Word Problems: From Theory to Strategies

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Which one does not belong?

Apples

John has 12 apples and gives 3 away. How many does he have now?

Flowers

John has 12 flowers and some died. He has 8 flowers left, how many died?

Toys

John has 12 toys and Mark has 8 toys. How many more toys does John have?

Why students struggle to solve word problems?

- **The scenario represented is outside the experience of a student.**
- **The vocabulary is unfamiliar.**
- **The numbers are larger than the student is able to compute mentally.**
- **The student was unable to identify the underlying schema of the problem.**

- **Students who have difficulties in math see every problem as a brand new problem.**

Research Paper: Bruun. F. (2013) Elementary Teachers' Perspectives of Mathematics Problem Solving Strategies. The Mathematics Educator 23(1) 45- 59

What have we been teaching?

- The Problem-Solving Process
  - Understanding the problem,
  - Devising a plan to solve the problem,
  - Implementing the plan, and
  - Reflecting on the problem.

Pólya, G. (1945). *How to Solve It*. Princeton University Press

Different Strategies to Solve Word Problems

- Draw a picture
- Choose an operation
- Make a table
- Act it out
- Work backward
- Guess, Test, Revise
- Work a simpler problem
- Make an organized list
- Find a pattern

NCTM Strategies

Strategy	%
Draw a picture	40%
Choose an operation	20%
Make a table or graph	17%
Act it out	9%
Work backwards	6%
Guess, test, revise	4%
Work a simpler problem	2%
Make an organized list	1%
Find a pattern	1%

Research: Bruun. F. (2013) Elementary Teachers' Perspectives of Mathematics Problem Solving Strategies. The Mathematics Educator 23(1) 45- 59

Draw a picture

Jenny has $12 in the bank. She earns $5 doing some jobs. How much money does she have now?

Does not show comprehension of the problem.

The girl was happy because she had money in the bank.

What does this show?

Word Problems connections

" One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connections with each other, and no connection at all with anything else… The teacher should encourage the students to imagine cases in which they could utilize again the procedure used, or apply the result obtained" Polya's *How to Solve It*(p.15-16).

Pólya, G. (1945). *How to Solve It*. Princeton University Press

The Problem-Solving Process

- Understanding the problem,
- Devising a plan to solve the problem,
  - Strategies we discussed – not as effective with struggling students.
- Implementing the plan, and
- Reflecting on the problem.

How Students Learn?

- Piaget's work on Schema
- Piaget (1952) defined a schema as the basic building block of intelligent behavior – a way of organizing knowledge!
- "A set of linked mental representations of the world"

Source: http://www.simplypsychology.org/piaget.html#schema

The Process of Intellectual Growth - Adaptation

- Assimilation
  - "Using existing schema to deal with new situations"
  - Using the same procedures, a student solves a different type of problem
- Accommodation
  - "Existing schema (knowledge) does not work, the schema is changed to deal with a new situation"
  - A student changes the procedures to solve a new problem when the old procedures did not work.
- Equilibration
  - The force to move from dis-equilibrium to equilibration accommodation

Source: : http://www.simplypsychology.org/piaget.html#schema

Implication for Education

- Piaget's constructivism and stage of development ideas support students using concrete models to represent math problems.

- Piaget's Schema model, he believes: " Children who have severely limited interactions with their environments simply will not have the opportunities to develop and reorganize their cognitive structures so as to achieve mature ways of thinking." (Cook)
  - Struggling students lack the cognitive maturity to comprehend abstract word problems.

- Hattie Ranking: Influences And Effect Sizes Related To Student Achievement
  - Piagetian program - 1.28 effect size
  - Second highest influence to student achievement.

Sources: Cook, J. and G (2005) *Child Development Principles & Perspectives*, Pearson & http://visible-learning.org/hattie-ranking-influences-effect-sizes-learning-achievement/

What Now!

We may need to reconsider what we teach?

Teach the structure of word problems!

What are the different types of word problems?

Addition/Subtraction Word Problems

Table 2: Addition and subtraction situations by grade level.

	Result Unknown	Change Unknown	Start Unknown
Add To	A bunnies sat on the grass. B more bunnies hopped there. How many bunnies are on the grass now? $A + B = \square$	A bunnies were sitting on the grass. Some more bunnies hopped there. Then there were C bunnies. How many bunnies hopped over to the first A bunnies? $A + \square = C$	Some bunnies were sitting on the grass. B more bunnies hopped there. Then there were C bunnies. How many bunnies were on the grass before? $\square + B = C$
Take From	C apples were on the table. I ate B apples. How many apples are on the table now? $C - B = \square$	C apples were on the table. I ate some apples. Then there were A apples. How many apples did I eat? $C - \square = A$	Some apples were on the table. I ate B apples. Then there were A apples. How many apples were on the table before? $\square - B = A$

	Total Unknown	Both Addends Unknown[1]	Addend Unknown[2]
Put Together /Take Apart	A red apples and B green apples are on the table. How many apples are on the table? $A + B = \square$	Grandma has C flowers. How many can she put in her red vase and how many in her blue vase? $C = \square + \square$	C apples are on the table. A are red and the rest are green. How many apples are green? $A + \square = C$ $C - A = \square$

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	*"How many more?" version.* Lucy has A apples. Julie has C apples. How many more apples does Julie have than Lucy? *"How many fewer?" version.* Lucy has A apples. Julie has C apples. How many fewer apples does Lucy have than Julie? $A + \square = C$ $C - A = \square$	*"More" version suggests operation.* Julie has B more apples than Lucy. Lucy has A apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have? $A + B = \square$	*"Fewer" version suggests operation.* Lucy has B fewer apples than Julie. Julie has C apples. How many apples does Lucy have? *"More" version suggests wrong operation.* Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have? $C - B = \square$ $\square + B = C$

Source: http://achievethecore.org/page/932/situation-types

Different Situations of Addition/Subtraction Word Problems

Schema Structure

Represents information unknown

Change

	RESULT UNKNOWN	CHANGE UNKNOWN	START UNKNOWN
ADD TO	*A* bunnies sat on the grass. *B* more bunnies hopped there. How many bunnies are on the grass now? A + B = ___ (k)	*A* bunnies were sitting on the grass. Some more bunnies joined them. Then there were *C* bunnies. How many bunnies joined the *A* bunnies? A + ___ = C (1)	Some bunnies were sitting on the grass. *B* more bunnies joined them. Then there were *C* bunnies. How many bunnies were on the grass before? ___ + B = C (2)
TAKE FROM	*C* apples were on the table. I ate *B* apples. How many apples are on the table now? C – B = ___ (k)	*C* apples were on the table. I ate some apples. There were *A* apples left. How many apples did I eat? C – ___ = A (1)	Some apples were on the table. I ate *B* apples. There were *A* apples left. How many apples were on the table before? ___ – B = A (2)

Combine

	TOTAL UNKNOWN	BOTH ADDENDS UNKNOWN	ADDEND UNKNOWN
PUT TOGETHER/ TAKE APART	*A* red apples and *B* green apples are on the table. How many apples are on the table? A + B = ___ (k)	Grandma has *C* flowers. How many can she put in her red vase and how many in her blue vase? C = ___ + ___ *OR* ___ + ___ = C (k)	*C* apples are on the table. *A* are red and the rest are green. How many apples are green? C – A = ___ *OR* A + ___ = C (1)

Compare

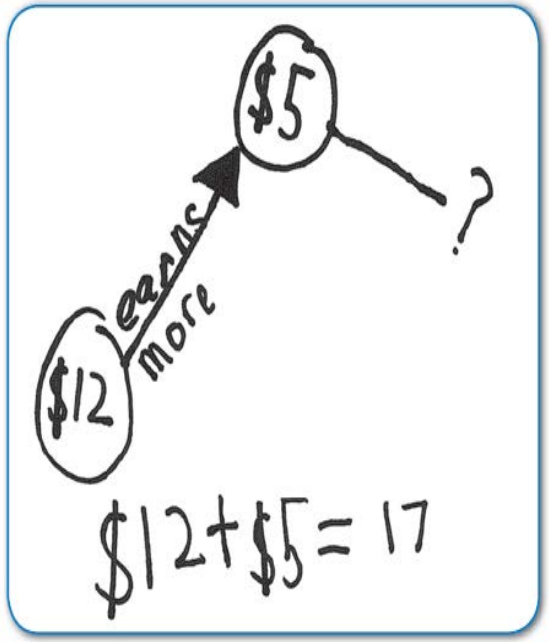
	DIFFERENCE UNKNOWN	BIGGER UNKNOWN	SMALLER UNKNOWN
COMPARE	**"How many more?" version.** Lucy has *A* apples. Jack has *C* apples. How many more apples does Jack have than Lucy? A + ___ = C *OR* C – A = ___ (1)	**"More" version suggests operation.** Jack has *B* more apples than Lucy. Lucy has *A* apples. How many apples does Jack have? A + B = ___ (1)	**"Fewer" version suggests operation.** Lucy has *B* fewer apples than Jack. Jack has *C* apples. How many apples does Lucy have? C – B = ___ *OR* ___ + B = C (1)

Schema Structures

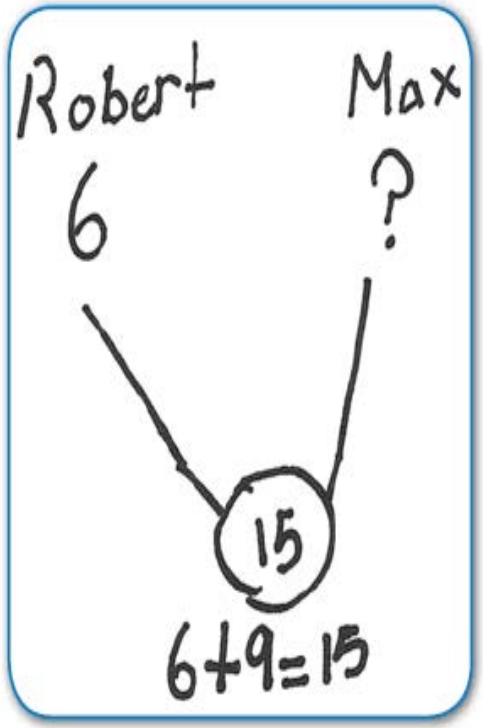
- **CHANGE**
  - Start with an amount which increases or decreases.
  - Has a beginning, change, and ending.
  - Verbs are critical to determine the change (increase or decrease).
- **COMBINE**
  - Starts with groups that are put together for a total.
  - Items in the groups are static.
  - The word problem shares information about two or more groups.
- **COMPARE**
  - Two amounts are compared to determine the difference.
  - Items in the groups are static.
  - The word problem has compare words (taller than, more than, less than….?).

Schematic Diagram

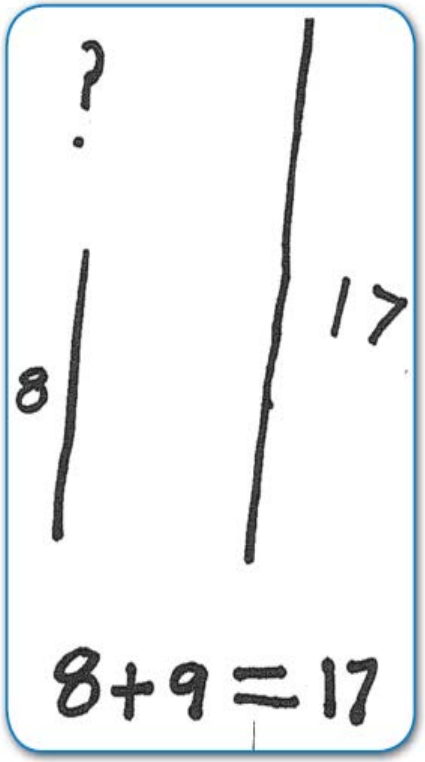
A schematic diagram is a visual tool to assist students in paring down information so that only the important structural information remains.



Change



Combine

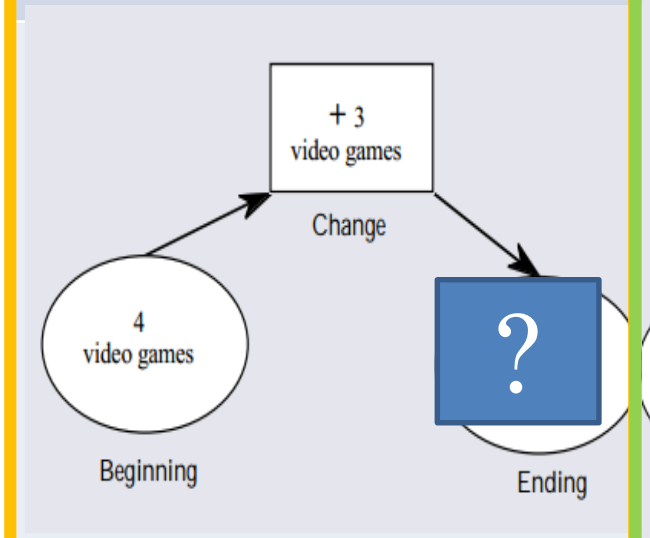


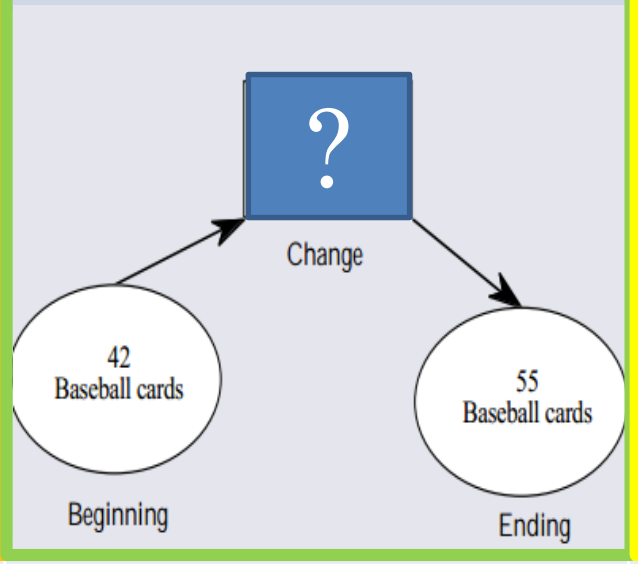
Compare

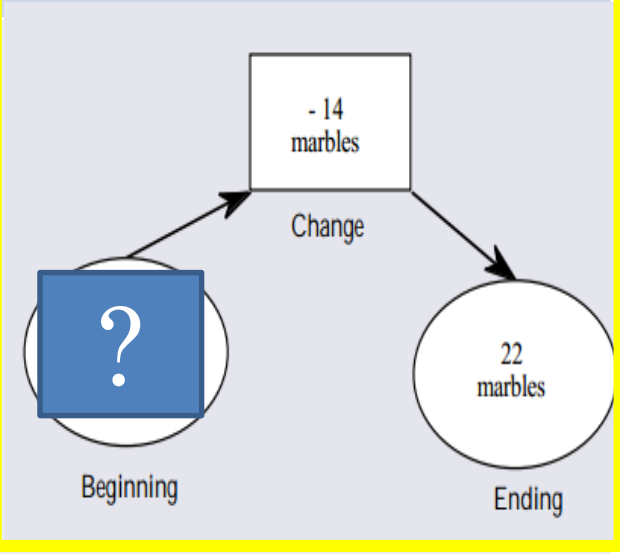
Source: van Klinken, E. (2012) *Word problem solving: a schema approach in year 3*. AMPC 17(1)

CHANGE Schema Structure

Result Unknown 4 + 3 = ?	Change Unknown 42 + ? = 55	Start Unknown ? – 14 = 22
Jane had 4 video games. Then her mother gave her 3 more video games for her birthday. How many video games does Jane have now?	Tom had 42 baseball cards. He buys more and now he has 55 baseball cards. How many cards did he buy?	James has a marble collection. He gives away 14 marbles and now he has 22 marbles. How many marbles did James have before he gave some away?



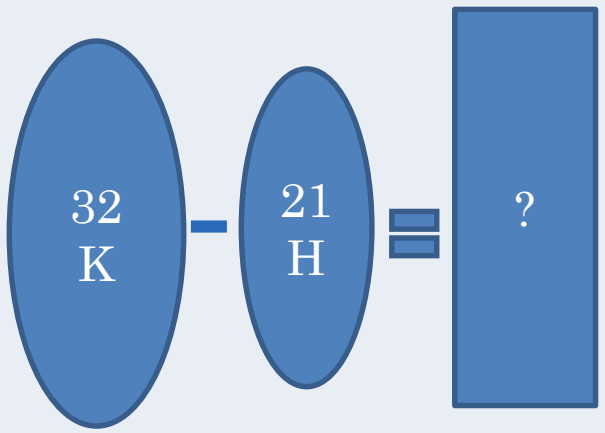
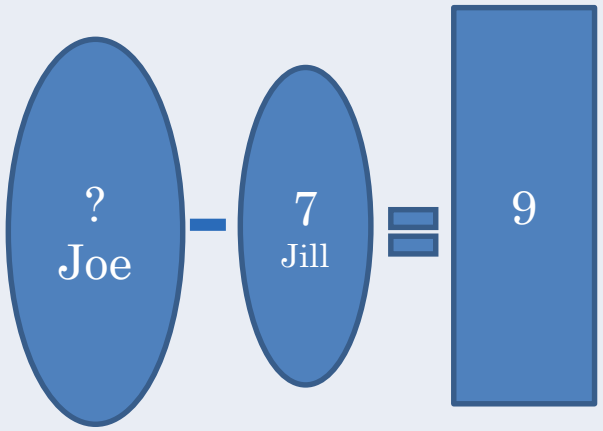
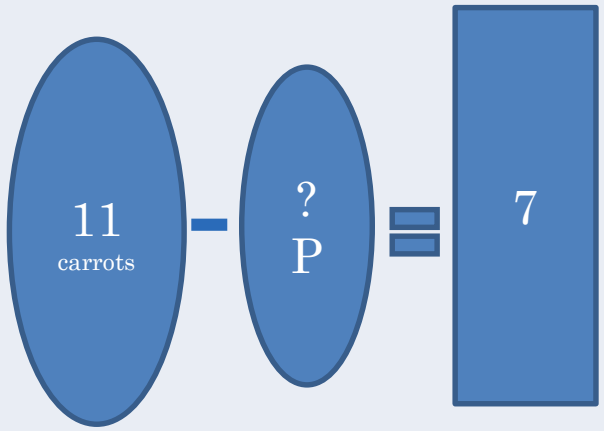




COMBINE Schema Structure

Total Unknown 22 + 19 + 27 = ?	Both Addends Unknown ? + ? = 15	Addend Unknown 45 + ? = 98
Students at Hillcrest Elementary took part in the school play. There were 22 third graders, 19 fourth graders, and 27 fifth graders in the play. How many total students were in the play?	Eleanor has 15 stuffed animals. Some are bears and some are dogs. How many of each kind of stuffed animals could she have?	In a survey, 98 people were asked what their favorite flower is and 45 named tulips. How many named another kind of flower?
3rd graders: 22 + 4th graders: 19 + 5th graders: 27 = All students (3rd, 4th, & 5th graders): ?	Bears: ? + Dogs: ? = Total number of stuffed animals: 15	Named tulip: 45 people + Named another kind of flower: ? people = Named tulip and another kind of flower: 98 people

COMPARE Schema Structure

Difference Unknown 32 – 21 = ?	Bigger Unknown ? – 7 = 9	Smaller Unknown 11 - ? = 7
The pet store is having a sale of 21 hamsters and 32 kittens. How many more kittens are on sale than the hamsters?	Jill is 7 years old. Joe is 9 years older than Jill. How old is Joe?	Steve picked 11 carrots. He picked 7 fewer green peppers than carrots. How many green peppers did Steve pick?
		

Multiplication and Division Situations

Structures

Common	UNKNOWN PRODUCT	GROUP SIZE UNKNOWN ("How many in each group?" – Division)	NUMBER OF GROUPS UNKNOWN ("How many groups?" – Division)
EQUAL GROUPS (3rd Grade)	There are 3 bags with 6 plums in each bag. How many plums are there in all?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag?	If 18 plums are to be packed 6 to a bag, then how many bags are needed?
ARRAYS (3rd Grade)	There are 3 rows of apples with 6 apples in each row. How many apples are there?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be?
COMPARE (4th & 5th Grade)	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat?
GENERAL	a x b = ?	a x ? = p and p ÷ a = ?	? x b = p and p ÷ b = ?

Multiplication and Division Situations

Structures

Measurement Examples **Measurement**	**UNKNOWN PRODUCT**	**GROUP SIZE UNKNOWN** ("How many in each group?" – Division)	**NUMBER OF GROUPS UNKNOWN** ("How many groups?" – Division)
EQUAL GROUPS **(3rd Grade)**	Groups that have the same number of equivalent items		
AREA **(3rd Grade)**	The space within an rectangular object.		
COMPARE **(4th & 5th Grade)**	Problems that contain two sets of items. One set contains a multiple copy of the other set.		
GENERAL	a x b = ?	a x ? = p and p ÷ a = ?	? x b = p and p ÷ b = ?

Multiplication & Division Word Problem Situations

Common

- Equal Groups
  - Groups that have the same number of equivalent items.
- For example:
  - I have 3 bags with 2 apples in each bag. How many apples do I have in all?

Measurement

- Equal Groups
  - Groups that have the same equivalent measurement.
- For example:
  - I need 3 strings 12 inches long for my project. How long of a string do I need to buy?

Multiplication & Division Word Problem Situations

Common

- Arrays
  - An arrangement of objects in equal rows.
- For example:
  - I planted 4 rows of 6 tomatoes. How many tomatoes did I plant in all?

Measurement

- Area
  - The space occupied by a rectangular object.
- For example:
  - My garden is 4 feet by 6 feet. How much space is my garden?

Multiplication & Division Word Problem Situations

Common

- Compare
  - Problems that contain two sets of items. One set contains a multiple copy of the other set.
- For example
  - Jad has 3 times as many pieces of candy than Kim. If Kim has 4 pieces of candy, how many pieces does Jad have?

Measurement

- Compare
  - Problems that contain two sets of items. One set contains a multiple measurement of the other set.
- For example
  - When the rubber band is stretched , it is 3 times as long as it was a rest. If the rubber band is 4 inches long at rest, how long is it stretched?

CSA Strategies

Concrete; Semi-Concrete; Abstract

CONCRETE: Students manipulate hands on, concrete material

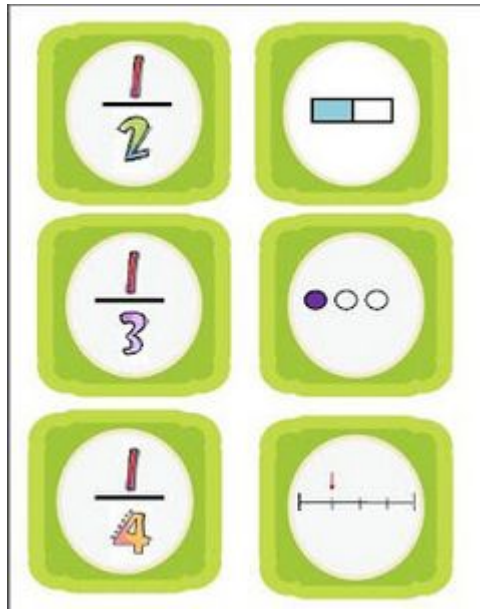
* Fraction Cubes/Cuisenaire Rods
* Base-10 Blocks
* Pattern Blocks
* Wiki Sticks
* Equation Balances

CSA STRATEGIES

CONCRETE; SEMI-CONCRETE; ABSTRACT

SEMI-CONCRETE: Students draw and create diagrams

- *Visual Representations
- *Tables
- *Graphs
- *Number Lines
- *Pictures



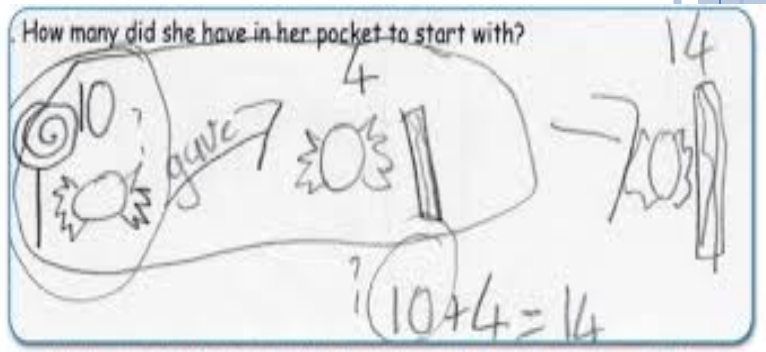


Figure 3. Visual representation of Question 6 in post-test. The student represents the important information and is able to write a matching number sentence. See Appendix A for details of the questions.

CSA Strategies

Concrete; Semi-Concrete; Abstract

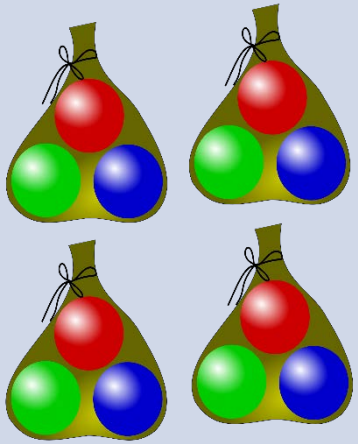
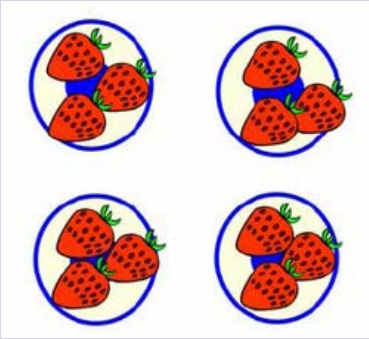
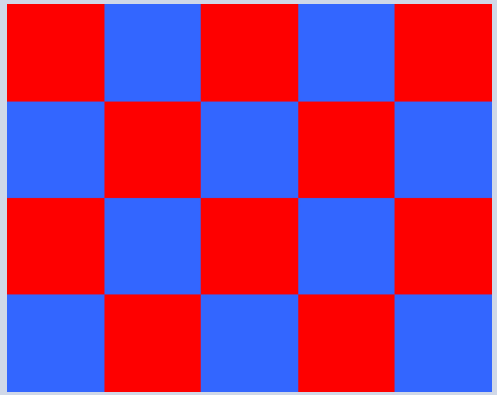
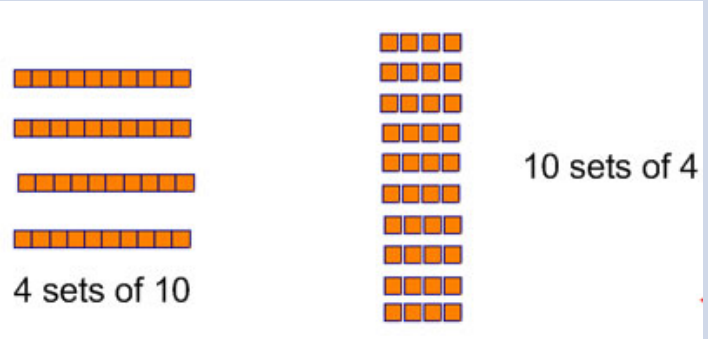
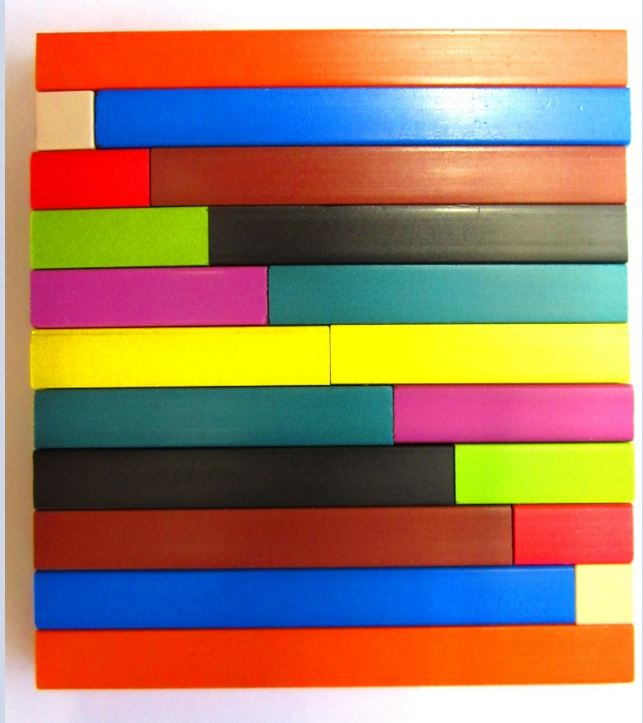
ABSTRACT: Students use numbers and mathematical symbols

* Formulas
* Equations
* Symbols
* Algorithms

x 4 Patterns				
4	8	12	16	20
24	28	32	36	40

8 x 5	45 ÷ 5
(4 x 2) x 5	(50-5) ÷ 5
4 x (2 x 5)	(50÷5) - (5÷5)
4 x 10	10-1
40	9

CSA STRATEGIES CONCRETE

Equal Groups	Arrays / Area	Compare
 	 	

CSA Strategies: Semi-Concrete

Types	
Groups	**Groups** Number of Groups: ___ X Number in each Group: ___ = Product: ___
Array	**Arrays** Number of Rows: ___ X Number in each row: ___ = Product: ___
Compare	**Compare** Product: ___ = Multiplier: ___ X Reference Set: ___

CSA Strategies: Abstract

Types	
Groups	3 baskets x 10 apples = 30 apples in all 3 x 10 = 30
Array	4 rows x 6 carrots = 24 carrots in all 4 x 6 = 24
Compare	28 is 4 times as much as 7 28 = 4 x 7

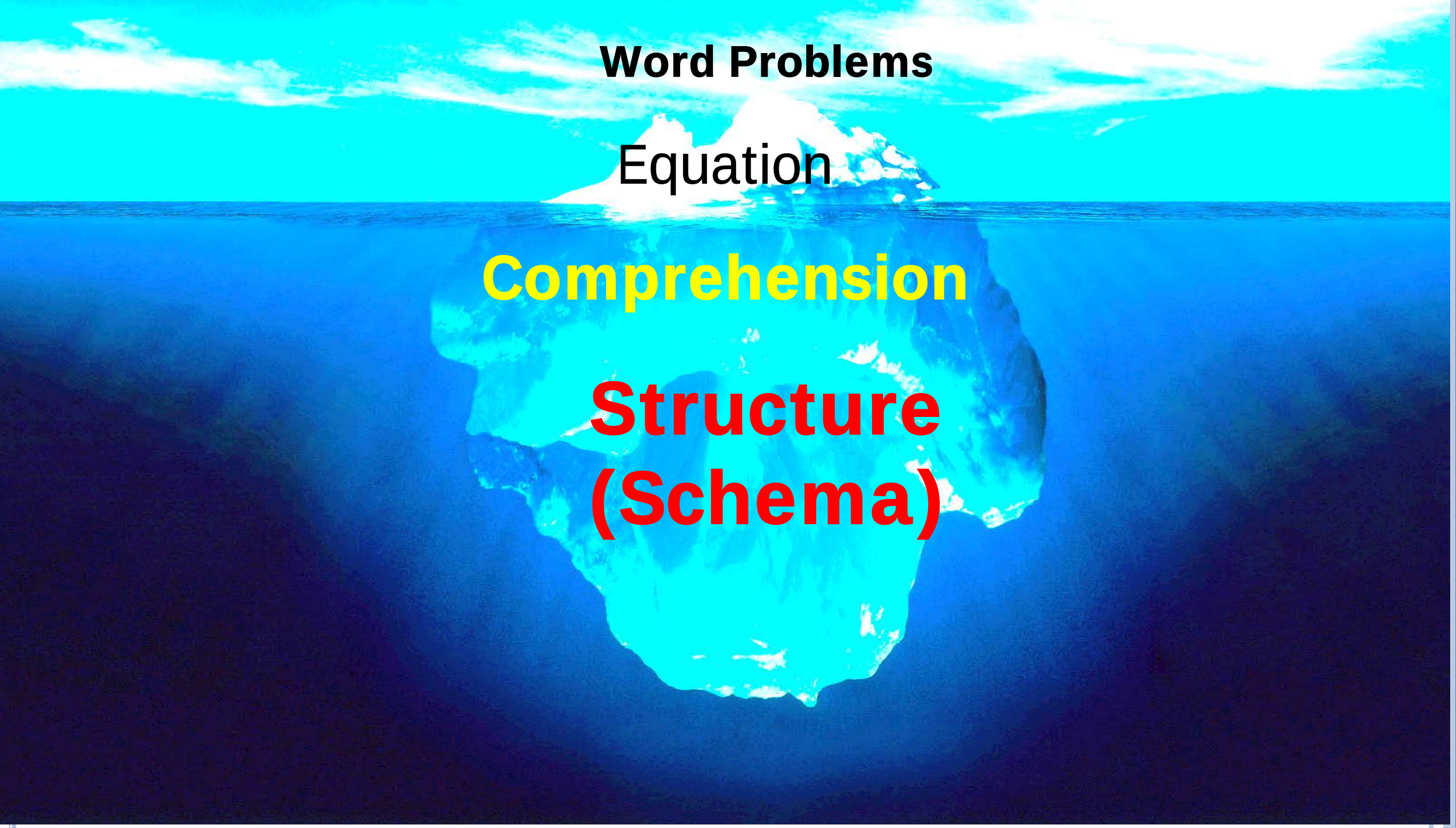
Schema-Based Instruction vs. General Strategy Instruction

Schema-Based Instruction	General Strategy Instruction
• Read to understand and identify the problem type	• Read to understand
• Draw and use the schema diagram to represent the problem	• Draw a picture / visual to represent the problem
• Transform the diagram to a math sentence and solve the problem	• Solve the problem
• Look back to check	• Look back to check

Schematic Based Instruction

- A Schema is a framework, outline or structure.
- Students are taught – Clearly and explicitly
  - To recognize problems as having a specific structure
  - To use specific graphic organizers (semi-concrete) to represent the problem
  - To apply a solution method that matches that problem type

Jitendra,A. *Meeting the needs of students with learning disabilities: the role of schema-based instruction*. NCTM 2008 Annual Convention


Word Problems
Equation
Comprehension
Structure
(Schema)