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The Mathematics of Sustainability

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The NC School of Science and Mathematics

NCTM Annual Conference

San Francisco, CA

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Chincoteague Bay
Field Station

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About North Carolina School of Science and Mathematics

NCSSM is the nation's first public residential high school focused on science, technology, engineering and math. Through a residential campus, extensive online offerings, and summer STEM enrichment programs, we challenge and inspire talented students from across the state.

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What is Math Modeling?

"Whether the problem is huge or little, the process of 'interaction' between the mathematics and real world is the same ... This entire process is what is called *mathematical modeling*."

Henry Pollack (2012). *Mathematical Modeling Handbook: Introduction*. COMAP.

Available at http://www.comap.com/modelingHB/

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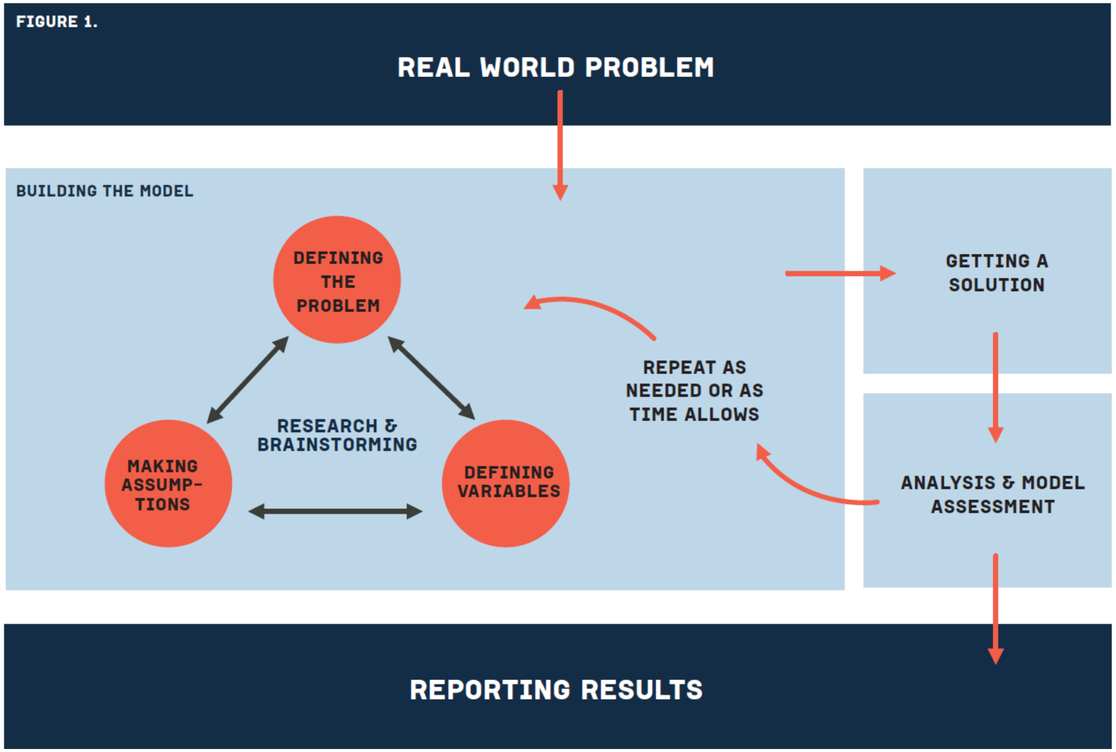
Modeling Cycle



From SIAM *Modeling Getting Started and Getting Solutions Handbook*

https://m3challenge.siam.org/resources/modeling-handbook

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Workshop Components

- Three different activities connecting mathematics and sustainability topics: Biodiversity in ocean populations and practices related to fish populations.
- MAA/PREP Workshop in Chincoteague Bay, VA

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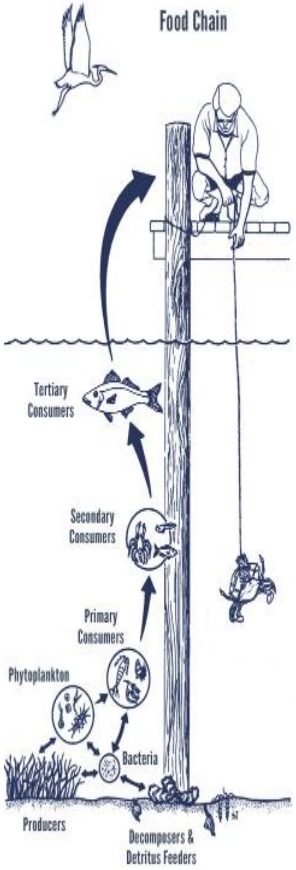
- Creation of lessons that *spawned* the creation of other lessons

All materials are available through the NCTM Conference site and our separate sites.

Who Goes There? Estimating Ocean Populations in Chincoteague Bay: Part I

- Appropriate for middle school or high school pre-algebra and algebra classes

- Focus is on quantitative reasoning (ratios and proportions) with algebraic extensions



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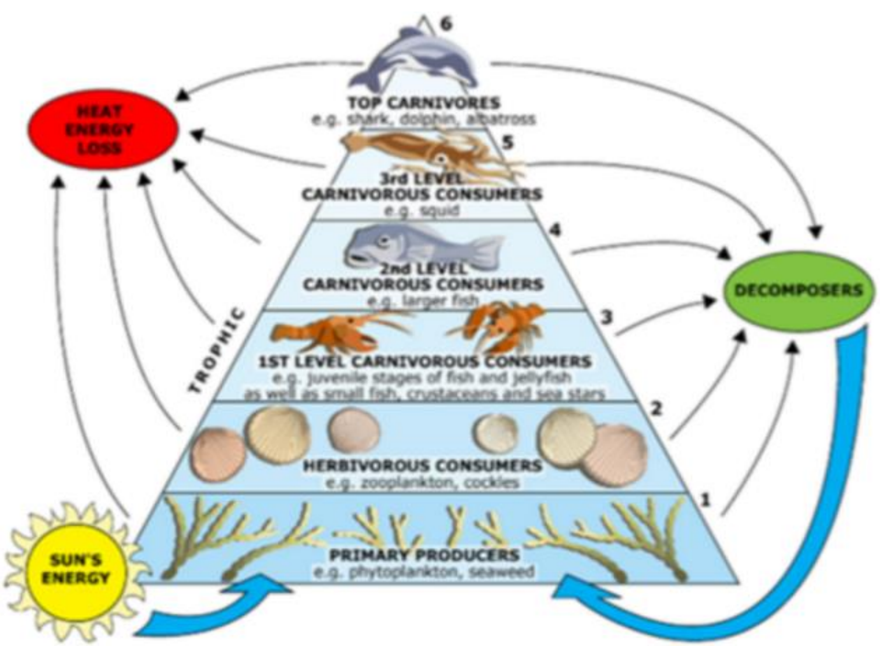
Part I

- Using data collected on the trawl, :
  - Classify organisms by trophic levels in the food
  - Calculate ratios of numbers of organisms in adjacent trophic levels as a measure of energy flow in the ecosystem.



Common Name	Number Collected	Trophic Level
Hermit crab	137	
Caprella	75	
Sand Shrimp A	40	
Bay Anchovy	34	
Sand Shrimp B	29	
Lady crab	6	
Mud crab	6	
Decorator crab	4	
Spotted Hake	4	
Snails	4	
Smooth Flounder	3	
Spider crab	3	
Sea Spider	3	
Purple Sea Urchin	2	
Sea Cucumber	2	
Caprella	2	
Squid	1	
Spiny Sea Urchin	1	
Starfish	1	
Dusky Pipefish	1	
Green Grass Shrimp	1	
Jonah Crab	1	
Mantis Shrimp	1	
Clam worm	1	
Common Shore Shrimp	1	
Caprella with Brood	1	

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Who Goes There? Estimating Ocean Populations in Chincoteague Bay: Part II

Extension Activity: Measuring Biodiversity in the Bay

Students explored Simpson's Diversity Index

Consider a sample collection of species in a body of water. Let ***S*** be the total number of species.

For a particular species, we can calculate the proportion of the population that that species represents. Let that proportion be ***p***.

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Simpson's Biodiversity Index

Suppose we have 50 species in our collection and we have 5 crabs.

Then the proportion of crabs in our collection is 5/50 =1/10 = 0.1.

Let p_i be the proportion of the *i*th population, then Simpson's Diversity index, D, is **defined** to be

$$D = {p_1}^2 + {p_2}^2 + {p_3}^2 + \ldots + {p_s}^2$$

Or in summation notation

$$D = \sum_{i=1}^{s} {p_i}^2$$

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What Questions Do you Have?

$$D = \sum_{i=1}^{s} p_i^{\,2}$$

- Creating examples for a sample with 100 organisms and 5 species, students calculated Simpson's Index.
- Designed to help students think about the possible values for D and make sense of these values in context.

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Some Sample Populations

Group 1	Group 2	Group 3	Group 4
# Species 1 = 20	# Species 1 = 5	# Species 1 = 15	# Species 1 = 13
# Species 2 = 20	# Species 2 = 5	# Species 2 = 15	# Species 2 = 17
# Species 3 = 20	# Species 3 = 5	# Species 3 = 15	# Species 3 = 38
# Species 4 = 20	# Species 4 = 5	# Species 4 = 15	# Species 4 = 21
# Species 5 = 20	# Species 5 = 80	# Species 5 = 40	# Species 5 = 11

$$D = \sum_{i=1}^{s} p_i^{\,2}$$

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Sample Calculations of Simpson Index

- Suppose we have 100 organisms from 5 different species. Let's calculate a sample Simpson index.

Group 1	Group 2	Group 3	Group 4
# Species 1 = 20	# Species 1 = 5	# Species 1 = 15	# Species 1 = 13
# Species 2 = 20	# Species 2 = 5	# Species 2 = 15	# Species 2 = 17
# Species 3 = 20	# Species 3 = 5	# Species 3 = 15	# Species 3 = 38
# Species 4 = 20	# Species 4 = 5	# Species 4 = 15	# Species 4 = 21
# Species 5 = 20	# Species 5 = 80	# Species 5 = 40	# Species 5 = 11
Simpson's Index = 0.2	Simpson's Index = 0.65	Simpson's Index = 0.25	Simpson's Index = 0.2464

For Group 4, $\sum_{i=1}^{5} p_i^2 = \left(\frac{13}{100}\right)^2 + \left(\frac{17}{100}\right)^2 + \left(\frac{38}{100}\right)^2 + \left(\frac{21}{100}\right)^2 + \left(\frac{11}{100}\right)^2 = 0.2464$

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Spreadsheet w/Scenarios

Can explore "best" case and "worst" case for a given number of species

Or change the number of species then compare across the samples.

5 Species, 100 samples		
Sample 1	Prop	Prop^2
20	0.2	0.04
20	0.2	0.04
20	0.2	0.04
20	0.2	0.04
20	0.2	0.04
Index		0.2

5 Species, 100 Samples		
Sample 5	Prop	Prop^2
1	0.01	0.0001
1	0.01	0.0001
1	0.01	0.0001
1	0.01	0.0001
96	0.96	0.9216
		0.922

10 Species, 100 Sample		
Sample 1		
10	0.1	0.01
10	0.1	0.01
10	0.1	0.01
10	0.1	0.01
10	0.1	0.01
10	0.1	0.01
10	0.1	0.01
10	0.1	0.01
10	0.1	0.01
10	0.1	0.01
Index		0.1

10 Species, 100 Samples		
Sample 5		
1	0.01	0.0001
1	0.01	0.0001
1	0.01	0.0001
1	0.01	0.0001
1	0.01	0.0001
1	0.01	0.0001
1	0.01	0.0001
1	0.01	0.0001
1	0.01	0.0001
91	0.91	0.8281
		0.829

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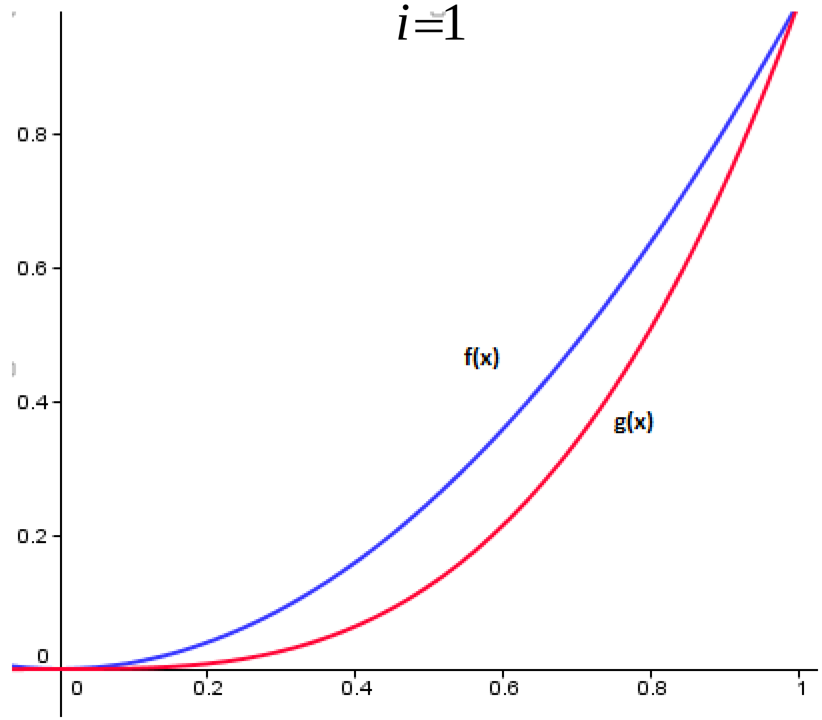
Why square the proportions?

$$D = \sum_{i=1}^{s} p_i^{\,2}$$

- Students are ask to think about why the proportions are squared before summing.
- Consider alternative of summing other powers of *p*.
- This leads to a discussion of the behavior of power functions on the interval [0,1].



$f(x) = x^2$ and $g(x) = x^3$

Data from Our Trawl

Students used the sample data from our trawl to calculate Simpson's Index and made some observations about the biodiversity of the bay.

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Simpson Index from our Data

We had 26 species and 324 organisms in our trawl sample.

Simpson Index Calculation:

$$\sum_{i=1}^{26} p_i^2 = \left(\frac{137}{364}\right)^2 + \left(\frac{75}{364}\right)^2 + \left(\frac{40}{364}\right)^2 + \cdots + \left(\frac{1}{364}\right)^2 = 0.2122$$

Common Name	Number Collected
Hermit crab	137
Caprella	75
Sand Shrimp A	40
Bay Anchovy	34
Sand Shrimp B	29
Lady crab	6
Mud crab	6
Decorator crab	4
Spotted Hake	4
Snails	4
Smooth Flounder	3
Spider crab	3
Sea Spider	3
Purple Sea Urchin	2
Sea Cucumber	2
Caprella	2
Squid	1
Spiny Sea Urchin	1
Starfish	1
Dusky Pipefish	1
Green Grass Shrimp	1
Jonah Crab	1
Mantis Shrimp	1
Clam worm	1
Common Shore Shrimp	1
Caprella with Brood	1

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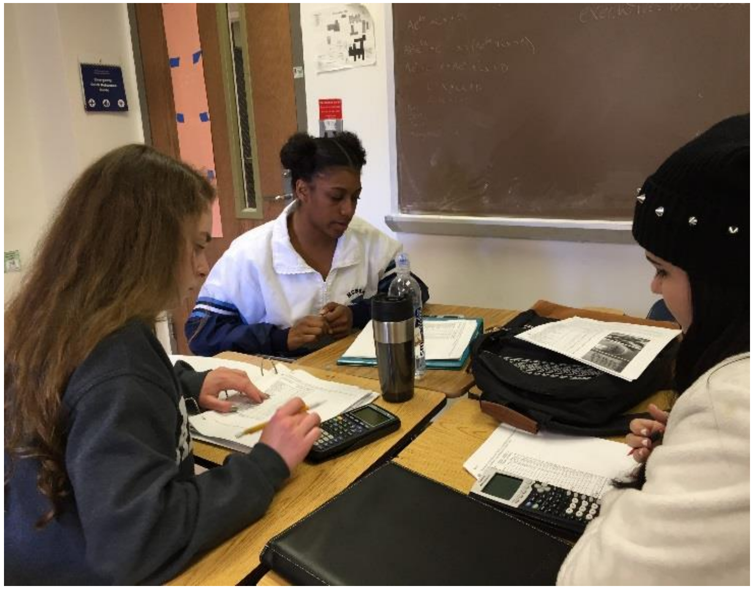
Student-Generated Extension

- Several groups were interested in determining the highest and lowest possible scores on the Simpson Index given the number of organisms from the trawl in the Chincoteague Bay.
- They calculated these indices and used them as baseline values to determine the amount of biodiversity in the bay.
- “The Simpson’s index of the Chincoteague Bay was calculated at .2122. We also calculated the minimum and the maximum indices given the types of organisms collected. The minimum was .0385 (the most diverse), and the maximum was .8675 (the least diverse). Given this range, we concluded that the life in the Chincoteague Bay is quite diverse.”

Writing Assignment and Reflections

"In the future, we believe it would be interesting to discuss what the biodiversity index means qualitatively. We know how to calculate it, and that a value closer to zero means that the ecosystem is more diverse, and a value closer to one means that it is less diverse. However, we never talked about what is considered a 'good' biodiversity index or if there is such a thing as a 'good' biodiversity."



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Student Suggestions for Activity

- Collaborating with other pods/reading their reports to see how they thought about and worked through this problem
- Collecting data ourselves (such as organisms from the bio pond) to create a more personal experience
- Comparing diversity levels in different bodies of water from different areas or comparing diversity levels over a span of a few years to see if it is getting worse or better”

Other Data Analysis Explorations & Collaboration

- Exploring more biodiversity topics with Biology and Ecology teachers at your school and local colleges and universities. SISL site from MAA/PREP http://serc.carleton.edu/sisl/index.html

- Exploring other measures of biodiversity: example

  Shannon-Wiener Index.

$$H = -\sum_{i=1}^{k} p_i \log p_i$$

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Weighing Your Fish With a Ruler

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Why Weigh a Fish With a Ruler?

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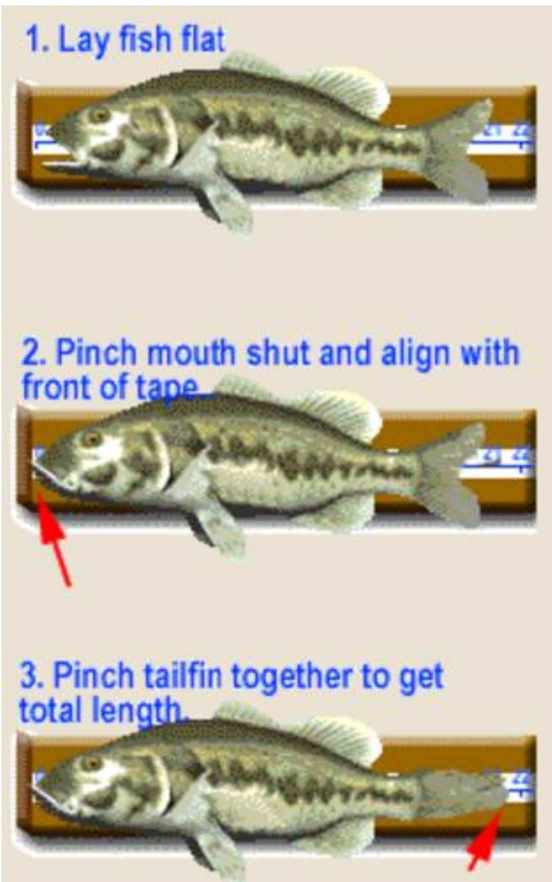
How to Measure

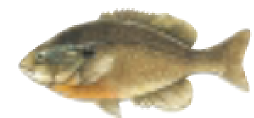


Pumpkinseed Sunfish

Inches	Weight
4	0.1
4.5	0.1
5	0.1
5.5	0.1
6	0.1
6.5	0.2
7	0.2
7.5	0.3
8	0.3
8.5	0.4
9	0.5
9.5	0.6
10	0.8

Bluegill

Inches	Weight
4	0.1
4.5	0.1
5	0.1
5.5	0.1
6	0.1
6.5	0.1
7	0.2
7.5	0.2
8	0.3
8.5	0.4
9	0.5
9.5	0.6
10	0.7
10.5	0.8
11	1.0
11.5	1.1
12	1.3
12.5	1.5
13	1.7



Red-Breasted Sunfish

Inches	Weight
3.5	0.1
4	0.1
4.5	0.1
5	0.1
5.5	0.1
6	0.1
6.5	0.1
7	0.2
7.5	0.2
8	0.3
8.5	0.4
9	0.5
9.5	0.6
10	0.7

Images from:
http://fishandboat.com/images/pages/fishin1/weightlength1.pdf and http://myfwc.com/fishing/freshwater/fishing-tips/measure/

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Data Analysis Explorations & Collaboration

- *Using a Ruler to Weigh Your Fish*

Students predict the weight of a fish based on its length.

Length (in)	Weight (oz)
7	4
7.5	5
8	7
8.5	8
9	10
9.5	12
10	14
10.5	17
11	20
11.5	23
12	26
12.5	30
13	35
13.5	39
14	44

Bluegill from [Lake Lanier](), Landrum, SC. (Caught & Released, June 14, 2004)
Photo from http://en.wikipedia.org/wiki/Bluegill#mediaviewer/File:Bluegill_sc.jpg

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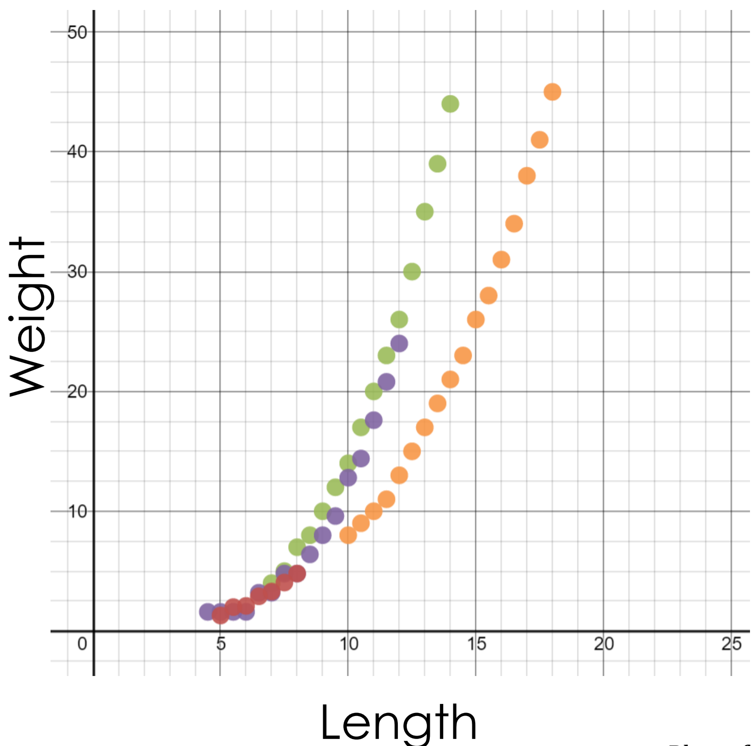

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Photo from http://en.wikipedia.org/wiki/Bluegill#mediaviewer/File:Bluegill_sc.jpg

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Data Analysis

- In order to model the relationship between length and weight, first need a method to linearize the data.
- Techniques used in many statistics courses
  - Great application of inverse functions for algebra and pre-calculus students

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Population Data

- If we assume population grows exponentially:

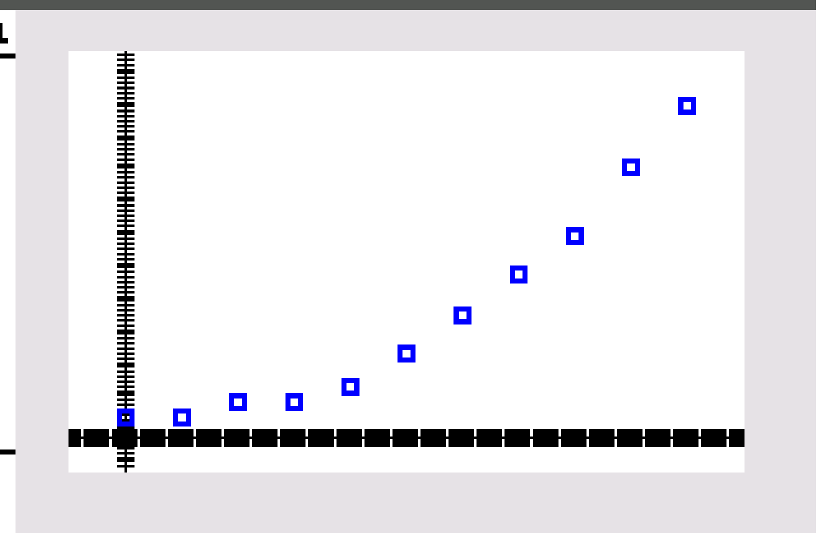
$$P(t) = a \cdot b^t$$

NORMAL FLOAT AUTO REAL RADIAN MP

L1	L2	L3	L4	L5	1
0	4.5	------	------	------	
20	4.3				
40	7.4				
60	7.9				
80	10.9				
100	17.8				
120	26				
140	34.7				
160	42.6				
180	57.5				
200	70.3				

L1(1)=0

NORMAL FLOAT AUTO REAL RADIAN MP



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Linearizing the Model

If:

$$P(t) = a \cdot b^t$$

Then:

$$\ln P = \ln(a \cdot b^t)$$
$$\ln P = \ln a + (\ln b)t$$

$\ln(P)$ is linear in t.

So the ordered pairs $(t, \ln P)$ should be linear.

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POPDATA in Calculator

NORMAL FLOAT AUTO REAL RADIAN MP

t L1	P L2	L3	L4	L5	1
0	4.5	------	------	------	
20	4.3				
40	7.4				
60	7.9				
80	10.9				
100	17.8				
120	26				
140	34.7				
160	42.6				
180	57.5				
200	70.3				

L1(1)=0

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Re-express Data in Lists

NORMAL FLOAT AUTO REAL RADIAN MP

L_1	L_2	L_3	L_4	L_5
0	4.5	------	---	
20	4.3			
40	7.4			
60	7.9			
80	10.9			
100	17.8			
120	26			
140	34.7			
160	42.6			
180	57.5			
200	70.3			

$L_3=\ln(L_2)$

NORMAL FLOAT AUTO REAL RADIAN MP

L_1	L_2	L_3	L_4	L_5
0	4.5	1.5041	------	------
20	4.3	1.4586		
40	7.4	2.0015		
60	7.9	2.0669		
80	10.9	2.3888		
100	17.8	2.8792		
120	26	3.2581		
140	34.7	3.5467		
160	42.6	3.7519		
180	57.5	4.0518		
200	70.3	4.2528		

$L_3(1)=1.504077396763$

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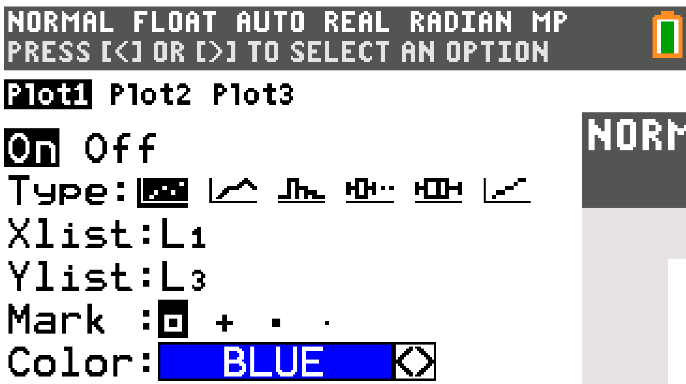

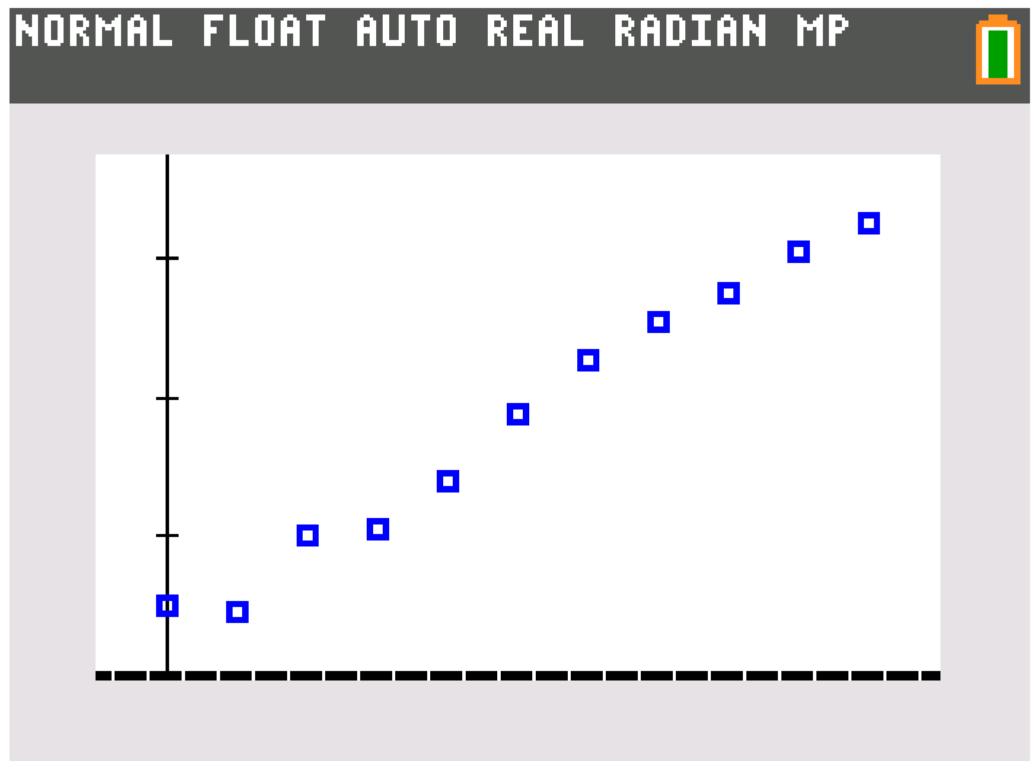

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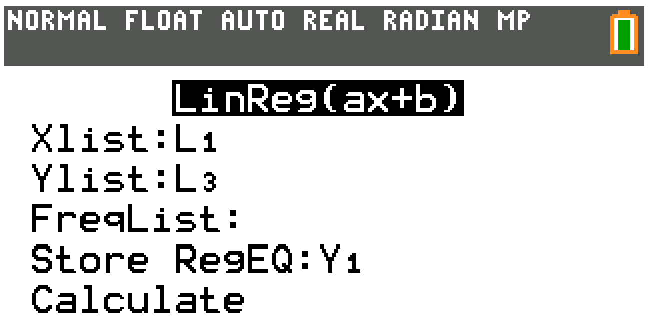
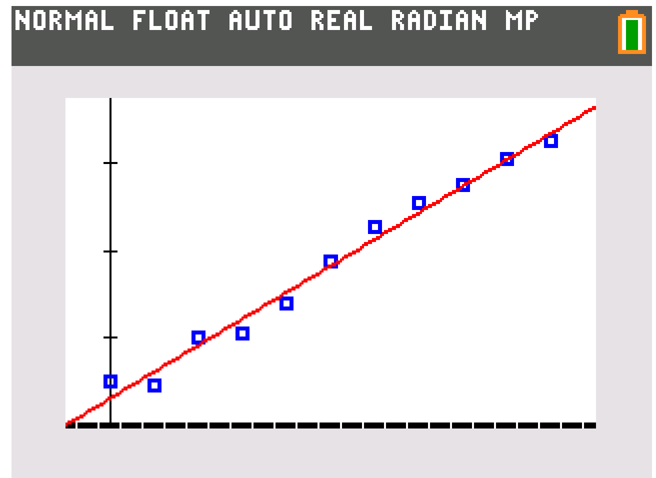
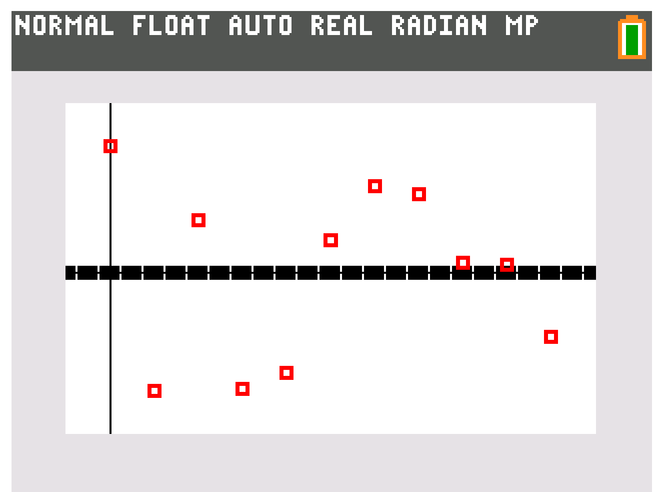




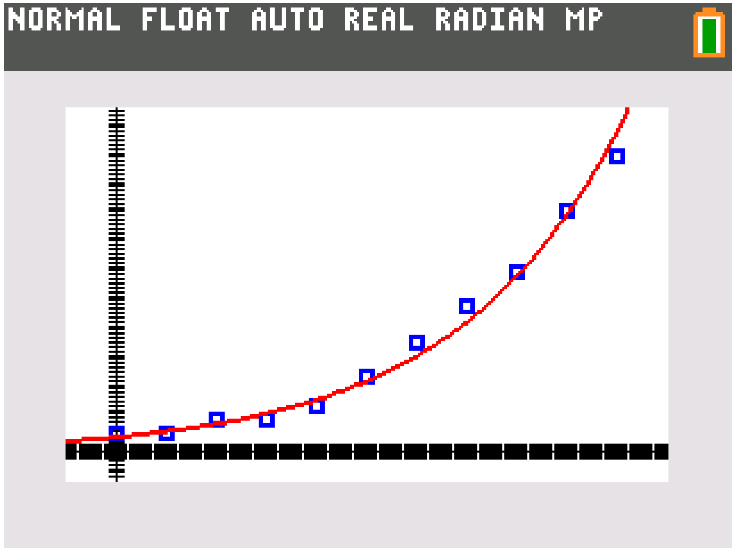


Getting the Model

- Linear Regression on Re-expressed Data:

$$\ln P = 0.015t + 1.134$$
$$P = e^{0.015t+1.134}$$
$$P = e^{1.134}e^{0.015t}$$

$$P(t) = 3.110e^{0.015t}$$



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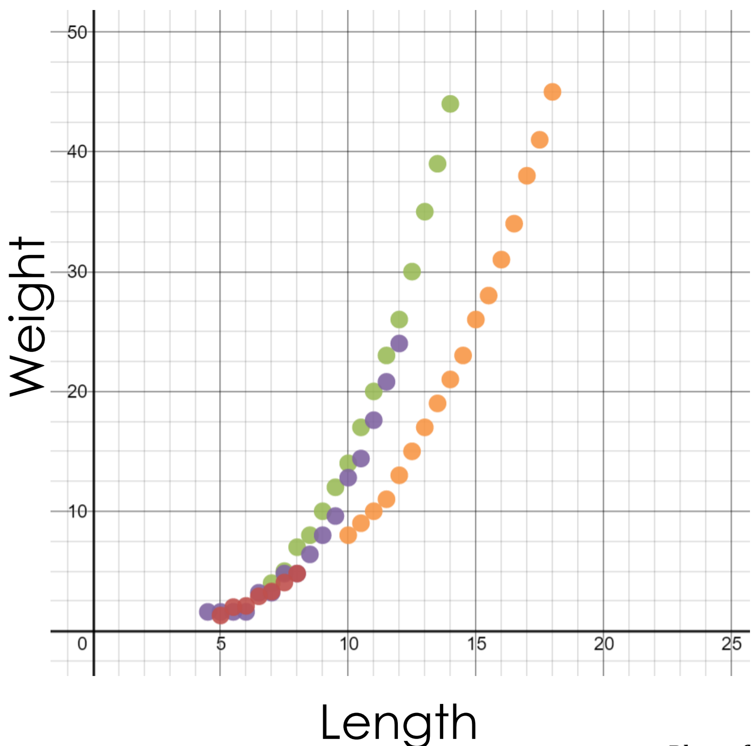
Back to Our Fish

- *Using a Ruler to Weigh Your Fish*

  Students predict the weight of a fish based on its length.



Bluegill from [Lake Lanier](), Landrum, SC. (Caught & Released, June 14, 2004)
Photo from http://en.wikipedia.org/wiki/Bluegill#mediaviewer/File:Bluegill_sc.jpg

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Linearizing the Model

If:

$$W(L) = a \cdot L^b$$

Then:

$$\ln(W) = \ln\left(a \cdot L^b\right)$$
$$\ln(W) = \ln(a) + b \cdot \ln(L)$$

$\ln(W)$ is linear in $\ln(L)$.

So the ordered pairs $(\ln(L), \ln(W))$ should be linear.

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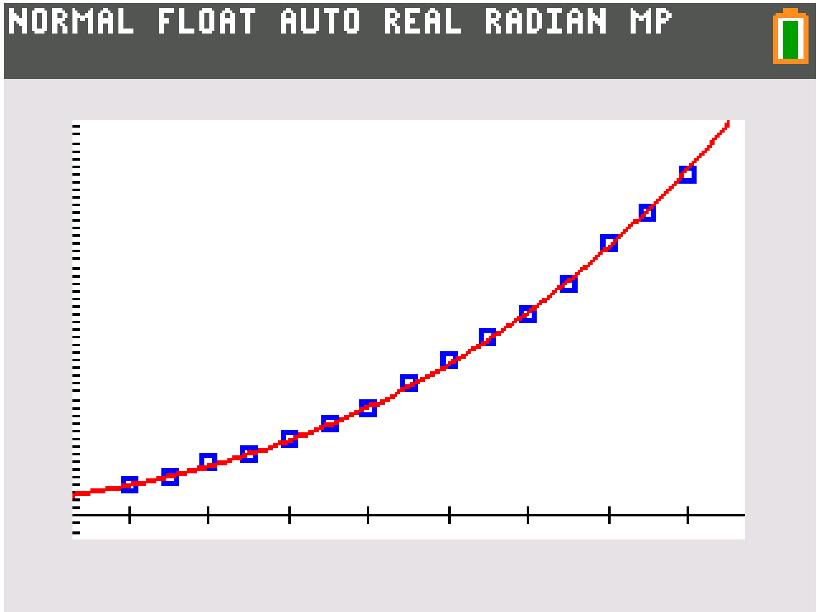
Final Model

NORMAL FLOAT AUTO REAL RADIAN MP

LinReg
y=ax+b
a=3.437107492
b=-5.267956451



$$\ln(W) = 3.437\ln(L) - 5.267$$

$$W = e^{3.437\ln(L) - 5.267}$$

$$W = e^{-5.267} \cdot e^{\ln\left(L^{3.437}\right)}$$

$$W = 0.00516 \cdot L^{3.437}$$

Other Data Analysis Explorations & Collaboration

Using log-log re-expression, we created a power function model for (length,weight) and can now predict the weight of a fish based on its length.

NOAA uses this method and you can find lots of data on the web and information on how this relationship varies depending on the type of fish.

$$\ln W = \ln a + b \ln L$$

where W = weight (kg), L = length (cm), a = y-intercept, and b = slope.

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Modeling Fish Population

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Using Recursion to Explore Fish Populations

The Fish and Wildlife Division monitors the trout population in a pond that is under its jurisdiction. Its research indicates that natural predators, together with pollution and fishing, are causing the trout population to decrease at a rate of 20% per month. The Division proposes to introduce additional trout each month to replenish the stream. Assume the current population is 100 fish.

Assume that the Division proposes adding 15 trout each month.

What will happen to the trout population over time?

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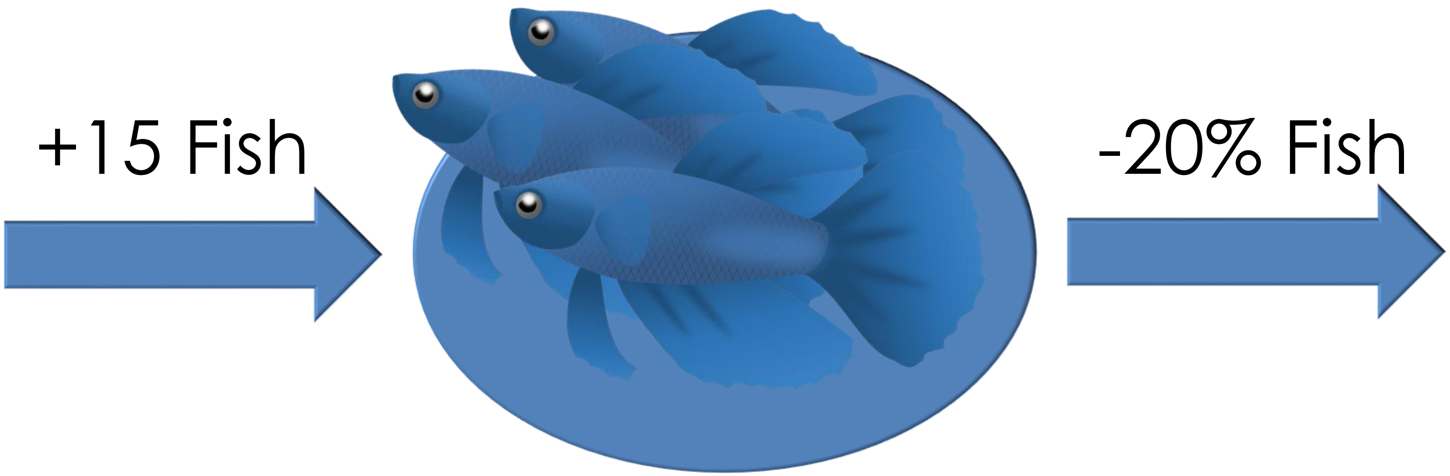
Diagram of the Process



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Simulation Using Paperclips

- Each table has a couple of boxes of paperclips
- Form groups of 3-4 people
- Simulate the fish population scenario with your group members
- Keep track of the number of fish in the pond at each iteration

Iteration	Number of Fish in Pond
0	100
1	100 – 20 + 15 = 95
2	
3	

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Iterate and Consider the Following

- What do you notice as your iterate?
- What to you think will happen in the "long-run"?
- If you were to graph the number of fish in the pond over time, what would the graph look like?
- What questions do you think your students might have?

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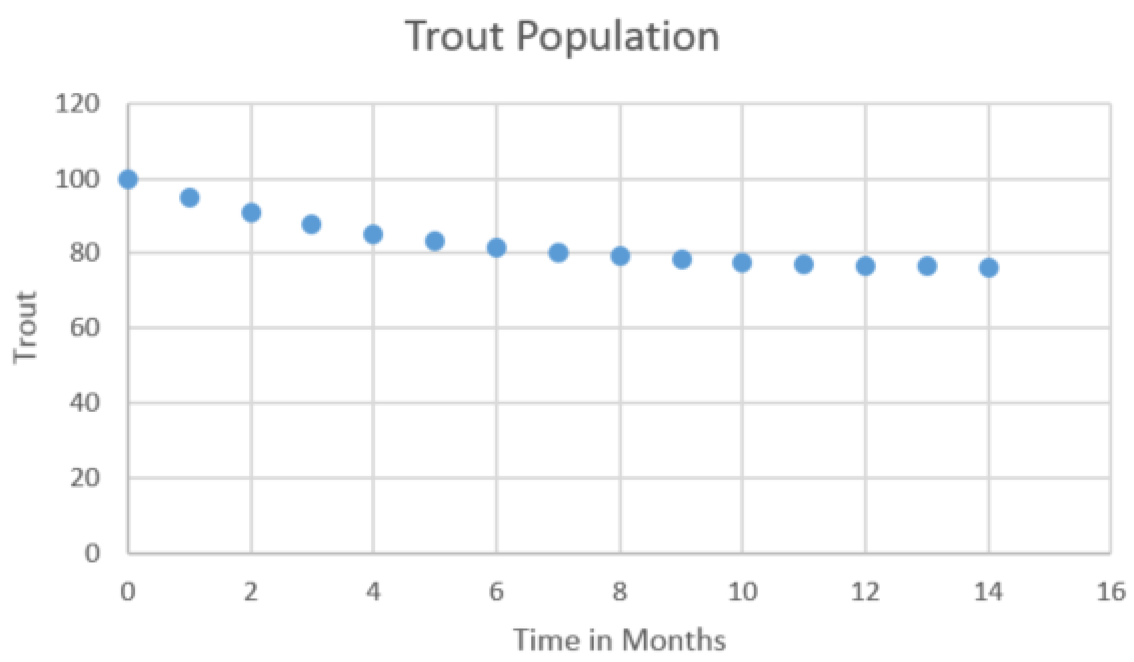
Graph Population Over time

What do you notice about the graph?



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Could we predict the equilibrium value without iterating?

- The system reaches equilibrium when the number of fish added to the pond equals the number of fish leaving the pond.

  How can we show this using mathematics?

- Or when the number of fish in the pond doesn't change.

  How can we show this using mathematics?

Fish Now = Fish Before – Fish Lost + Fish Gained

Or we can use recursive notation…

What other scenarios would you like to consider?

- What if we change the number of fish added to the pond?
- What if the population loses a different percentage of fish each month?
- What if we start with a different number of fish?

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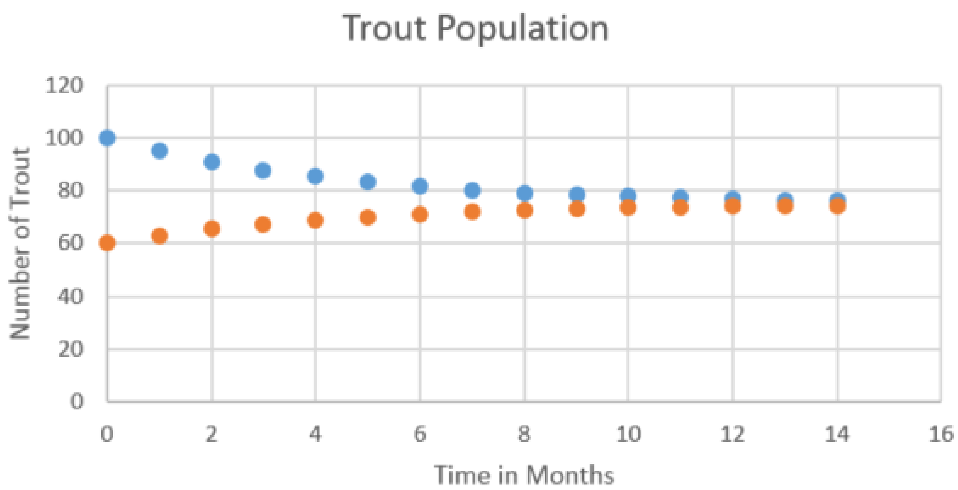
Graphs of Other Scenarios

Initial Pop = 60

Percent lost = 20

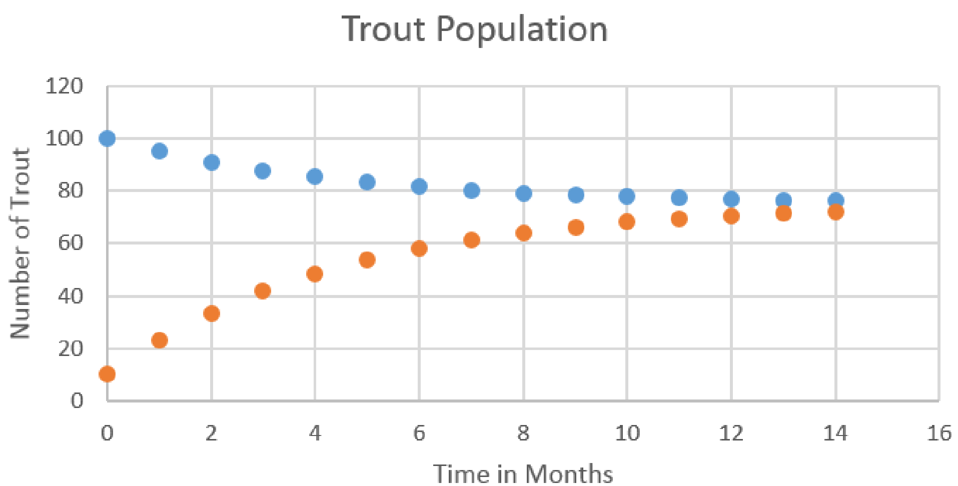
Fish Added = 15



Initial Pop = 10

Percent lost = 20

Fish Added = 15



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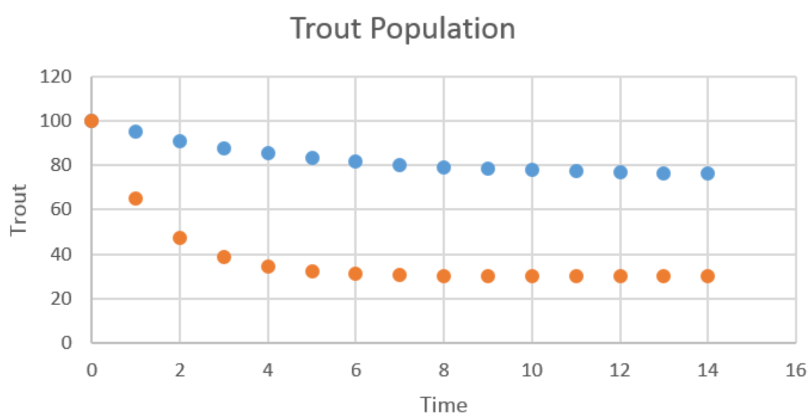
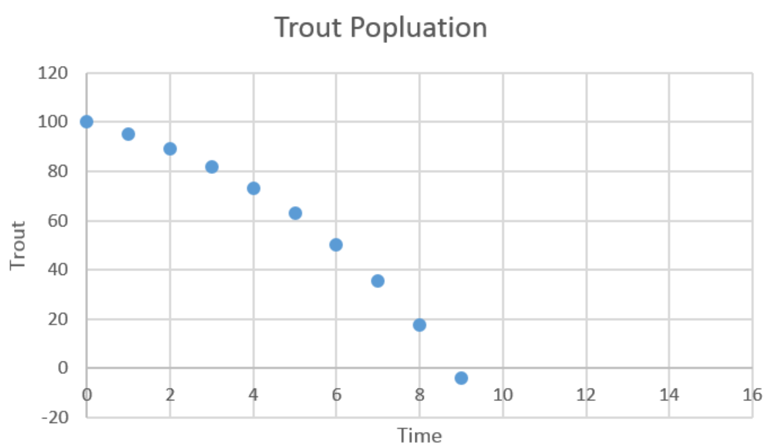
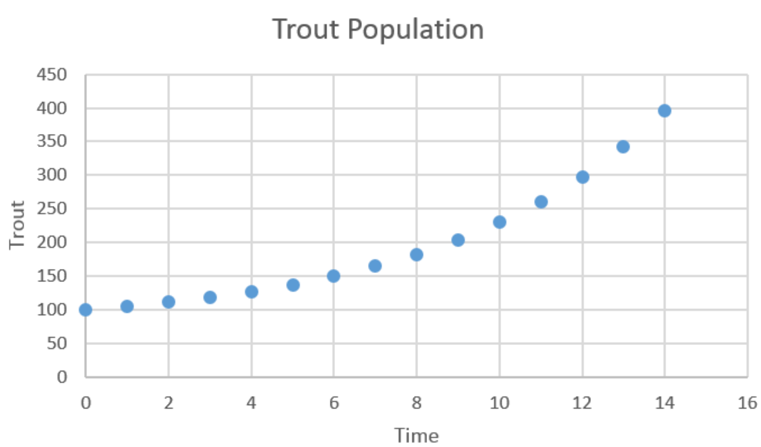
Can you create a scenario for the graph?







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What is Math Modeling?

"Whether the problem is huge or little, the process of 'interaction' between the mathematics and real world is the same … This entire process is what is called *mathematical modeling.*"

Henry Pollack (2012). *Mathematical Modeling Handbook: Introduction.* COMAP.

Available at http://www.comap.com/modelingHB/

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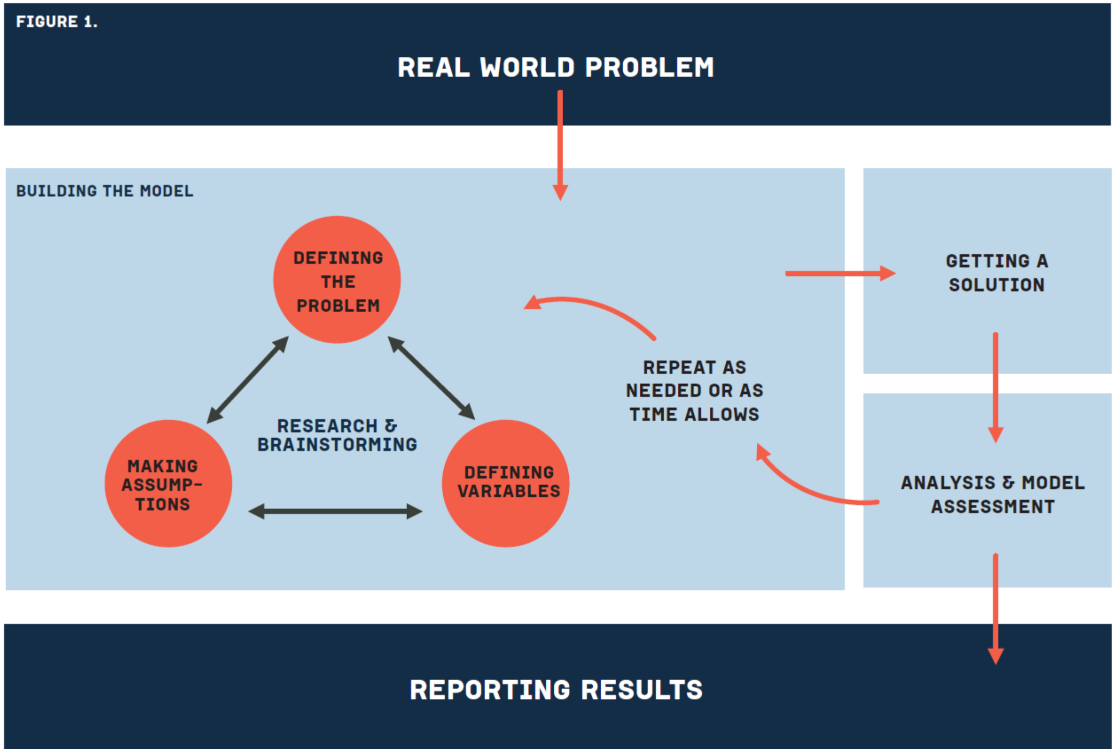
Modeling Cycle



From SIAM *Modeling Getting Started and Getting Solutions Handbook*

https://m3challenge.siam.org/resources/modeling-handbook

Helps in Explaining Cycle/Process

- **Defining the Problem Statement** Real-world problems can be broad and complex. It's important to refine the conceptual idea into a concise problem statement which will indicate exactly what the output of your model will be.

- **Making Assumptions** Early in your work, it may seem that a problem is too complex to make any progress. That is why it is necessary to make assumptions to help simplify the problem and sharpen the focus. During this process you reduce the number of factors affecting your model, thereby deciding which factors are most important.

- **Defining Variables** What are the primary factors influencing the phenomenon you are trying to understand? Can you list those factors as quantifiable variables with specified units? You may need to distinguish between independent variables, dependent variables, and model parameters. In understanding these ideas better, you will be able both to define model inputs and to create mathematical relationships, which ultimately establish the model itself.

- **Getting a Solution** What can you learn from your model? Does it answer the question you originally asked? Determining a solution may involve pencil-and-paper calculations, evaluating a function, running simulations, or solving an equation, depending on the type of model you developed. It might be helpful to use software or some other computational technology.

- **Analysis and Model Assessment** In the end, one must step back and analyze the results to assess the quality of the model. What are the strengths and weaknesses of the model? Are there certain situations when the model doesn't work? How sensitive is the model if you alter the assumptions or change model parameters values? Is it possible to make (or at least point out) possible improvements?

- **Reporting the Results** Your model might be awesome, but no one will ever know unless you are able to explain how to use or implement it. You may be asked to provide unbiased results or to be an advocate for a particular stakeholder, so pay attention to your point of view. Include your results in a summary or abstract at the beginning of your report.

From Henry Pollack

"When you use mathematics to understand a situation in the real world, and then use it to take action or even to predict the future, both the real-world situation and the ensuing mathematics are taken seriously."

From *Introduction: What is Mathematical Modeling*

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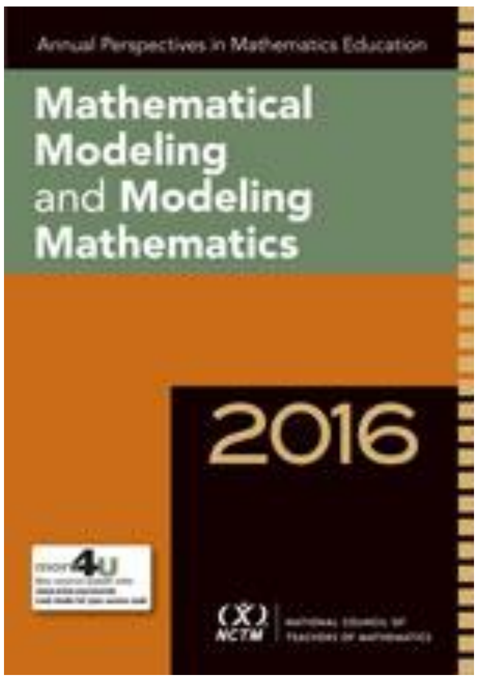
Annual Perspectives in Math Ed 2016



Moving from Remembering to Thinking: The Power of Mathematical Modeling

Dan Teague
12:30pm – 1:30pm
301, Moscone
Presentation #156

North Carolina School of Science and Mathematics

Conferences, etc…

- Anja S. Greer Summer Conference at Phillips Exeter Academy

  June 26 - July 1, 2016

https://www.exeter.edu/summer_programs/7325.aspx

- Knowles Teaching Fellows Program (KSTF)

http://kstf.org/

- Duke Masters of Arts in Teaching
  https://educationprogram.duke.edu/graduate

Questions, Comments, Discussion

Taylor Gibson, gibson@ncssm.edu

Maria Hernandez, hernandez@ncssm.edu

- Website w/Handouts

- NCSSM
  Teaching Contemporary
  Math Conference
  January 27-28, 2017
  www.ncssm.edu/tcmconference