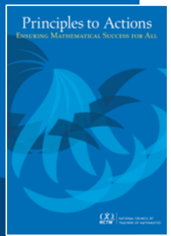

Strategies and Tasks to Build Procedural Fluency from Conceptual Understanding

Please take a handout.

Sit near someone for partner discussion

Diane J. Briars
President
National Council of Teachers of Mathematics
dbriars@nctm.org

2015 NCTM Nashville Regional Conference
November 19, 2015

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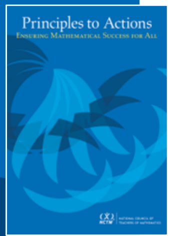

FYI

Electronic copies of slides are available
by request
dbriars@nctm.org

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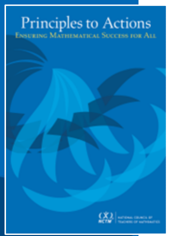
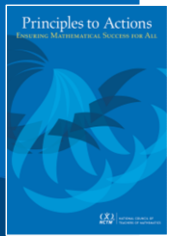


Key Questions

- What is procedural fluency?
- What instructional practices promote students' development of procedural fluency?
- How can I implement these practices in my classroom?



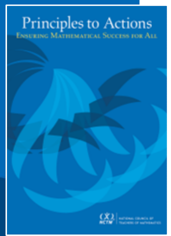
Discuss with a Shoulder Partner

What does it mean to be fluent with computational procedures?

What is Procedural Fluency?

Which students demonstrate procedural fluency? Evidence for your answer?

- Alan
- Ana
- Marissa



Alan

1000 - 98

100 - 18

https://mathreasoninginventory.com/Home/VideoLibrary

Source: The Marilyn Burns Math Reasoning Inventory

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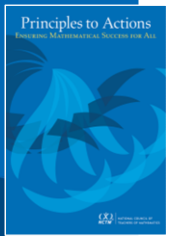

Ana

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99 + 17

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Which students demonstrate procedural fluency? Evidence for your answer?

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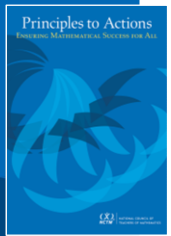

Procedural Fluency

- **Efficiency**—can carry out easily, keep track of subproblems, and make use of intermediate results to solve the problem.
- **Accuracy**—reliably produces the correct answer.
- **Flexibility**—knows more than one approach, chooses appropriate strategy, and can use one method to solve and another method to double-check.

Marissa

295 students, 25 on each bus

Source: The Marilyn Burns Math Reasoning Inventory

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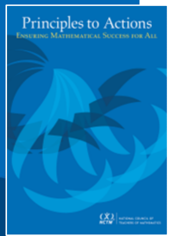

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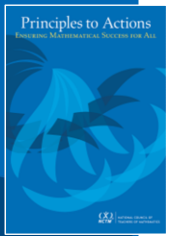
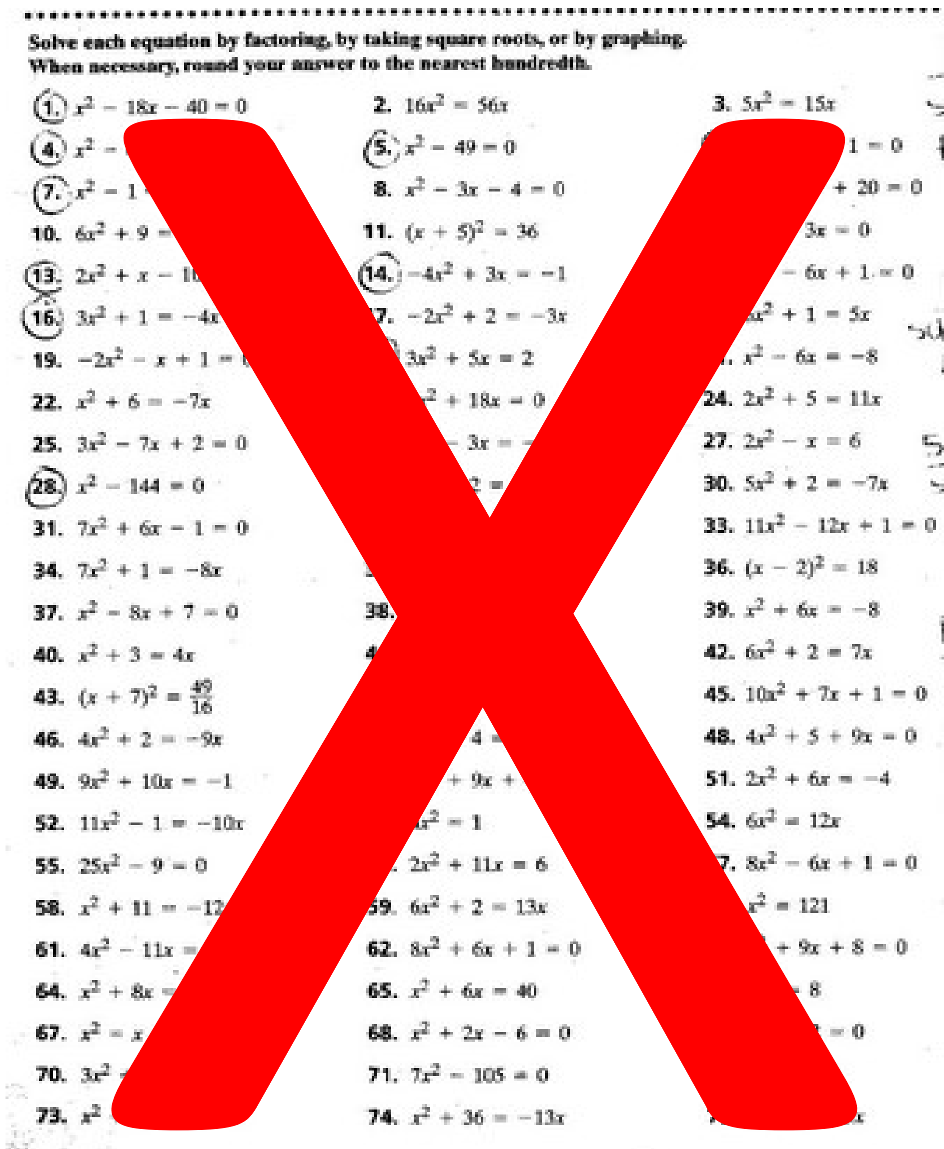
- **Appropriately**—knows when to apply a particular procedure.



How Can We Develop Students' Proficiency?



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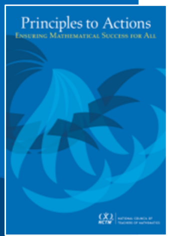
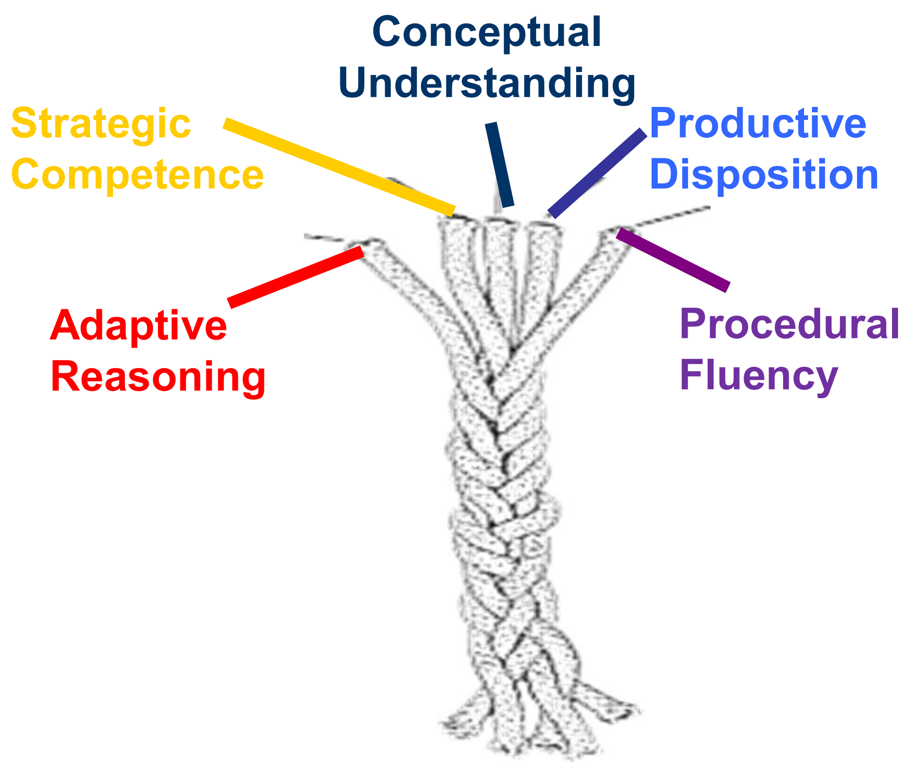
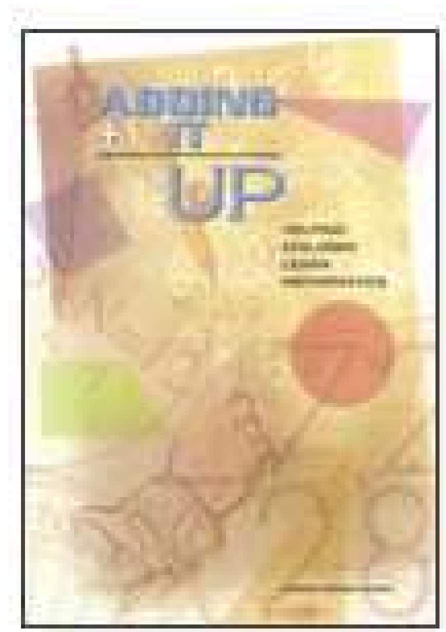


Strands of Mathematical Proficiency

- ***Conceptual Understanding*** – comprehension of mathematical concepts, operations, and relations
- ***Procedural Fluency*** – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- ***Strategic Competence*** – ability to formulate, represent, and solve mathematical problems
- ***Adaptive Reasoning*** – capacity for logical thought, reflection, explanation, and justification
- ***Productive Disposition*** – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

Strands of Mathematical Proficiency





NRC (2001). *Adding It Up*. Washington, D.C.: National Academies Press.

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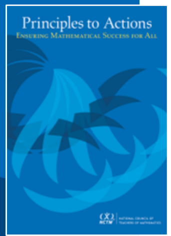

What Research Tells Us

- When procedures are connected with the underlying concepts, students have better retention of the procedures and are more able to apply them in new situations
- Informal methods → general methods → formal algorithms is more effective than rote instruction.
- Engaging students in solving challenging problems is essential to build conceptual understanding.

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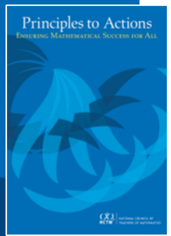

Principles to Actions: Ensuring Mathematics Success for All

Guiding Principles for School Mathematics

1. **Teaching and Learning**
2. **Access and Equity**
3. **Curriculum**
4. **Tools and Technology**
5. **Assessment**
6. **Professionalism**

➔ Essential Elements of Effective Math Programs

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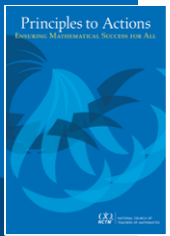
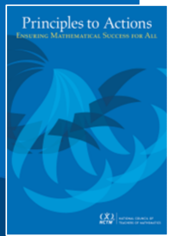


Teaching and Learning Principle

An excellent mathematics program requires effective teaching that engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically.



Effective
Mathematics Teaching Practices

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8. **Elicit and use evidence** of student thinking.

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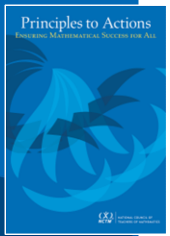

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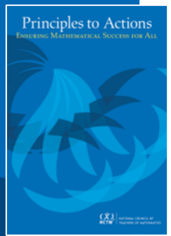
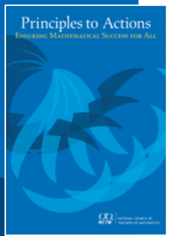


Build Procedural Fluency from Conceptual Understanding

Procedural Fluency should:

- Build on a foundation of conceptual understanding;
- Result in generalized methods for solving problems; and
- Enable students to flexibly choose among methods to solve contextual and mathematical problems.



"Fluency builds from initial exploration and discussion of number concepts to using informal reasoning strategies based on meanings and properties of the operations to the eventual use of general methods as tools in solving problems."

Principles to Actions (NCTM, 2014, p. 42)

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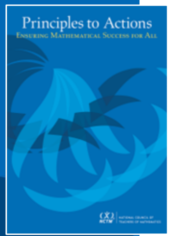
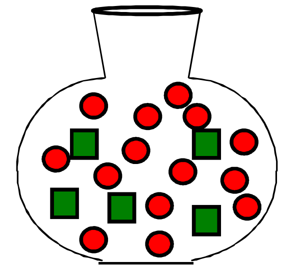


Candy Jar Problem



A candy jar contains 5 Jolly Ranchers (squares) and 13 Jawbreakers (circles). Suppose you had a new candy jar with the same ratio of Jolly Ranchers to Jawbreakers, but it contained 100 Jolly Ranchers. How many Jawbreakers would you have? Explain how you know.

- Please work this problem as if you were a seventh grader.
- When done, share your work with a neighbor.
- What mathematical content does this task support?

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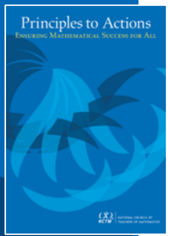
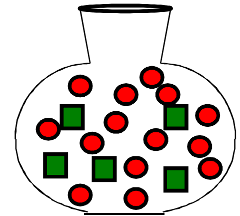


Candy Jar Problem



Group 1 (incorrect, additive)

100 JRs is 95 more than the 5 we started with. So we will need 95 more JBs than the 13 we started with.

5 JRs + 95 JRs = 100 JRs
13 JBs + 95 JBs = 108 JBs

Groups 3 and 5 (scale factor)

You had to multiply the five JRs by 20 to get 100, so you'd also have to multiply the 13 JBs by 20 to get 260.

(× 20)
5 JRs → 100 JRs
13 JBs → 260 JBs
(× 20)

Groups 4 and 7 (scaling up)

JR	JB	JR	JB
5	13	55	143
10	26	60	156
15	39	65	169
20	52	70	182
25	65	75	195
30	78	80	208
35	91	85	221
40	104	90	234
45	117	95	247
50	130	100	260

Group 2 (unit rate)

Since the ratio is 5 JRs for 13 JBs, we divided 13 by 5 and got 2.6. So that would mean that for every 1 JR there are 2.6 JBs. So then you just multiply 2.6 by 100.

(× 100)
1 JR → 100 JRs
2.6 JBs → 260 JBs
(× 100)

Group 6 (scaling up)

JRs	5	10	20	40	80	100
JBs	13	26	52	104	208	260

We started by doubling both the number of JRs and JBs. But then when we got to 80 JRs, we didn't want to double it anymore because we wanted to end up at 100 JRs, and doubling 80 would give us too many. So we noticed that if we added 20 JRs:52 JBs and 80 JRs:208 JBs, we would get 100 JRs:260 JBs.

Group 8 (scaling up)

We drew 100 JRs in groups of 5. Then we put 13 JBs with each group of 5 JRs. We then counted the number of JBs and found we had used 260 of them.

Procedural Fluency

Extending learning from the original Candy Jar Task:

$$\frac{5}{13} = \frac{127}{x}$$

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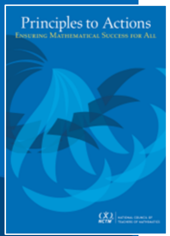

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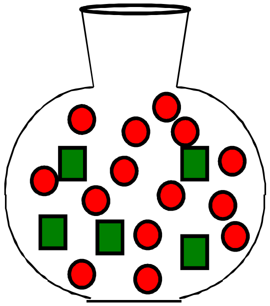

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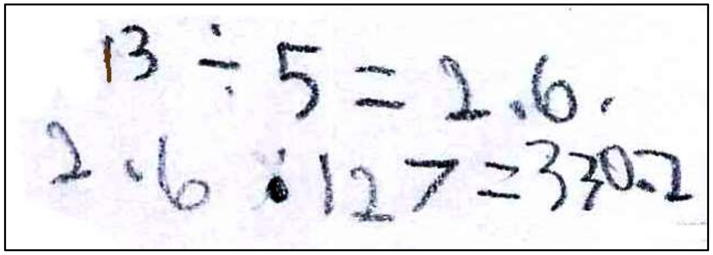

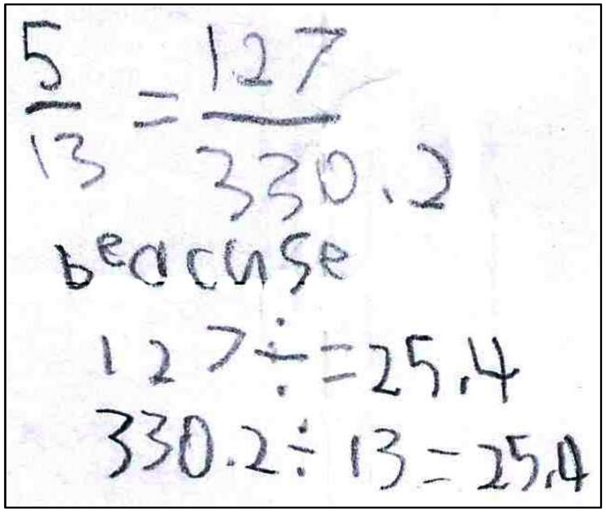

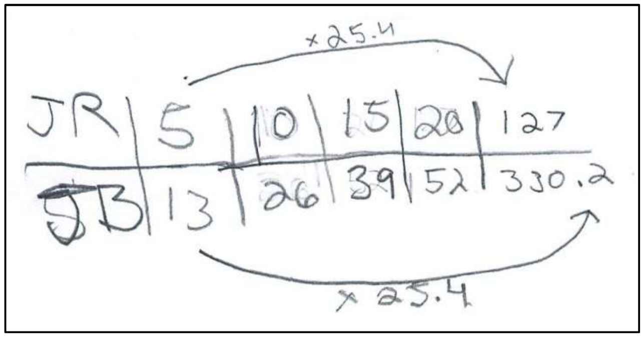

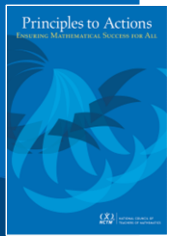

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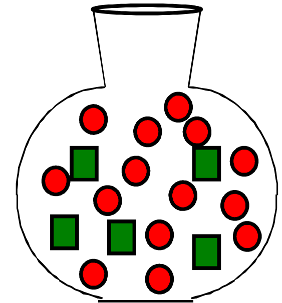
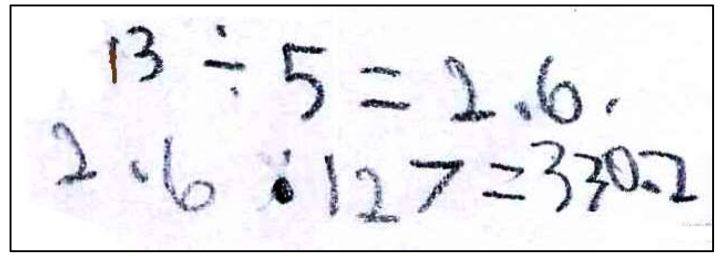
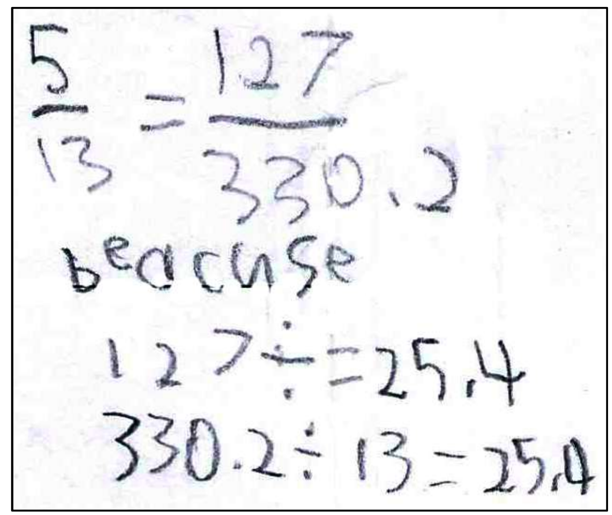


Unit Rate

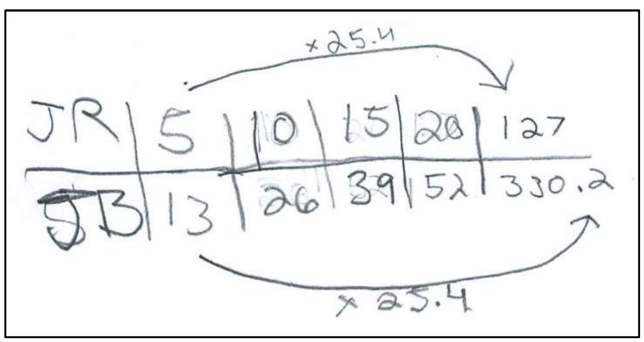
$$\frac{5}{13} = \frac{127}{x}$$



Scale Factor



Scaling Up



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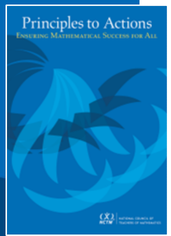

Building Procedural Fluency

<u>Finding the Missing Value</u>

Find the value of the unknown in each of the proportions shown below.

$$\frac{5}{2} = \frac{y}{10} \qquad \frac{a}{24} = \frac{7}{8}$$

$$\frac{n}{8} = \frac{3}{12} \qquad \frac{30}{6} = \frac{b}{7}$$

$$\frac{5}{20} = \frac{3}{d} \qquad \frac{3}{x} = \frac{4}{28}$$

What might we expect students to be able to do when presented with a missing value problem, after they have had the opportunity to develop a set of strategies through solving a variety of contextual problems like the Candy Jar Task?

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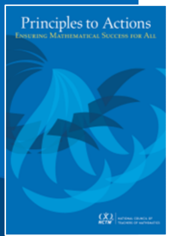

Developing Procedural Fluency

1. Develop conceptual understanding building on students' informal knowledge
2. Develop informal strategies to solve problems
3. Refine informal strategies to develop fluency with standard methods and procedures (algorithms)

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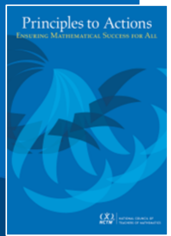

Build Procedural Fluency from Conceptual Understanding

What are teachers doing?

Providing students with opportunities to use their own reasoning strategies and methods for solving problems.

Asking students to discuss and explain why the procedures that they are using work to solve particular problems.

Connecting student-generated strategies and methods to more efficient procedures as appropriate.

Using visual models to support students' understanding of general methods.

Providing students with opportunities for distributed practice of procedures.

What are students doing?

Making sure that they understand and can explain the mathematical basis for the procedures that they are using.

Demonstrating flexible use of strategies and methods while reflecting on which procedures seem to work best for specific types of problems.

Determining whether specific approaches generalize to a broad class of problems.

Striving to use procedures appropriately and efficiently.

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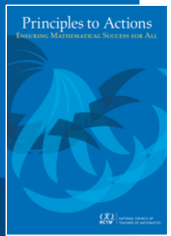

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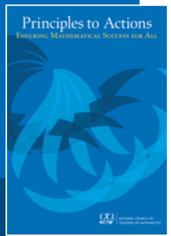

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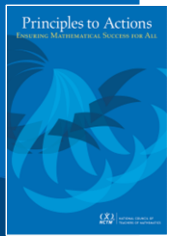


Implement Tasks that Promote Reasoning and Problem Solving

Mathematical tasks should:

- Provide opportunities for students to engage in exploration or encourage students to use procedures in ways that are connected to concepts and understanding;
- Build on students' current understanding; and
- Have multiple entry points.

Dividing Fractions—Book 1

4.1 Preparing Food

There are times when the amounts given in a division situation are not whole numbers but fractions. First, you need to understand what division of fractions means. Then you can learn how to calculate quotients when the divisor or the dividend, or both, is a fraction.

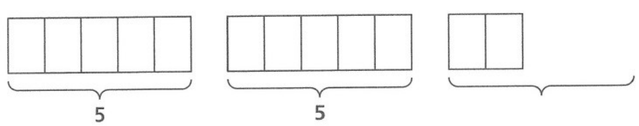
When you do the division $12 \div 5$, what does the answer mean?

The answer should tell you how many fives are in 12 wholes. Because there is not a whole number of fives in 12, you might write:

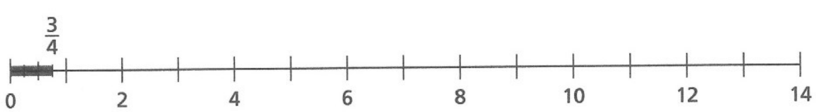
$$12 \div 5 = 2\frac{2}{5}$$

Now the question is, what does the *fractional part* of the answer mean?

The answer means you can make 2 fives and $\frac{2}{5}$ *of another five.*



Suppose you ask, "How many $\frac{3}{4}$'s are in 14?" You can write this as a division problem, $14 \div \frac{3}{4}$.



Can you make a whole number of $\frac{3}{4}$'s out of 14 wholes?

If not, what does the fractional part of the answer mean?

As you work through the problems in this investigation, keep these two questions in mind.

What does the answer to a division problem mean?

What does the fractional part of the answer to a division problem mean?

Problem 4.1 Dividing a Whole Number by a Fraction

Use written explanations or diagrams to show your reasoning for each part. Write a number sentence showing your calculation(s).

A. Naylah plans to make small cheese pizzas to sell at a school fundraiser. She has nine bars of cheese. How many pizzas can she make if each pizza needs the given amount of cheese?

1. $\frac{1}{3}$ bar **2.** $\frac{1}{4}$ bar **3.** $\frac{1}{5}$ bar

4. $\frac{1}{6}$ bar **5.** $\frac{1}{7}$ bar **6.** $\frac{1}{8}$ bar

B. Frank also has nine bars of cheese. How many pizzas can he make if each pizza needs the given amount of cheese?

1. $\frac{1}{3}$ bar **2.** $\frac{2}{3}$ bar **3.** $\frac{3}{3}$ bar **4.** $\frac{4}{3}$ bar

5. The answer to part (2) is a mixed number. What does the fractional part of the answer mean?

C. Use what you learned from Questions A and B to complete the following calculations.

1. $12 \div \frac{1}{3}$ **2.** $12 \div \frac{2}{3}$ **3.** $12 \div \frac{5}{3}$

4. $12 \div \frac{1}{6}$ **5.** $12 \div \frac{5}{6}$ **6.** $12 \div \frac{7}{6}$

7. The answer to part (3) is a mixed number. What does the fractional part of the answer mean in the context of cheese pizzas?

D. 1. Explain why $8 \div \frac{1}{3} = 24$ and $8 \div \frac{2}{3} = 12$.

2. Why is the answer to $8 \div \frac{2}{3}$ exactly half the answer to $8 \div \frac{1}{3}$?

E. Write an algorithm that seems to make sense for dividing any whole number by any fraction.

F. Write a story problem that can be solved using $12 \div \frac{2}{3}$. Explain why the calculation matches the story.

Dividing Fractions—Book 2

7-4 Dividing Whole Numbers by Fractions

You'll Learn ...

■ to divide a whole number by a fraction

... How It's Used

Structural engineers divide whole numbers by fractions when building tunnels.

Vocabulary

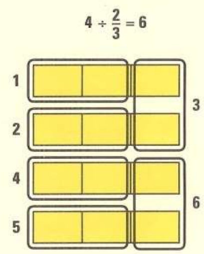
reciprocal

▶ **Lesson Link** In the last section, you learned to multiply whole numbers by fractions. Now you'll divide whole numbers by fractions. ◀

Explore — Dividing Whole Numbers by Fractions

Circles and Strips Forever

Dividing a Whole Number by a Fraction



- Draw a number of strips equal to the whole number.
- Divide the strips into equal pieces. The number of pieces in each strip should be equal to the fraction denominator.
- Circle groups of equal pieces. The number of pieces in each circled group should equal the numerator.
- Describe the number of groups circled.

1. Model these problems.

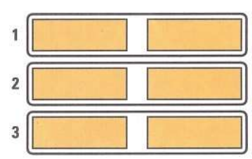
a. $6 \div \frac{2}{3}$ **b.** $7 \div \frac{1}{2}$ **c.** $5 \div \frac{5}{6}$ **d.** $4 \div \frac{3}{6}$ **e.** $2 \div \frac{2}{7}$

2. When you divide a whole number by a fraction less than 1, is the quotient larger or smaller than the original whole number? Why?

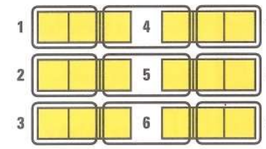
3. Will $3 \div \frac{2}{5}$ have a whole-number answer? Explain.

Learn — Dividing Whole Numbers by Fractions

You can think of division as taking a given amount and breaking it down into groups of a certain size. For example, 6 ÷ 2 can be modeled as 6 loaves of bread divided into groups of 2. The quotient, 3, is the number of groups you have.



You can think of dividing by fractions in the same way. For example, $6 \div \frac{2}{3}$ is the same as 6 loaves of bread divided into groups of $\frac{2}{3}$. The number of groups you have, 9, is the quotient.



Notice that to find the answer, you first found the number of thirds by multiplying the number of loaves, 6, by the denominator, 3. Then, you divided the number of thirds by the numerator, 2.

$6 \div \frac{2}{3} = 6 \times 3 \div 2 = 9$

Dividing by a fraction is the same as multiplying by its **reciprocal**. Reciprocals are numbers whose numerators and denominators have been switched. When two numbers are reciprocals, their product is 1.

Dividing	Multiplying by reciprocal
$6 \div \frac{2}{3} = 9$	$6 \times \frac{3}{2} = \frac{6}{1} \times \frac{3}{2}$
	$= \frac{18}{2}$
	$= 9$

Remember

The numerator is the number on top of a fraction. The denominator is the number on the bottom. [Page 287]

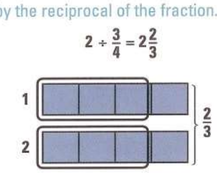
Examples

1 Divide: $2 \div \frac{3}{4}$

$2 \div \frac{3}{4} = \frac{2}{1} \times \frac{4}{3}$ Multiply by the reciprocal of the fraction.

$= \frac{2 \times 4}{1 \times 3}$

$= \frac{8}{3}$ or $2\frac{2}{3}$



2 1 *nail* = $\frac{9}{4}$ in. of cloth. Find the length of 5 in. of cloth in nails.

$5 \div \frac{9}{4} = \frac{5}{1} \times \frac{4}{9}$ Multiply by the reciprocal.

$= \frac{20}{9}$ or $2\frac{2}{9}$ Simplify.

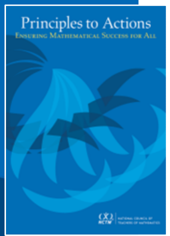
A 5-inch piece of cloth is $2\frac{2}{9}$ nails long.

Try It

Divide. **a.** $4 \div \frac{3}{5}$ **b.** $1 \div \frac{4}{7}$ **c.** $10 \div \frac{17}{4}$ **d.** $3 \div \frac{3}{5}$

DID YOU KNOW?

Three measurements used primarily for cloth include the *nail*, the *finger*, and the *span*. A *finger* is equal to $4\frac{1}{2}$ inches. A *span* is equal to 9 inches.



Dividing Fractions—Book 1

Problem 4.1 Dividing a Whole Number by a Fraction

Use written explanations or diagrams to show your reasoning for each part. Write a number sentence showing your calculation(s).

A. Naylah plans to make small cheese pizzas to sell at a school fundraiser. She has nine bars of cheese. How many pizzas can she make if each pizza needs the given amount of cheese?

1. $\frac{1}{3}$ bar **2.** $\frac{1}{4}$ bar **3.** $\frac{1}{5}$ bar

4. $\frac{1}{6}$ bar **5.** $\frac{1}{7}$ bar **6.** $\frac{1}{8}$ bar

B. Frank also has nine bars of cheese. How many pizzas can he make if each pizza needs the given amount of cheese?

1. $\frac{1}{3}$ bar **2.** $\frac{2}{3}$ bar **3.** $\frac{3}{3}$ bar **4.** $\frac{4}{3}$ bar

5. The answer to part (2) is a mixed number. What does the fractional part of the answer mean?

C. Use what you learned from Questions A and B to complete the following calculations.

1. $12 \div \frac{1}{3}$ **2.** $12 \div \frac{2}{3}$ **3.** $12 \div \frac{5}{3}$

4. $12 \div \frac{1}{6}$ **5.** $12 \div \frac{5}{6}$ **6.** $12 \div \frac{7}{6}$

7. The answer to part (3) is a mixed number. What does the fractional part of the answer mean in the context of cheese pizzas?

D. **1.** Explain why $8 \div \frac{1}{3} = 24$ and $8 \div \frac{2}{3} = 12$.

2. Why is the answer to $8 \div \frac{2}{3}$ exactly half the answer to $8 \div \frac{1}{3}$?

E. Write an algorithm that seems to make sense for dividing any whole number by any fraction.

F. Write a story problem that can be solved using $12 \div \frac{2}{3}$. Explain why the calculation matches the story.

Dividing Fractions—Book 2

7-4 Dividing Whole Numbers by Fractions

You'll Learn ...

- to divide a whole number by a fraction

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reciprocal

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Explore Dividing Whole Numbers by Fractions

Circles and Strips Forever

Dividing a Whole Number by a Fraction

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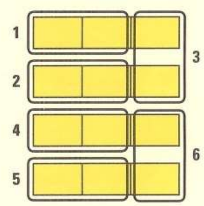
$4 \div \frac{2}{3} = 6$



1. Model these problems.

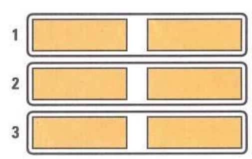
   a. $6 \div \frac{2}{3}$ **b.** $7 \div \frac{1}{2}$ **c.** $5 \div \frac{5}{6}$ **d.** $4 \div \frac{3}{6}$ **e.** $2 \div \frac{2}{7}$

2. When you divide a whole number by a fraction less than 1, is the quotient larger or smaller than the original whole number? Why?

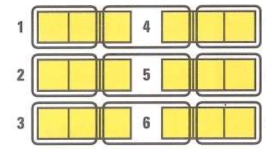
3. Will $3 \div \frac{2}{5}$ have a whole-number answer? Explain.

Learn Dividing Whole Numbers by Fractions

You can think of division as taking a given amount and breaking it down into groups of a certain size. For example, $6 \div 2$ can be modeled as 6 loaves of bread divided into groups of 2. The quotient, 3, is the number of groups you have.



You can think of dividing by fractions in the same way. For example, $6 \div \frac{2}{3}$ is the same as 6 loaves of bread divided into groups of $\frac{2}{3}$. The number of groups you have, 9, is the quotient.



Notice that to find the answer, you first found the number of thirds by multiplying the number of loaves, 6, by the denominator, 3. Then, you divided the number of thirds by the numerator, 2.

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Remember

The numerator is the number on top of a fraction. The denominator is the number on the bottom. [Page 287]

Examples

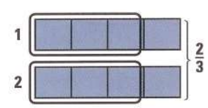
1 Divide: $2 \div \frac{3}{4}$

$2 \div \frac{3}{4} = \frac{2}{1} \times \frac{4}{3}$ Multiply by the reciprocal of the fraction.

$= \frac{2 \times 4}{1 \times 3}$

$= \frac{8}{3}$ or $2\frac{2}{3}$

$2 \div \frac{3}{4} = 2\frac{2}{3}$



2 1 *nail* = $\frac{9}{4}$ in. of cloth. Find the length of 5 in. of cloth in nails.

$5 \div \frac{9}{4} = \frac{5}{1} \times \frac{4}{9}$ Multiply by the reciprocal.

$= \frac{20}{9}$ or $2\frac{2}{9}$ Simplify.

A 5-inch piece of cloth is $2\frac{2}{9}$ nails long.

Try It

Divide. **a.** $4 \div \frac{3}{5}$ **b.** $1 \div \frac{4}{7}$ **c.** $10 \div \frac{17}{4}$ **d.** $3 \div \frac{3}{5}$

DID YOU KNOW?

Three measurements used primarily for cloth include the *nail*, the *finger*, and the *span*. A *finger* is equal to $4\frac{1}{2}$ inches. A *span* is equal to 9 inches.

Dividing Fractions—Standard Algorithm

4.4 Writing a Division Algorithm

You are ready now to develop an algorithm for dividing fractions. To get started, you will break division problems into categories and write steps for each kind of problem. Then you can see whether there is one "big" algorithm that will solve them all.

Problem 4.4 Writing a Division Algorithm

A. **1.** Find the quotients in each group below.

Group 1	Group 2	Group 3	Group 4
$\frac{1}{3} \div 9$	$12 \div \frac{1}{6}$	$\frac{5}{6} \div \frac{1}{12}$	$5 \div 1\frac{1}{2}$
$\frac{1}{6} \div 12$	$5 \div \frac{2}{3}$	$\frac{3}{4} \div \frac{3}{4}$	$\frac{1}{2} \div 3\frac{2}{3}$
$\frac{3}{5} \div 6$	$3 \div \frac{2}{5}$	$\frac{9}{5} \div \frac{1}{2}$	$3\frac{1}{3} \div \frac{2}{3}$

2. Describe what the problems in each group have in common.

3. Make up one new problem that fits in each group.

4. Write an algorithm that works for dividing *any* two fractions, including mixed numbers. Test your algorithm on the problems in the table. If necessary, change your algorithm until you think it will work all the time.

B. Use your algorithm to divide.

1. $9 \div \frac{4}{5}$ **2.** $1\frac{7}{8} \div 3$ **3.** $1\frac{2}{3} \div \frac{1}{5}$ **4.** $2\frac{5}{6} \div 1\frac{1}{3}$

C. Here is a multiplication-division fact family for whole numbers:

$5 \times 8 = 40$ $8 \times 5 = 40$ $40 \div 5 = 8$ $40 \div 8 = 5$

1. Complete this multiplication-division fact family for fractions.

$$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$

2. Check the division answers by using your algorithm.

D. For each number sentence, find a value for N that makes the sentence true. If needed, use fact families.

1. $\frac{2}{3} \div \frac{4}{5} = N$ **2.** $\frac{3}{4} \div N = \frac{7}{8}$ **3.** $N \div \frac{1}{4} = 3$

Dividing Fractions by Fractions

▶ Lesson Link In the last lesson, you learned to divide whole numbers by fractions. Now you'll divide fractions by fractions. ◀

Explore: Dividing Fractions by Fractions

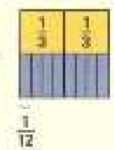
Materials: Fraction Bars®

Wish Upon a Bar

Dividing a Fraction by a Fraction

$\frac{2}{3} \div \frac{1}{12} = 8$

- Using a Fraction Bar®, draw and label the first fraction.
- Under that, use a Fraction Bar® to draw as many diagrams of the second fraction as will fit.
- Describe the number of diagrams below the first fraction.



1. Model each problem.

a. $\frac{3}{6} \div \frac{1}{12}$ **b.** $\frac{1}{2} \div \frac{1}{4}$ **c.** $\frac{2}{3} \div \frac{1}{6}$ **d.** $\frac{2}{4} \div \frac{2}{12}$

2. When you divide a fraction by a fraction less than 1, why is the answer bigger than the fraction you started with?

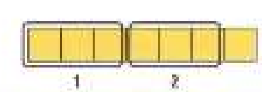
3. How is dividing a fraction by a fraction similar to dividing a whole number by a fraction?

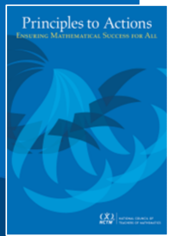
4. Can you use Fraction Bars® to divide $\frac{1}{2} \div \frac{1}{5}$? Explain.

Learn: Dividing Fractions by Fractions

When you divide a whole number by a fraction, you get the same result as if you had multiplied the whole number by the fraction's reciprocal. This is also true when you divide a fraction by a fraction.

Dividing	Multiplying by Reciprocal
$\frac{6}{7} \div \frac{3}{7} = 2$	$\frac{6}{7} \times \frac{7}{3} = \frac{42}{21}$ or 2





Developing Procedural Fluency Whole Number Multiplication

1. Develop conceptual understanding building on students' informal knowledge
2. Develop informal strategies to solve problems
3. Refine informal strategies to develop fluency with standard methods and procedures (algorithms)

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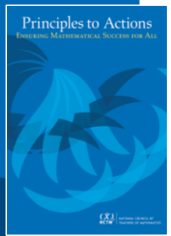
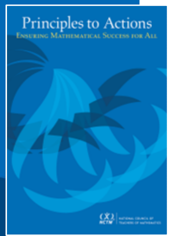


	Numbers and Operations in Base Ten
1	Use place value understanding and properties of operations to add and subtract.
2	Use place value understanding and properties of operations to add and subtract.
3	Use place value understanding and properties of operations to perform multi-digit arithmetic. *A range of algorithms may be used.*
4	Use place value understanding and properties of operations to perform multi-digit arithmetic. *Fluently add and subtract multi-digit whole numbers using the standard algorithm.*
5	Perform operations with multi-digit whole numbers and with decimals to hundredths. *Fluently multiply multi-digit whole numbers using the standard algorithm.*
6	Compute fluently with multi-digit numbers and find common factors and multiples. *Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.*



	Numbers and Operations in Base Ten
1	Use place value understanding and properties of operations to add and subtract.
2	Use place value understanding and properties of operations to add and subtract.
3	Use place value understanding and properties of operations to perform multi-digit arithmetic. ***A range of algorithms may be used.***
4	Use place value understanding and properties of operations to perform multi-digit arithmetic. ***Fluently add and subtract multi-digit whole numbers using the standard algorithm.***
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The Band Concert

The third-grade class is responsible for setting up the chairs for their spring band concert. In preparation, they need to determine the total number of chairs that will be needed and ask the school's engineer to retrieve that many chairs from the central storage area.

The class needs to set up 7 rows of chairs with 20 chairs in each row, leaving space for a center aisle.

How many chairs does the school's engineer need to retrieve from the central storage area?

Make a quick sketch or diagram of the situation.

How might Grade 3 students approach this task?

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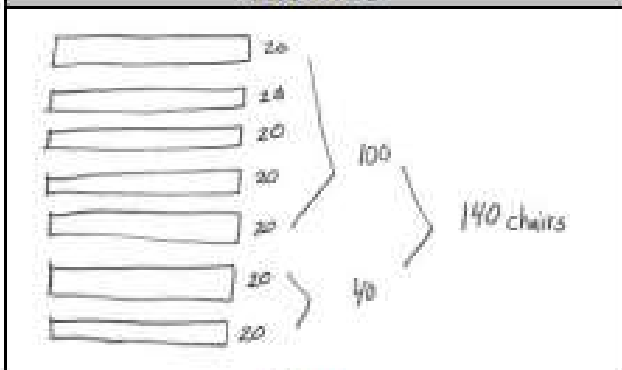
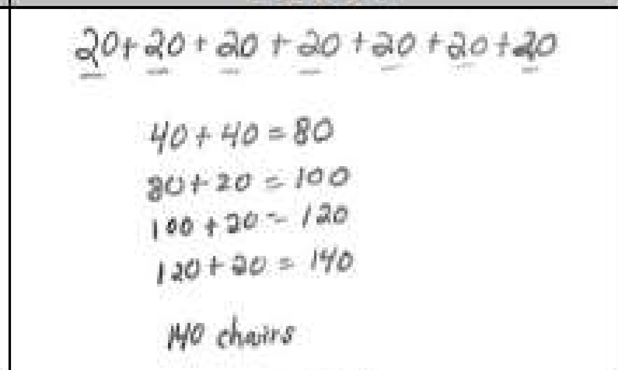
The Band Concert

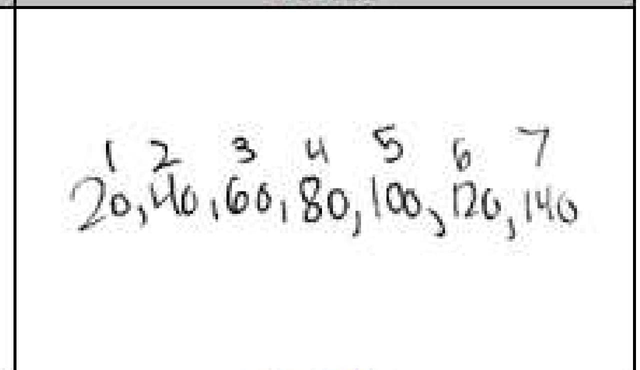
Jasmine



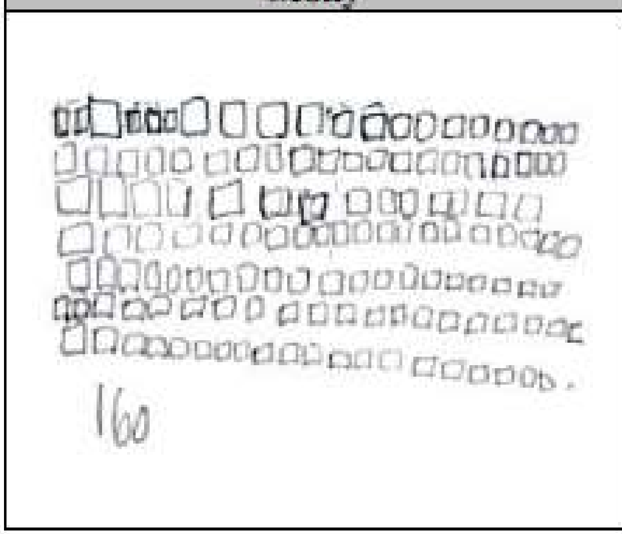
Kenneth



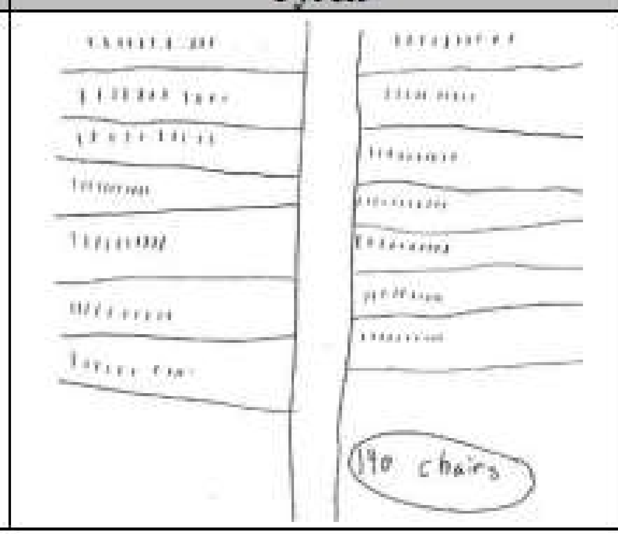
Teresa



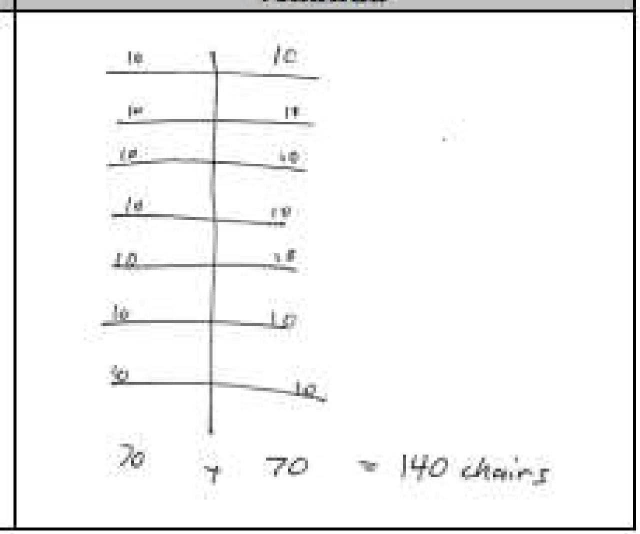
Molly

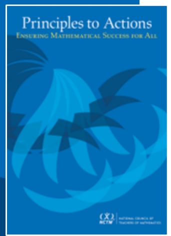


Tyrell



Ananda





Starting Point to Build Procedural Fluency

The Band Concert

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The class needs to set up 7 rows of chairs with 20 chairs in each row, leaving space for a center aisle.

How many chairs does the school's engineer need to retrieve from the central storage area?

Find the product

5 x 20 =

6 x 80 =

4 x 70 =

3 x 50 =

9 x 20 =

2 x 60 =

8 x 30 =

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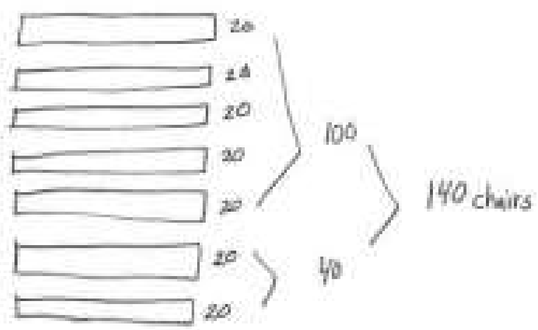
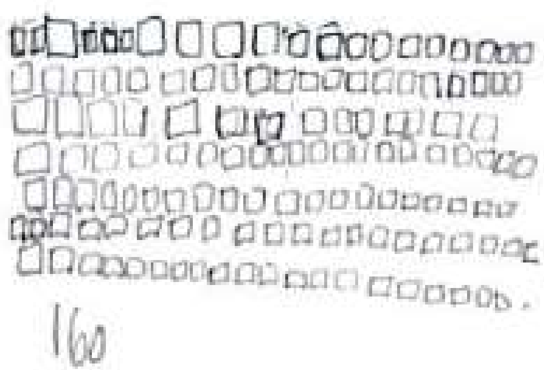
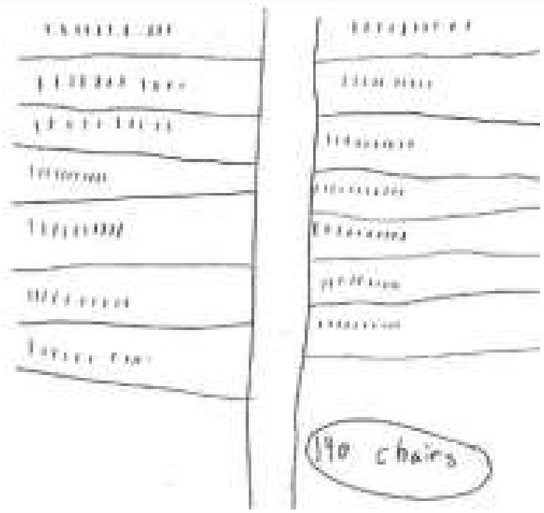
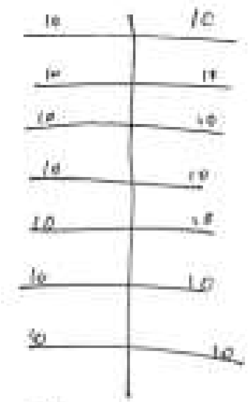
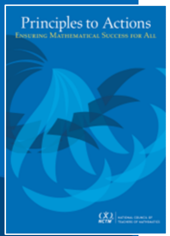
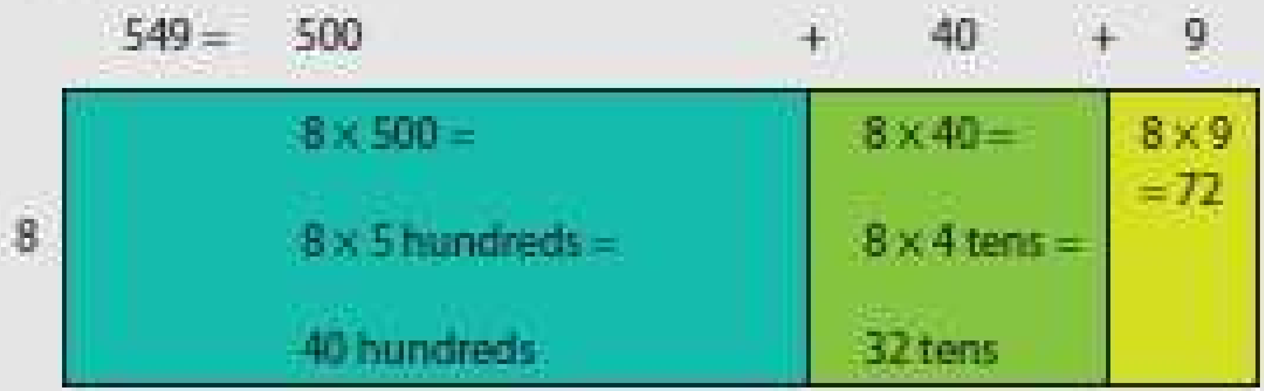
The Band Concert

Jasmine	Kenneth	Teresa
	20+20+20+20+20+20+20 40+40=80 80+20=100 100+20=120 120+20=140 140 chairs	1 2 3 4 5 6 7 20, 40, 60, 80, 100, 120, 140
Molly	**Tyrell**	**Ananda**
		



Multiplication Algorithms

Computation of 8×549 connected with an area model



Each part of the region above corresponds to one of the terms in the computation below.

$$\begin{aligned} 8 \times 549 &= 8 \times (500 + 40 + 9) \\ &= 8 \times 500 + 8 \times 40 + 8 \times 9. \end{aligned}$$

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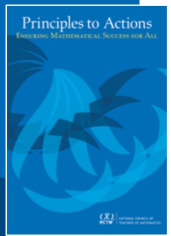
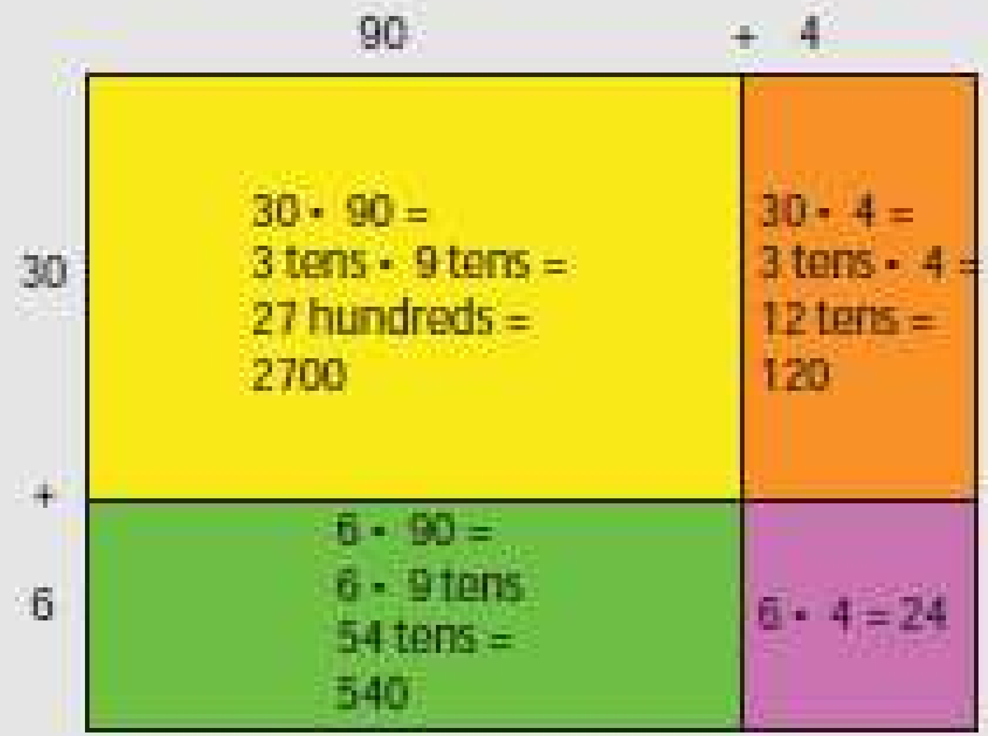


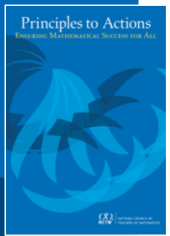
Multiplication Algorithms

Computation of 36 × 94 connected with an area model

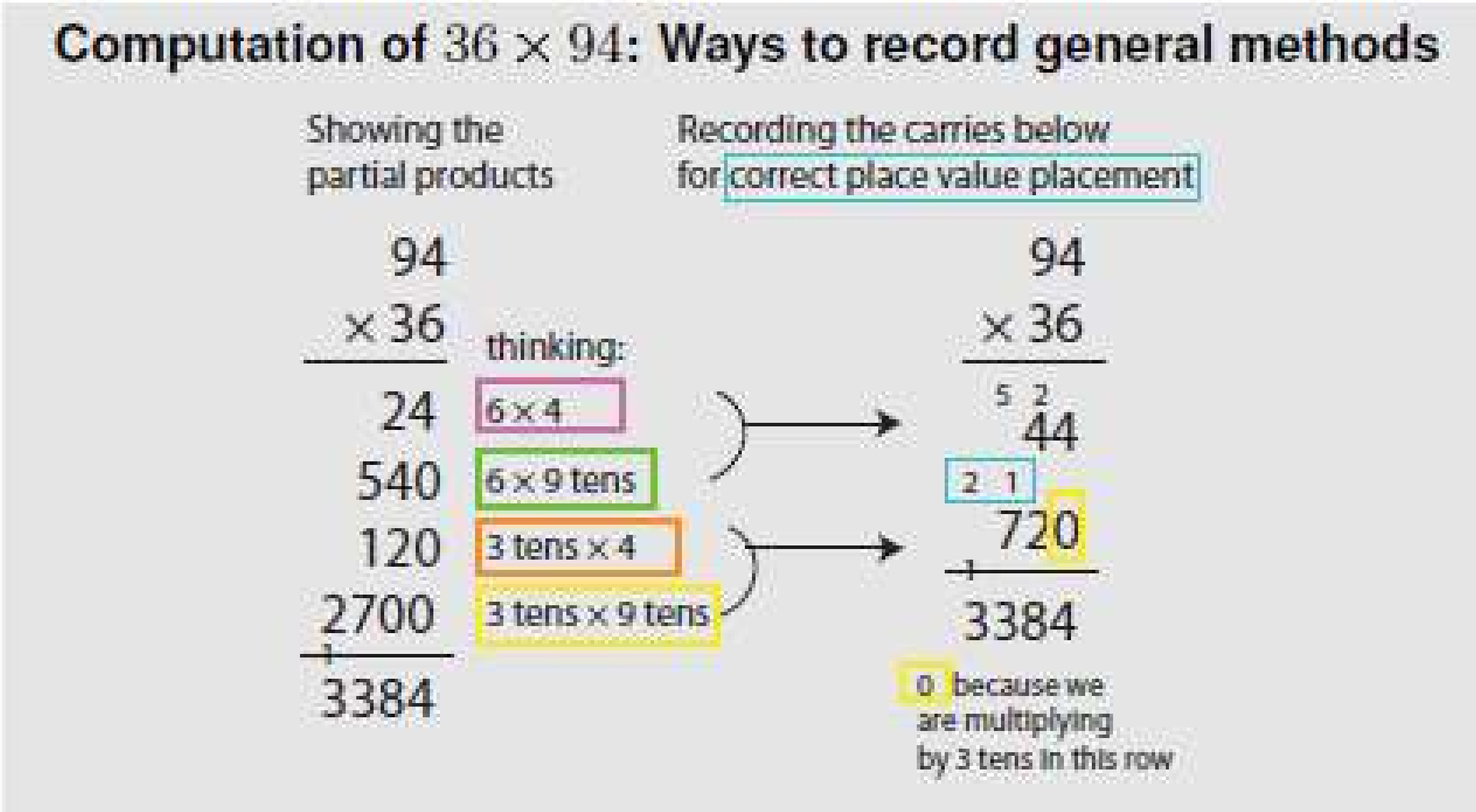


The products of like base-ten units are shown as parts of a rectangular region.

CCSS Numbers and Operations in Base-Ten Progression, April 2011



Multiplication Algorithms



CCSS Numbers and Operations in Base-Ten Progression, April 2012

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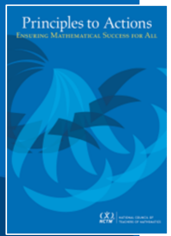
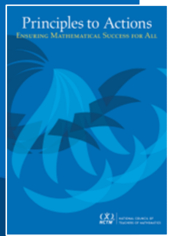


What is Meant by "Standard Algorithm?"

"In mathematics, an algorithm is defined by its steps and not by the way those steps are recorded in writing. With this in mind, minor variations in methods of recording standard algorithms are acceptable."

Fuson & Beckmann, 2013; NCSM Journal, p. 14



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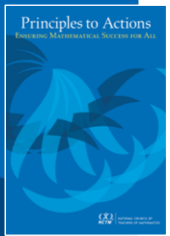

Multiplication Algorithms

$$\begin{array}{r} 1 \\ 2 \\ 94 \\ \times\, 36 \\ \hline 564 \\ {}^{1}282 \\ \hline 3384 \end{array}$$

← From $30 \times 4 = 120$. The 1 is 1 hundred, not 1 ten.

$$\begin{array}{rcl} 94 & = & 90 + 4 \\ \times\, 36 & = & 30 + 6 \\ \hline 6 \times 4 & = & 24 \\ 6 \times 90 & = & 540 \\ 30 \times 4 & = & 120 \\ 30 \times 90 & = & 2700 \\ \hline & & {}^{1} \\ & & 3384 \end{array}$$

Fuson & Beckmann, 2013, p. 25

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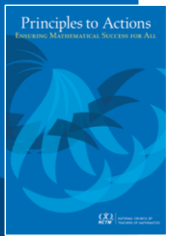

Multiplication Algorithms

$$
\begin{array}{r}
\scriptstyle 1 \\
\scriptstyle 2 \\
94 \\
\times\ 36 \\
\hline
564 \\
{}^{1}282 \\
\hline
3384
\end{array}
\qquad\qquad
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94 \\
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\hline
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540 \\
120 \\
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\hline
{}_{1}\ 3384
\end{array}
$$

Fuson & Beckmann, 2013, p. 25

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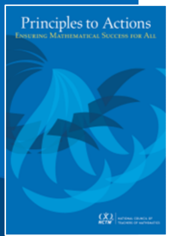


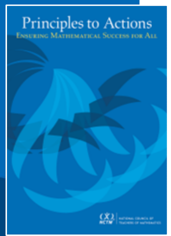
A Functions Approach: Building Upon Informal Knowledge

$y = 12x + 10$

- Solve for y when x = 3, 10, 100.
- Solve $70 = 12x + 10$

U.S. Shirts charges $12 per shirt plus $10 set-up charge for custom printing.

- What is the total cost of an order for 3 shirts?
- What is the total cost of an order for 10 shirts?
- What is the total cost of an order for 100 shirts?
- A customer spends $70 on T-shirts. How many shirts did the customer buy?



Procedural Fluency

- **Efficiency**—can carry out easily, keep track of subproblems, and make use of intermediate results to solve the problem.
- **Accuracy**—reliably produces the correct answer.
- **Flexibility**—knows more than one approach, chooses appropriate strategy, and can use one method to solve and another method to double-check.
- **Appropriately**—knows when to apply a particular procedure.

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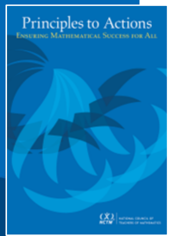

Build Procedural Fluency from Conceptual Understanding

What are teachers doing?

Providing students with opportunities to use their own reasoning strategies and methods for solving problems.

Asking students to discuss and explain why the procedures that they are using work to solve particular problems.

Connecting student-generated strategies and methods to more efficient procedures as appropriate.

Using visual models to support students' understanding of general methods.

Providing students with opportunities for distributed practice of procedures.

What are students doing?

Making sure that they understand and can explain the mathematical basis for the procedures that they are using.

Demonstrating flexible use of strategies and methods while reflecting on which procedures seem to work best for specific types of problems.

Determining whether specific approaches generalize to a broad class of problems.

Striving to use procedures appropriately and efficiently.

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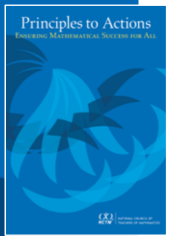

Effective Mathematics Teaching Practices

1. Establish mathematics **goals** to focus learning.
2. Implement **tasks** that promote reasoning and problem solving.
3. Use and connect mathematical **representations.**
4. Facilitate meaningful mathematical **discourse**.
5. Pose purposeful **questions**.
6. Build **procedural fluency** from conceptual understanding.
7. Support **productive struggle** in learning mathematics.
8. **Elicit and use evidence** of student thinking.

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The Title Is Principles to *Actions*

Your Actions?

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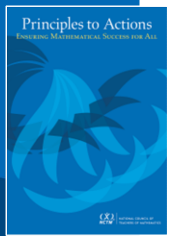

Thank You!

Diane Briars
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