NCTM Nashville 2015
Presentation 130
Slices of Calculus
Volumes of Known Cross Sections Models

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- Starting with Triangles, Squares, and Circles, fit shapes inside other shapes and compare relative Side Lengths and Areas
- Find the Volume of a solid with a SQUARE on SIDE cross section
- Compare that $\text{Volume}_{\text{square}}$ with Volumes with
  - an EQUILATERAL TRIANGLE cross section
  - an ISOSCELES RIGHT TRIANGLE on LEG cross section
  - a SEMICIRCLE cross section
  - a RECTANGLE with height h cross section
- Discuss Volumes with variations on these basic cross sections, such as
  - a SQUARE on diagonal cross section
  - a CIRCLE cross section
  - HORIZONTAL versus VERTICAL slices
- Build Volumes of Known Cross Section Model

Question:
How do the volumes of different known cross sections compare?

The ratio of Areas of an Equilateral Triangle to a Square is $\frac{\sqrt{3}}{4}:1$

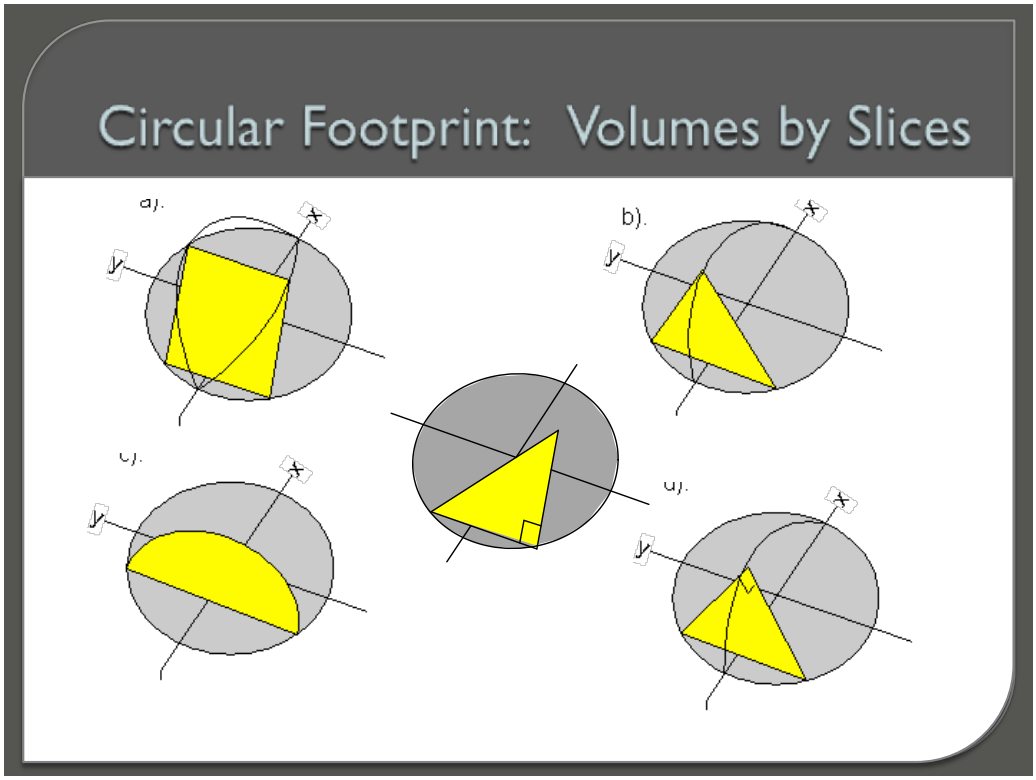
The ratio of Areas of an Isosceles Right Triangle on a leg to a Square is $\frac{1}{2}:1$

The ratio of Areas of an Isosceles Rt Triangle on its hypotenuse to a Sq is $\frac{1}{4}:1$

The ratio of Areas of a Circle to a Square is $\frac{\pi}{4}:1$

The ratio of Areas of a Semicircle to a Square is $\frac{\pi}{8}:1$

Question:
How do the Volumes of Known Cross Sections on a circular 'footprint' compare?

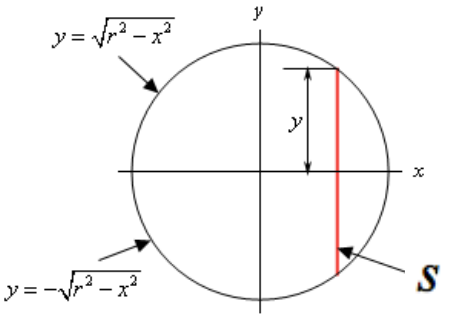


Given a 'footprint' designated by the area between the curves,

$\sqrt{1-x^2}$ and $-\sqrt{1-x^2}$

that intersect the x-axis at,

$x=\pm 1$



Side of a square is

$$s=\sqrt{1-x^2}-\left(-\sqrt{1-x^2}\right)=2y=2\sqrt{1-x^2}$$

Answer:

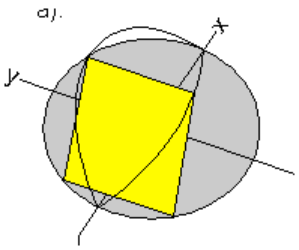
1. Volume of a solid with SQUARE ON SIDE cross sections

$$V_{square} = \int_a^b s^2 dx$$

$$V_{square} = \int_{-1}^{1} \left(2\sqrt{1-x^2}\right)^2 dx = 4\int_{-1}^{1} (1-x^2)dx$$

$$= \left(4x - \frac{4}{3}x^3\right)\Big|_{-1}^{1} = \left(4 - \frac{4}{3}\right) - \left(-4 + \frac{4}{3}\right) = 8 - \frac{8}{3}$$

$$= \frac{16}{3}$$

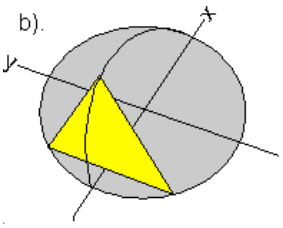


2. Volume of a solid with EQUILATERAL TRIANGLE cross sections

$$V_{equilateral\,triangle} = \frac{\sqrt{3}}{4} V_{square} = \frac{\sqrt{3}}{4}\int_a^b s^2 dx$$

$$V_{equilateral\,triangle} = \frac{\sqrt{3}}{4}\int_{-1}^{1} \left(2\sqrt{1-x^2}\right)^2 dx$$

$$= \frac{\sqrt{3}}{4}\cdot\frac{16}{3} \qquad = \frac{4\sqrt{3}}{3}$$

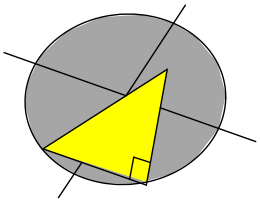


3. Volume of a solid with ISOSCELES RIGHT TRIANGLE ON LEG cross sections

$$V_{isosceles\,right\,triangle\,leg} = \frac{1}{2} V_{square} = \frac{1}{2}\int_a^b s^2 dx$$

$$V_{isosceles\,right\,triangle\,leg} = \frac{1}{2}\int_{-1}^{1} \left(2\sqrt{1-x^2}\right)^2 dx$$

$$= \frac{1}{2}\cdot\frac{16}{3} \qquad = \frac{8}{3}$$

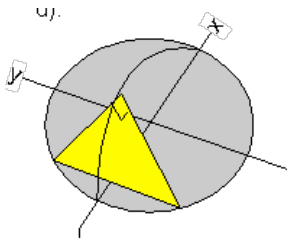


4. Volume of a solid with ISOSCELES RIGHT TRIANGLE ON HYPOTENUSE cross sections

$$V_{isosceles\,right\,triangle\,hyp} = \frac{1}{4} V_{square} = \frac{1}{4}\int_a^b s^2 dx$$

$$V_{isosceles\,right\,triangle\,hyp} = \frac{1}{4}\int_{-1}^{1} \left(2\sqrt{1-x^2}\right)^2 dx$$

$$= \frac{1}{4}\cdot\frac{16}{3} \qquad = \frac{4}{3}$$

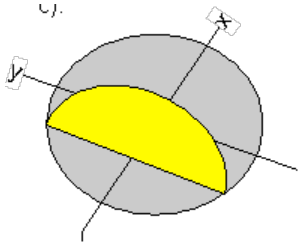


5. Volume of a solid with SEMICIRCLE cross sections

$$V_{semicircle} = \frac{\pi}{8} V_{square} = \frac{\pi}{8}\int_a^b s^2 dx$$

$$V_{semicircle} = \frac{\pi}{8}\int_{-1}^{1} \left(2\sqrt{1-x^2}\right)^2 dx$$

$$= \frac{\pi}{8}\cdot\frac{16}{3} \qquad = \frac{2\pi}{3}$$

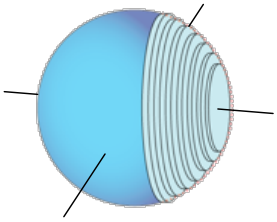


6. Volume of a solid with CIRCLE cross sections

$$V_{circle} = \frac{\pi}{4} V_{square} = \frac{\pi}{4}\int_a^b s^2 dx$$

$$V_{circle} = \frac{\pi}{4}\int_{-1}^{1} \left(2\sqrt{1-x^2}\right)^2 dx$$

$$= \frac{\pi}{4}\cdot\frac{16}{3} \qquad = \frac{4\pi}{3}$$



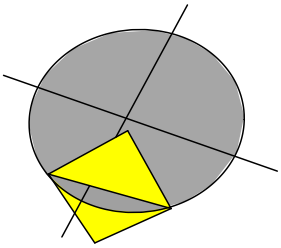
7. Volume of a solid with SQUARES ON the DIAGONAL cross sections

$$V_{square\,on\,diagonal} = \frac{1}{2} V_{square} = \frac{1}{2}\int_a^b s^2 dx$$

$$V_{square\,on\,diagonal} = \frac{1}{2}\int_{-1}^{1} \left(2\sqrt{1-x^2}\right)^2 dx$$

$$= \frac{1}{2}\cdot\frac{16}{3} \qquad = \frac{8}{3}$$

$$= 2V_{isosceles\,right\,triangle\,on\,hyp}$$

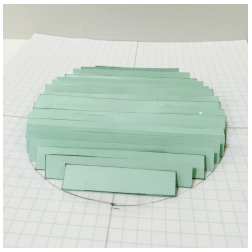


8. Volume of a solid with RECTANGLE of HEIGHT h cross sections

$$V_{rectangle} = \int_a^b s \cdot h \, dx$$

$$V_{rectangle} = \int_{-1}^{1} \left(2\sqrt{1-x^2}\right) \cdot h \, dx$$

$$= \pi h$$



9. Volume of a solid with RECTANGLE of HEIGHT h cross sections perpendicular to the y-axis instead (so horizontal slices)

Will the shape be different?

Will the volume change?

$$V_{rectangle} = \int_{-1}^{1} \left(2\sqrt{1-y^2}\right) \cdot h \, dy$$

What if the footprint was not symmetric?

Summary

Comparison of Volumes:

$$V_{square} = \int_a^b s^2 dx$$

- $V_{equilateral\ triangle} = \frac{\sqrt{3}}{4} V_{square}$
- $V_{isosceles\ right\ triangle\ leg} = \frac{1}{2} V_{square}$
- $V_{isosceles\ right\ triangle\ hyp} = \frac{1}{4} V_{square}$
- $V_{semicircle} = \frac{\pi}{8} V_{square}$
- $V_{circle} = \frac{\pi}{4} V_{square}$
- $V_{square\ on\ diagonal} = \frac{1}{2} V_{square}$