WORKING WITH LINEAR EQUATIONS: A DIFFERENT APPROACH

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CLIFTON WINGARD
PROFESSOR OF MATHEMATICS
DELTA STATE UNIVERSITY
CLEVELAND, MISSISSIPPI
cwingard@deltastate.edu

MICAH HICKMAN
MATHEMATICS TEACHER
MANDEVILLE HIGH SCHOOL
MANDEVILLE, LOUISIANA

The Box Method

The procedure outlined here for finding the equation of a line eliminates some of the manipulations that the students find difficult. The entire procedure is based on the definition of the slope of a line and also based on the point-slope formula for linear equations. However, neither of these formulas is directly seen.

Example 1: Given the two points on a line, write the equation of the line in standard form.

(-3, 4) and (5, 7)

Solution:

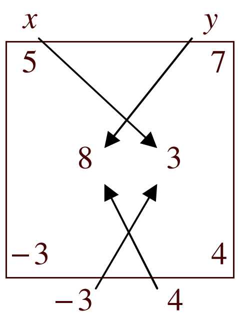
Step 1: Create an array as shown by entering the two ordered pairs in the box with the second one also written below the box. (It is helpful if the pair with the larger y-value is put on top, but it is not necessary.)

x			*y*
5			7
−3			4
	−3	4	

Step 2: Subtract the numbers in the columns in the box and enter those numbers in the box.

x			*y*
5			7
	8	3	
−3			4
	−3	4	

Step 3: Multiply diagonally as demonstrated below to yield the equation. It is important to note that the terms are subtracted on each side of the equation. Because of this, the order is important.



$3x - 8y = -3(3) - 4(8)$

Step 4: Perform any necessary arithmetic to simplify terms, and the result will be the equation of the line in standard form.

$3x - 8y = -9 - 32$
$3x - 8y = -41$

The process also works when finding equations of vertical lines which have no slope.

The Proportion Method

The reasoning behind the method: The slope of a line is a ratio of the change in the y-value to the change in the x-value. A proportion is created, and the usual method of solving a proportion by cross multiplication is used. (It should be noted that the methods demonstrated here do not work for finding the equation of a vertical line.)

Example 2: Find the equation of the line that passes through the points (3, 4) and (6, -4).

Solution: We set up a proportion using the slope formula using both points $\left(m = \frac{y_2 - y_1}{x_2 - x_1} \right)$ and then again using an arbitrary point (x, y) and one of the ordered pairs.

$$\frac{y-4}{x-3} = \frac{-4-4}{6-3}$$

$$\frac{y-4}{x-3} = \frac{-8}{3}$$

Then cross-multiply to yield

$$3(y - 4) = -8(x - 3)$$
$$3y - 12 = -8x + 24$$

The equation can be manipulated to give standard form $8x + 3y = 36$ or slope-intercept form

$$3y = -8x + 36$$
$$y = -\frac{8}{3}x + 12$$

Example 3: Write the equation of the line with slope $m = \frac{4}{5}$ and that contains the point (-3, 7).

Solution: Set up the proportion using the given slope ratio and the slope using the arbitrary point (x, y) and the given ordered pair.

$$\frac{y-7}{x-(-3)} = \frac{4}{5}$$

$$\frac{y-7}{x+3} = \frac{4}{5}$$

Cross-multiplication yields

$$5(y - 7) = 4(x + 3)$$
$$5y - 35 = 4x + 12$$

The equation can be manipulated to give standard form $4x - 5y = -47$ or slope-intercept form

$$5y = 4x + 47$$
$$y = \frac{4}{5}x + \frac{47}{5}$$

Alternative Method

The method demonstrated here is related to the method demonstrated in the previous two examples. Here, the idea that the slope is a ratio of the change in y to the change in x is used. An equation in standard form is generated, and substitution into the equation yields the constant C. A common formula exhibited in textbooks shows that the slope of a line $Ax + By = C$ is $m = -\frac{A}{B}$. This method uses a proportion to "undo" this formula to obtain the coefficients of *x* and *y* in the appropriate places.

Example 4: Find the equation of the line with slope $m = -\frac{4}{7}$ that contains the point (-5, 8).

Solution: Set up a proportion relating the slope to the basic variables *x* and *y*.

$$\frac{y}{x} = \frac{-4}{7}$$

Cross-multiplication yields

$$7y = -4x$$

Any line with this slope will have an equation of the form $4x + 7y = C$.

By substitution of the ordered pair into this equation for *x* and *y* will yield the value of the constant C.

$$4(-5) + 7(8) = C$$
$$-20 + 56 = C$$
$$36 = C$$

The equation of the line is $4x + 7y = 36$ which can then be put in slope-intercept form as needed.

$$7y = -4x + 36$$
$$y = -\frac{4}{7}x + \frac{36}{7}$$

Solving Systems of Equations

Example 5: Find the solution of the system of equations $\begin{cases} 3x+4y=9 \\ 2x+5y=13 \end{cases}$

Step 1: Isolate the x term in each equation.

$3x = 9 - 4y$
$2x = 13 - 5y$

Step 2: Multiply an equation by the coefficient of x in the other equation.

(multiply by 2) $3x = 9 - 4y \quad \rightarrow \quad 6x = 18 - 8y$
(multiply by 3) $2x = 13 - 5y \quad \rightarrow \quad 6x = 39 - 15y$

Step 3: Set the equations equal to each other.

$18 - 8y = 39 - 15y$

Step 4: Solve for y.

$7y = 21$
$y = 3$

Steps 5 – 8: Repeat the process for the other variable or substitute the value in Step 4 into one of the equations to find the value of the remaining variable.

Step 5: $4y = 9 - 3x$
$5y = 13 - 2x$

Step 6: (multiply by 5) $4y = 9 - 3x \quad \rightarrow \quad 20y = 45 - 15x$
(multiply by 4) $5y = 13 - 2x \quad \rightarrow \quad 20y = 52 - 8x$

Step 7: $45 - 15x = 52 - 8x$

Step 8: $-7x = 7$
$x = -1$

The solution of the system of equations is the ordered pair (-1, 3). Because the two lines intersect at a single point, the system is described as *consistent and independent*.

In the case of parallel lines, there is no solution. The system is described as *inconsistent*. The above process develops as follows.

Example 6: Find the solution of the system of equations $\begin{cases} 3x - 2y = 6 \\ 6x - 4y = 24 \end{cases}$

Step 1: Solving for x yields the following:

$3x = 6 + 2y$
$6x = 24 + 4y$

Step 2: (multiply by 6) $3x = 6 + 2y \rightarrow 18x = 36 + 12y$
(multiply by 3) $6x = 24 + 4y \rightarrow 18x = 72 + 12y$

Step 3: $36 + 12y = 72 + 12y$

Step 4: $36 = 72$
Because this is a false statement, we know that the system has no solution.

For the case of coinciding lines, there will be an infinite number of solutions. Here, there is not a unique solution, and it is said that the system is *consistent and dependent*.

Example 7: Find the solution of the system of equations $\begin{cases} 2x + 5y = 21 \\ y = -0.4x + 4.2 \end{cases}$

Step 1: Solving for x yields the following:

$2x = 21 - 5y$
$-0.4x = y - 4.2$

Step 2: (multiply by –0.4) $2x = 21 - 5y \rightarrow -0.8x = -8.4 + 2.0y$
(multiply by 2) $-0.4x = y - 4.2 \rightarrow -0.8x = 2y - 8.4$

Step 3: $-8.4 + 2.0y = 2y - 8.4$

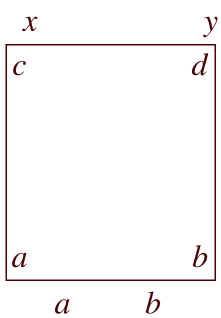
Step 4: $0 = 0$
Because this is a true statement, we know that the system has an infinite number of solutions. The solutions may be represented as points of the form $(x, -0.4x + 4.2)$ where the y-coordinate is the expression that results from solving one of the equations for y.

Justification of Writing the Equation of a Line Using the Box Method

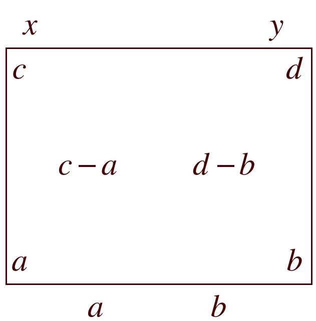
In general, the student is asked to write the equation of a line passing through two points. In this justification, we will perform the manipulations using symbols and show that it is equivalent to the traditional method of using the point-slope formula.

Find the equation of the line that passes through the points (a, b) and (c, d).

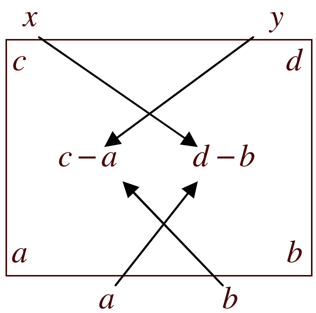
Step 1: Create an array as shown by entering the two ordered pairs in the box with the second one also written below the box. (It is helpful if the pair with the larger *y*-value is put on top, but it is not necessary.)



Step 2: Subtract the numbers in the columns in the box and enter those numbers in the box.



Step 3: Multiply diagonally as demonstrated below to yield the equation. It is important to note that the terms are subtracted on each side of the equation. Because of this, the order is important.



$$(d - b)x - (c - a)y = a(d - b) - b(c - a)$$

Following this step, we will show that this is equivalent to the point-slope formulation of the line.

$$(d - b)x - a(d - b) = (c - a)y - b(c - a)$$

$$(x - a)(d - b) = (y - b)(c - a)$$

$\left(y-b\right)=\dfrac{\left(d-b\right)}{\left(c-a\right)}\left(x-a\right)$ where the slope of the line is $m=\dfrac{\left(d-b\right)}{\left(c-a\right)}$.