage problems involving fractions, and decimals. Problem types included: 1) finding the unknown part of a number represented by the percent of a whole number, 2) finding the unknown whole number when the percent and the part were known, and 3) finding the unknown percent when the part and the whole numbers were known and 4) combinations of the three types. These problems included both de-contextual and contextual based problems. Two example problems are provided below:

- De-contextual: 24 is 60% of what number?
- Contextual: Three candidates participated in a school election. Bianca received ¼ of the votes, Chelsea received 0.30 of the votes, and Francisco received the rest of the votes. What percent of the votes did Francisco receive?

Teaching Approaches

For research purposes, two teaching approaches also guided the design of instructional activities within modules. These two approaches are explained below.

Traditional Algorithmic (TA) Approach. The TA teaching approach relied heavily on the use of common textbook algorithms and mnemonic devices for solving percent problems (Bennett et al., 2007). For example, students were given the mnemonic device, “is over of;

percent over 100" and instruction/practice with setting up proportions for the various problem types (see Table 1).

Table 1: Proportion Algorithm Set-up by the Major Problem Type

Problem Type	Example Problem	Proportion Algorithm Set-Up
Finding the percent of a number	15% of 240 = n	$\frac{15}{100} = \frac{n}{240}$
Finding the percent one number is of another	p% of 240 = 18	$\frac{p}{100} = \frac{18}{240}$
Finding a number when the percent is known	15% of n = 18	$\frac{15}{100} = \frac{18}{n}$

During the TA approach students were provided with direct instruction on how to solve problems using algorithms. The steps usually followed during the algorithmic approach were: 1) identify the problem type, 2) set up a proportion to represent the problem, 3) solve the problem, and 4) check and verify solution. During TA lessons, students were provided with guided, independent, and group practice on how to solve these problems step-by-step. In the TA approach, emphasis was placed on using the proportion set-up algorithm (see Equation 1). Using the mnemonic device, students set up proportions for problems where *is*, *of*, and *percent* were cue words to guide them in their placement of the known and unknown values.

$$\frac{is}{of} = \frac{\%}{100} \qquad \text{(Equation 1)}$$

An example problem that students could solve with this algorithm is, "What is 20% of 4000?" Students would use "what is" to place an unknown variable (x) in the first numerator. They would use "of" to place the 4000 in the denominator. Finally, they would place 20 (%) over 100 (see Equation 2). To solve the unknown value in the proportion, students used the already familiar “cross multiply and divide” algorithm from a previous unit. Basically, students would multiply 20 times 4000 and 100 times x. The result would be the equation *100x=8000*. Students would then divide 8000 by 100 to find their solution.

$$\frac{x}{4000} = \frac{20}{100} \qquad \text{(Equation 2)}$$

Multiple Representations (MR) Approach. The MR approach included opportunities for exploring multiple representations and multiple solution methods, and mathematical communication. In the MR approach students learned and were expected to solve percent problems by using various representations and models, explaining their methods in writing, discussing strategies with their peers in groups, and rationalizing methods verbally with their teacher and peers. Examples of representations and models used by students included chunking, number lines, double number lines, percent bars, ratio tables, and writing equations (See Figure 1). While students were introduced to these MR approaches, students were encouraged to explore, create their own models, and given choice with which representation they could use to solve problems. A major aspect of the MR approach was that students justified the representation they used. An additional aspect of the MR approach was that students were encouraged to estimate and self-evaluate whether their answers made mathematical sense. For example, one MR method, “chunking”, was used to calculate or estimate percentages by using the benchmarks, 10% and 1%. To calculate 32% of 200, students could add three times 10% of 200 (or 20) plus two times 1% of 200 (or 2) to get the correct answer 64.

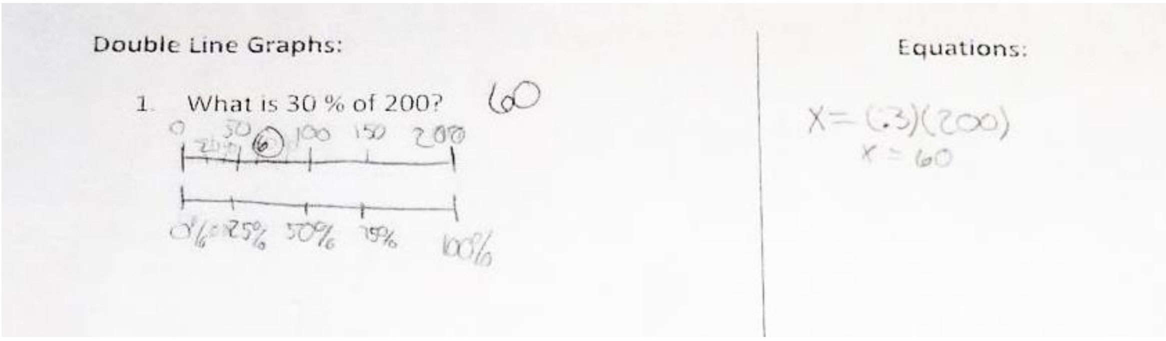

Figure 1: Example of using MR approach methods for solving percent problems

Design

This study used a quasi-experimental design using two intact groups, two pre-algebra classes which students had already been assigned to based on their individual scheduling needs. Students from both groups received each teaching approach (i.e. MR and TA approach) however they were offered in different order dependent on the treatment group that students were placed into.

Treatment Groups

Group 1, the MR first group (N=22), experienced MR lessons first in Module 1 and then TA lessons in Module 2. Group 2, the TA first group (N=21), received TA lessons in Module 1 and then MR lessons in Module 2. The counterbalanced design was used to assess whether the order of teaching approach impacted student learning outcomes but also to ensure that both groups had equal opportunities to experience both teaching approaches.

Instruments & Measures

Data were obtained through performance tests, scales, and observations during the spring 2013 semester. These instruments are described below:

Pre/Post Knowledge Test: In order to measure students' abilities to solve problems involving fractions, decimals, and percentages, show their work, and explain their thinking process locally developed parallel assessments for pre and post knowledge assessments were used. Each test consisted of 10 open response items. These included 2 de-contextual based and 8 contextual-based problems. To differentiate student abilities to solve and show their mathematical processes and sense-making for solving, a rubric was used to score students' open-ended responses on each item. With this rubric emphasis was placed on the approaches that students took to solve the problem however, the approach that students took was not prescribed (i.e. students could use TA approaches or MR approaches). Responses were scored based on the appropriateness of students' approaches and the correctness of solutions obtained by those approaches. For incorrect or no approach shown students received 0 or 1 points (0 for incorrect responses, 1 for correct responses) and 2 or 3 points for correct approaches (2 for incorrect responses, 3 for correct responses). The maximum score on each of pre and post-knowledge tests was 30 points. The open response format of the items and the associated rubric allowed researchers to gather deeper information on students' reasoning and sense-making in contrast to commonly used dichotomously scored multiple choice items which often conceal those details. Pre and post knowledge assessments were administered before and after the treatments. The purpose of these instruments was to examine the impact of treatment conditions on students' abilities to solve and to investigate whether treatment conditions impacted students differently.

Confidence Scale: This measure consisted of a 5-point likert scale item (1=No confidence – 5-Very high confidence) and asked students "How much confidence do you have with solving mathematics problems that involve fractions, decimals, and percents?". This scale was administered before and after the treatments. The purpose of this scale was to assess whether the treatment conditions had an impact on students' confidence in their abilities to solve.

On Task Behaviors Observation Form: An adapted version of the *Basic 5 Observation Form* (Sprick, Knight, Reinke, & McKale, 2006) was used as a measure of students' on task behaviors. On task behaviors, related to the task at hand during the observation, and these included students writing or taking notes, tracking the teacher with their eyes, talking with their partners about relevant topics, asking questions, drawing MR models, and/or following directions. Examples of observable off-task behaviors included: off-topic conversations, (audible) up and out of seat or tipping over in chair while laughing, not tracking the teacher or staring into space, and/or not following directions. During the observation, for five minutes, the trained observer focused on different students every 5 seconds. If the student were on-task the observer would tally a "+" on the form. Alternatively, a "-" was tallied if an off-task behavior was observed. A total of sixty tallies were made during each observation session. The percentage of on task behaviors was calculated by dividing the total number of on-task tallies by the set total number of tallies (60) and multiplying by 100. For each module, groups 1 and 2 received a total percentage of on task behaviors. This form was used to measure and assess students' on-task behaviors within their treatment groups while learning with each distinctive teaching approach (e.g. MR or TA).

Teacher Reflection Notes: Each day after class, the teacher wrote down her reflections on how the lesson went and students reactions to the lesson. To organize these notes, the teacher used a reflection notes template that consisted of a blank table with three columns: module day, group 1 notes, and group 2 notes; and rows for each module day. The purpose of the reflection notes was to gather additional data to support and further explain students' on task behaviors within their treatment groups while learning with each distinctive teaching approach (e.g. MR or TA).

Procedures

For both groups, instruction over percents, fractions, and decimals was broken into two modules. Before Module 1, students completed a Pre-Confidence Scale and a Pre-Knowledge Test. During Module 1 students experienced MR or TA lessons depending on the group that they were in. On day three of Module 1, an outside observer (trained instructional coach from the school district) completed the On Task Behaviors Observation Form for each group. On day 7, students began Module 2. During Module 2, groups were presented with the other teaching approach (i.e. Group 1 experienced TA lessons and Group 2 experienced MR lessons). On day 9, a trained observer recompleted the On Task Behaviors Observation Form for each group. Following the completion of Module 2, students fill out a Post-Knowledge Test and Post-Confidence Scale on day 11. For each module day students were in class for about 50 minutes.

Data Analysis

The scores from all instruments were entered and analyzed using SPSS v21. Inferential statistics were then used to examine the impact of the teaching approach order on student learning outcomes (i.e. abilities to solve and confidence in abilities), and the impact of teaching approach on student engagement. Data analyses included ANCOVAs, Chi-square tests, and qualitative data analyses. These are described in Table 2 below.

Table 2: Summary of research questions, variables, and data analyses

Research Questions/Variables	Instruments	Data Analysis
Does the order of teaching approaches (MR first versus TA first) impact student learning outcomes (SLO1, SLO2) when working with math problems that involve fractions, percents, and decimals?		
SLO1). Abilities to solve	Pre/Post Knowledge Tests	ANCOVA with pre-knowledge as a covariate
SLO2). Confidence in abilities to solve	Pre/Post Confidence Scales	ANCOVA with pre-confidence as a covariate
Does the teaching approach (MR versus TA) impact students' on task behaviors (SLO3)?:		
SLO3). On Task Behaviors while learning	On Task Behaviors Observation Form Teacher Reflection Notes	McNemar Chi square Qualitative data analysis

Limitations

Methodological limitations for this study include the small sample size and the short duration of treatment conditions. Consequently, while results may provide valuable insights, they are suggestive and may not generalize to all middle school student populations. Specifically, the use of a sample of convenience may limit the study to middle school pre-algebra students. Further because the duration of treatment groups was short (i.e. eleven days), students may not have been exposed to the treatments long enough for them to have an impact on their student learning outcomes. Nevertheless, numerous results were found that provide opportunities for insight and application.

Results

The goal of this study was to investigate and compare the impact of the order of two teaching approaches (e.g. multiple representation (MR) instruction and traditional algorithmic (TA) instruction) on students' abilities to solve math problems that involve fractions, decimals, and percents (SLO1), and their confidence in their abilities to solve these types of problems (SLO2). In addition, this study sought to examine whether the teaching approaches (e.g. MR versus TA) impacted on task behaviors while learning (SLO3). The results of these investigations are provided below.

Order of Teaching Approach: Impact on *Students' Abilities to Solve (SLO1)*

Using pre-knowledge score as a covariate, an ANCOVA was used with group (e.g. MR first vs TA first) as a between-subjects factor and ability to solve as the dependent measure. Although Group 1 (MR first) had a slightly higher post-knowledge tests (M=23.36, SD=6.63) than Group 2 (TA first) on post-tests (M=22.33, SD=5.79), no significant differences on ability to solve were found between the two groups (F(1,40)=1.01, p=0.32). The covariate, pre-knowledge, was significant (p=0.03) indicating that treatment groups differed in prior knowledge (see Table 3).

Table 3: ANCOVA Results for Teaching Approach Order on Ability to Solve

Source	*SS*	*df*	*MS*	*F*	*p*
Pre-knowledge	188.96	1	100.18	5.38	0.03*
Group	35.32	1	108.24	1.01	0.32
Error	1404.80	40	1.32		
Total	1605.16	42			

*Significant covariate p<0.05

Order of Teaching Approach: Impact on *Students' Confidence in Their Abilities to Solve (SLO2)*

A second ANCOVA was used with group as a between-subjects factor, confidence to solve as the dependent measure, and pre-confidence score as a covariate. Although Group 2 (TA first) had slightly higher post-confidence scores (*M=4.30, SD=0.57*) than Group 1 (MR first) (*M=3.91, SD=0.971*), no significant differences on confidence to solve were found between the two groups (F(1,39)=.80, p=0.38). The covariate, pre-confidence, was significant (p=0.003) indicating that treatment groups differed in prior knowledge (see Table 4).

Table 4: ANCOVA Results for Confidence to Solve

Source	*SS*	*df*	*MS*	*F*	*p*
pre-confidence	5.32	1	5.319	10.02	0.003*
Group	0.42	1	0.423	0.797	0.38
Error	20.70	39	0.531		
Total	732.00	42			

*Significant covariate p<0.05.

As a follow up investigation, two separate paired sample t-tests were used to examine the difference between pre and post confidence scores. Both paired sample t-tests were statistically significant for both groups indicating that Group 1 (PreC-M=3.50, PreC-SD=0.74; PostC-M=3.91, PostC-SD=0.97), *t(21)=-2.25, p=0.04;* and Group 2 (PreC-M=3.90, PreC-SD=0.85; PostC-M=4.30, PostC-SD=0.57), *t(19)=-2.18, p=0.04*, showed increases in their confidence to solve. Effect sizes, calculated by using the within-subject calculator for means and standard deviations method presented by Morris and DeShon (2002), indicated moderate effect sizes for both Group 1 (*d*= 0.494) and Group 2 (*d*= 0.509). Findings suggest similar positive increases in students' confidence in their abilities to solve among both groups.

Teaching Approach: Impact on *Students' On Task Behaviors (SLO3)*

In order to explore whether there was a relationship between students' engagement and each distinctive teaching approach used (i.e. MR approach versus TA approach), a McNemar Chi square test was performed. Results indicated a significant relationship was found indicating that students exhibited higher levels of engagement when TA approaches were used (Group 1: 78%; Group 2: 80%) compared to when MR approaches were used (Group 1: 55%; Group 2: 57%) ($\chi^2(1, N = 240) = 11.358, p =.001$). Additionally, teacher written reflections were examined to further explore student behaviors during the implementation. Similarly, reflection notes suggested that in students exhibited a higher percentage of on task behaviors when TA approaches were used. Student's perceptions towards TA instruction were also more favorable compared to MR approaches. For example, students in Group 1 struggled with the MR methods in Module 1, but when they were introduced to algorithms in Module 2, they were more on task and engaged. Teacher reflection notes also suggested that the teacher perceived that "students felt like the algorithms helped them fit together the pieces from what they had learned or struggled with during modeling [MR approach]." Further, the teacher indicated that she also experienced difficulty with motivating Group 2 students [TA first] to engage in the modeling [MR] activities having already received instruction on algorithms. The teacher states that students in Group 2 (TA first) "already knew a method for solving the problem, and during Module 2 they asked whether they could use proportion algorithm instead." This preference may be due to the novelty of the MR approach as the teacher noted that students "complained about the newness of the modeling approaches [MR]" and vocally expressed that they "experienced difficulty with trying something new."

Discussions

The goal of this study was to test the hypotheses that students taught with the multiple representation (MR) approach first before traditional algorithmic (TA) approaches would improve their learning, confidence to solve, and that when students were taught with MR approaches they would be more engaged in their learning. To test these hypotheses, we used two treatment groups: Group 1 received MR approaches first and Group 2 received TA approaches first. The primary goal of this study was not to test which teaching approach is better but rather to investigate how teachers can best organize these two approaches to better meet the learning needs of students and get the most effective student learning outcomes.

No significant differences were found between the two treatment groups (MR first versus TA first) in terms of abilities to solve on post-tests however, results from follow up paired sample t-tests suggested a larger gain in abilities to solve for students introduced to MR approaches before TA approaches. In this study, students in Group 1 (MR first) began with lower abilities to solve (e.g. pre-knowledge) than Group 2 and yielded higher abilities to solve on post-tests when compared to Group 2 (TA first). This finding suggests that students may improve their learning if they are first introduced to MR approaches that were designed to emphasize mathematical sense making and justification. After being presented with various MR representations students may then be more ready to supplement those initial problem solving skills with more efficient algorithms (Van de Walle & Lovin, 2006; Donovan & Bransford, 2005). The sequence of MR first is also aligned with a number of empirical studies that support

Bruner's theory (Bruner, 1966) which recommends that instruction be sequenced from grounded representations to more abstract ones with numbers and symbols (Bransford, Brown, & Cocking, 1999; Goldstone & Son, 2005; Koedinger & Anderson, 1998; Moreno & Mayer, 1999; Nathan, Kintsch, & Young, 1992; Nathan & Koedinger, 2000). While the results of this study suggest support for the MR first approach, it is important to highlight that much empirical support has also been found for the procedures-first approach (Briars & Siegler, 1984; Byrnes & Wasik, 1991; Frye, Braisby, Love, Maroudas, & Nicholls, 1989; Fuson, 1988; Hiebert & Wearne, 1996). Future research therefore should continue to further investigate how both TA and MR teaching approaches can be integrated to best support student learning outcomes and with other mathematical content.

Results further suggest that, in terms of teaching, using MR approaches before TA approaches may also be advantageous for teachers. In this study, the teacher noted that although most students struggled and were challenged by MR approaches, it was especially difficult to engage TA first students who had already been presented with traditional algorithmic approaches on how to solve given problems. Furthermore, the teacher noted that TA first students expressed that they did not find much value for exploring alternative methods for solving problems having already been introduced to efficient algorithms that could be used. This also caused the teacher a few challenges. Although the two approaches were presented in different order, observed students' on task behaviors were also similar for MR first students. During Module 2 where MR first students were presented with TA approaches, students were observed to be more on task. Teacher reflections also noted that students even expressed preference for step-by-step algorithmic approaches. While there may be many reasons for these results, this may be attributed to the ease of using algorithms. Furthermore, students' preferences may have been due to students' familiarity with this type of teaching approach which is often emphasized in U.S. mathematics classrooms and textbooks (National Research Council [NRC], 2001; Lee, Brown, & Orrill, 2011; Ma, 1999, 2010; Van de Walle & Louvin, 2006). Further, a major assumption of using MR approaches is that they will lead to increased student on task behaviors because students are required to make sense of their various representations and models, explain their methods in writing, discuss strategies with their peers in groups, and rationalize their methods verbally with their teacher and peers. Unfortunately, as this study suggests, this may not always be the case. Future research should continue to investigate students' on task behaviors but also students' preferences for various problem solving strategies and tools. Furthermore, it is important to highlight the importance of teacher knowledge and preferences as well. Previous research has found that students often use the same tools and models that their teachers use (Cai, 2004; Cai & Lester, 2005). If teachers stress TA approaches over MR approaches this may explain students' preferences and comfort with TA approaches. This also leads to implications of how teachers are prepared. Unfortunately, if teachers have been taught prepared and trained solely with TA approaches (Wu, 2001), they often also teach mathematics the way they were taught mathematics. As students find MR approaches challenging, research has also found that teachers may also lack the deep mathematical understanding and proficiency to use MR approaches (Ma,1999). Future research should continue to investigate, not only how students can be supported but also how their teachers can be better prepared to support to help their students. More research is needed on how to provide teachers with guidelines on how to scaffold student learning with MR approaches.

In terms of confidence to solve, Newton and Sands (2012) found that by engaging in exploring alternate strategies and justifying their choices students gained in their confidence. Results of this study found that students' confidence was positively impacted by both teaching approach combinations (i.e. MR first and TA first) and both impacted students similarly. Future studies should however continue to investigate whether students' confidence increased by learning multiple approaches and further investigate whether student engagement has an impact on confidence and student abilities to solve. Furthermore, future studies could include student interviews that could provide an additional source of data to capture student confidence.

Conclusions

Many leaders in mathematics education today support the idea that students must have a balance of both conceptual understanding and procedural fluency in all areas of mathematics (Capraro & Joffrion, 2006; Fennell et al., 2007; NCTM, 2000, 2006). While conceptual knowledge should not be elevated above procedural knowledge (Howe, 1999), teaching approaches that help students with conceptual understanding are critical especially with newer mathematical standards that require as much attention to be given towards conceptual understanding as to procedural fluency (NRC, 2001). With these new standards teachers and students may be encouraged to partake in teaching approaches that they may not be familiar with, such as the use of multiple representations and mathematical models. With that being the case, as was evident in this study, more research is needed on how to support students with these new approaches and how to organize their instruction in light of the traditional instructional practices that students have been accustomed to.

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