Functions for ALL: Toward a Rigorous and Thorough Understanding

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What is a function?

"A function is a social gathering. No, seriously, it's a table where each x-value corresponds to exactly one y-value."

"An equation that can be graphed"

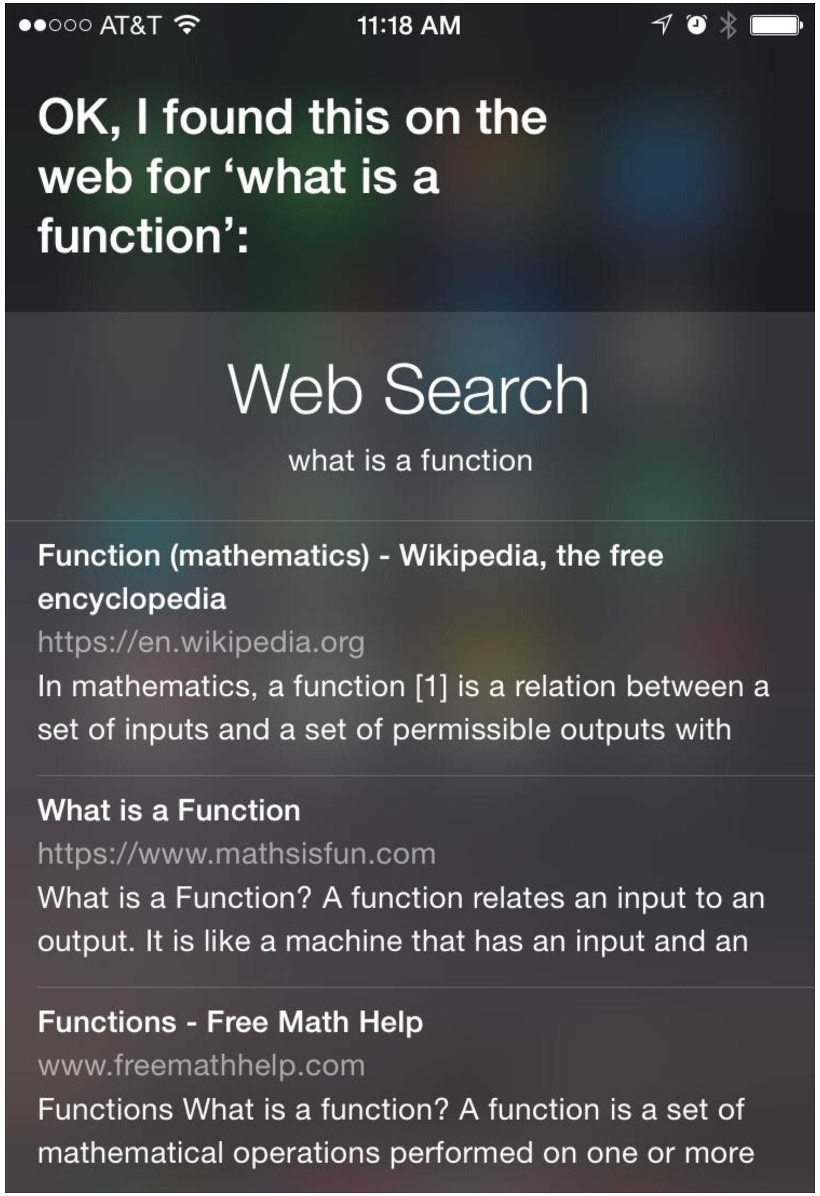
"A function is an equation with no repeating domains."

"Specifically, it's an equation where you have one input and you get just one single output for that one input."

"An equation where each input has one output"

"It's a relation between the inputs and the outputs."

"Wait a minute, let me ask Siri."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11:18 AM
OK, I found this on the web for ‘what is a function’:
Web Search
what is a function
Function (mathematics) - Wikipedia, the free encyclopedia
https://en.wikipedia.org
In mathematics, a function [1] is a relation between a set of inputs and a set of permissible outputs with
What is a Function
https://www.mathsisfun.com
What is a Function? A function relates an input to an output. It is like a machine that has an input and an
Functions - Free Math Help
www.freemathhelp.com
Functions What is a function? A function is a set of mathematical operations performed on one or more

The Experts Said…

"If now a unique finite *y* correspondin
that when *x* ranges continuously over t
continuously, then *y* is called a *continu*
at all necessary here that *y* be given
throughout the entire interval, and it
dependence expressed using mathem
undergo changes

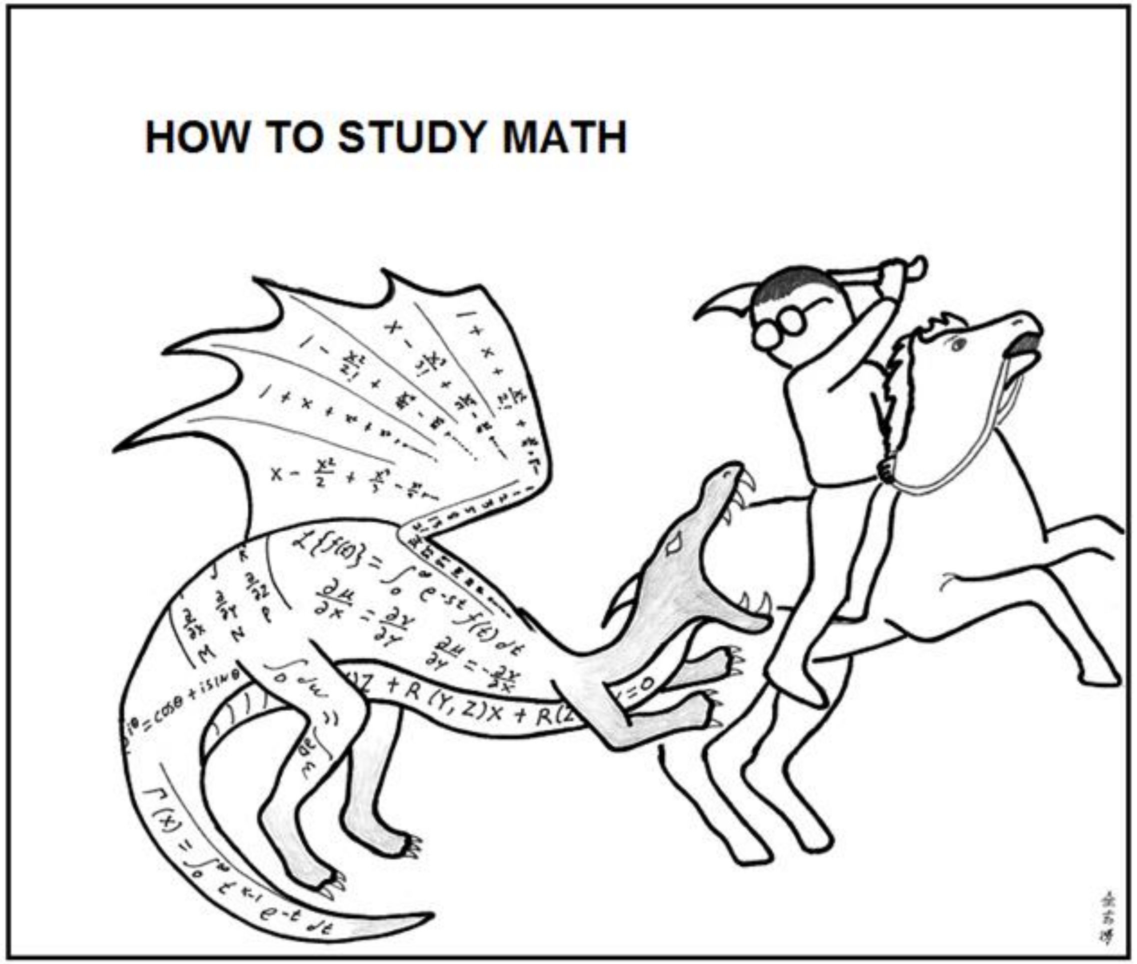
"Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if for all $x \in E$ there exists a unique $y \in F$ which is in the given relation with x." – Bourbaki, 1939

"In general the function $f(x)$ represents a succession of values or ordinates each of which is arbitrary…" – Fourier, 1822

"For half a century we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible the honest functions which serve some purpose." – Poincare, 1899

"A definition or theorem is not just a static statement, it is a weapon for deducing truth."

- Charles Wells, 2016



Don't just read it; fight it!

--- Paul R. Halmos

http://abstrusegoose.com/353

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Standards for Mathematical Practice

- **MP.3 Construct Viable Arguments**
  - Mathematically proficient students **understand** and **use** stated assumptions, **definitions**, and previously established results in **constructing arguments**.
- **MP.6 Attend to Precision**
  - Mathematically proficient students try to communicate precisely to others. They try to **use clear definitions** in discussion with others and in their own **reasoning**. By the time they reach high school they have learned to examine claims and **make explicit use of definitions**.

The Importance of Definitions

"Teachers can help students see that some words that are used in everyday language, such as similar, factor, area, or function are used in mathematics with different or more-precise meanings. This observation is the foundation for understanding the concept of mathematical definition."

NCTM *Principles and Standards for School Mathematics*, 2000

"A function is a social gathering. No, seriously, it's a table where each x-value corresponds to exactly one y-value."

The Concept Image vs. The Concept Definition

A function is …
- a graph,
- a table,
- a set of ordered pairs,
- a rule or equation
- a way to find the y-value

FUNCTION. A *function* is a correspondence between two sets, X and Y, in which each element of X is matched to one and only one element of Y. The set X is called the *domain of the function*.

The concept image is "built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures."

The concept definition may be learned "in a rote fashion or more meaningfully … related to a greater or lesser degree to the concept as a whole." (Tall and Vinner, 1981)

Students struggle to use definitions to make arguments …

x	-2	-2	0	5	6	5	-1	3	8
$f(x)$	0	2	4	5	7	9	11	10	12

No this is not a function, on a graph this table's points would fail the pencil test.

x	-2	-2	0	5	6	5	-1	3	8
$f(x)$	0	2	4	5	7	9	11	10	12

This is not a function because there are multiple ranges scheduled to one domaine

x	-2	-2	0	5	6	5	-1	3	8
$f(x)$	0	2	4	5	7	9	11	10	12

NO because x=5 happens twice with 2 different answers making it not a function.

CCSS Functions Progressions

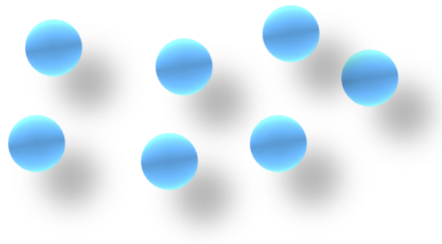
The essential question when investigating functions is:

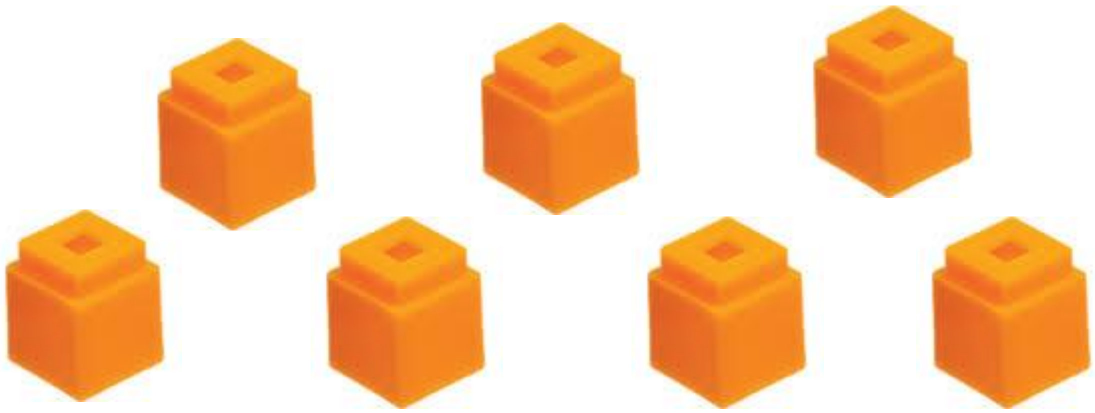
“Does each element of the domain correspond to exactly one element in the range?”

Function Concepts in Grades K–5

Correspondence (Grade K Application Problem)

Pretend your linking cubes are little baskets. Use your clay to make as many balls as there are baskets. Check your work by putting a ball in each basket. Do you have just enough? Score 1 point for every basket you made!





Function Concepts in Grades K–5

Fluency—Counting by Units

8	16	24	32	40	48	56	64	72	80
1 eight	2 eights	3 eights	4 eights	5 eights	6 eights	7 eights	8 eights	9 eights	10 eights

1 eight is assigned to 8 ones because $1 \times 8 = 8$.
2 eights are assigned to 16 ones because $2 \times 8 = 16$.

Function Concepts in Grades K–5

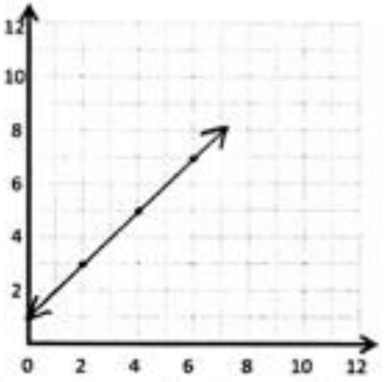
- Grade 4: Equal Measurements

Feet	Inches
1	12
2	24
3	36
4	48
5	60
6	72
7	84
8	96
9	108
10	120

- Grade 5: Graph Ordered Pairs and Write a Rule

1. Complete the chart. Then, plot the points on the coordinate plane below.

x	y	(x, y)
0	1	(0, 1)
2	3	(2, 3)
4	5	(4, 5)
6	7	(6, 7)



a. Use a straightedge to draw a line connecting these points.

b. Write a rule showing the relationship between the x- and y-coordinates of points on the line.

y is 1 more than x.

c. Name 2 other points that are on this line.

(7, 8) (9, 10)

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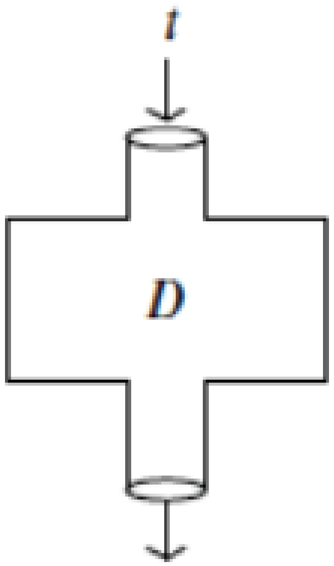
Functions in Grade 8

- *Definition:* A **function** is a correspondence between a set (whose elements are called inputs) and another set (whose elements are called outputs) such that each input corresponds to one and only one output.

Input/Output Machine



Distance traveled in *t* seconds

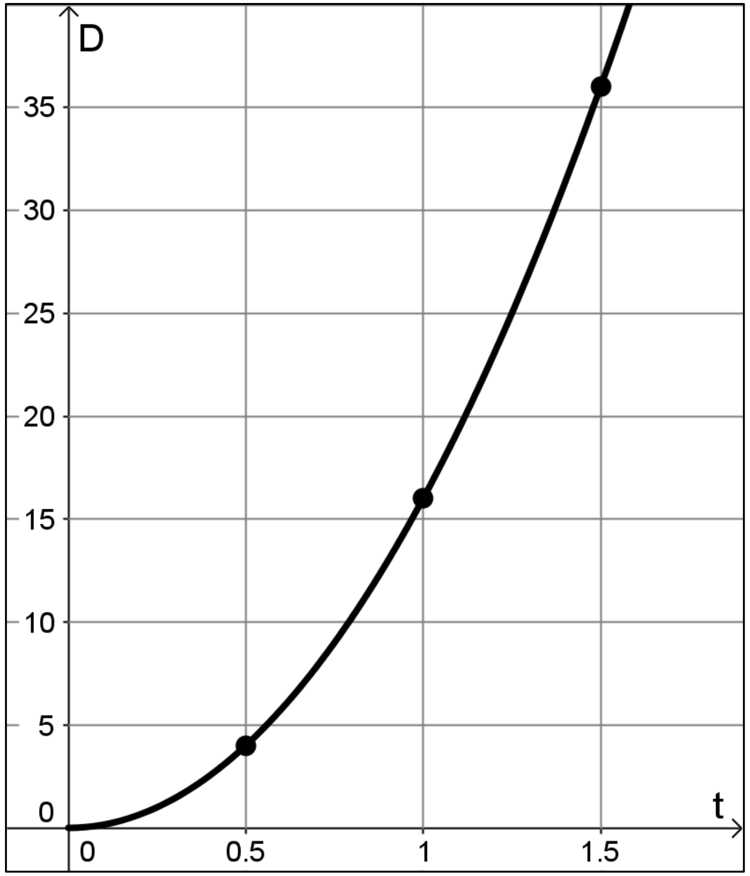
t (sec)	D (feet)
0.5	4
1	16
1.5	36

Functions in Grade 8

- Multiple representations emphasized

t (sec)	D (feet)
0.5	4
1	16
1.5	36

$$D = 16t^2$$



- Some functions have algebraic rules and some do not.

Functions in Grade 8

Eureka Math Grade 8 Module 5 End of Module Assessment Item 1

a. We define x as a year between 2008 and 2013 and y as the total number of smartphones sold that year, in millions. The table shows values of x and corresponding y values.

Year (x)	2008	2009	2010	2011	2012	2013
Number of smartphones in millions (y)	3.7	17.3	42.4	90	125	153.2

i. How many smartphones were sold in 2009?

ii. In which year were 90 million smartphones sold?

iii. Is y a function of x? Explain why or why not.

b. Randy began completing the table below to represent a particular linear function. Write an equation to represent the function he was using and complete the table for him.

Input (x)	-3	-1	0	1/2	1	2	3
Output (y)	-5		4				13

c. Create the graph of the function in part (b).

d. At NYU in 2013, the cost of the weekly meal plan options could be described as a function of the number of meals. Is the cost of the meal plan a linear or nonlinear function? Explain.

8 meals: \$125/week 10 meals: \$135/week 12 meals: \$155/week 21 meals: \$220/week

iscrete or not discrete.

. The graph of a

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scatter plots and using

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Algebra 1: Sequences are Functions

1. Write the first three terms of the sequence. Is it arithmetic or geometric?

$$A(n+1) = \frac{1}{2}A(n) \text{ for } n \geq 1 \text{ and } A(1) = 4$$

2. Identify the sequence $14, 11, 8, 5, \ldots$ as arithmetic or geometric. Explain your answer, and write an explicit formula.

Eureka Math Definition of a Function

FUNCTION: A *function* is a correspondence between two sets, X and Y, in which each element of X is matched to one and only one element of Y. The set X is called the *domain of the function*.

The notation $f: X \to Y$ is used to name the function and describes both X and Y. If x is an element in the domain X of a function $f: X \to Y$, then x is matched to an element of Y called $f(x)$. We say $f(x)$ is the value in Y that denotes the *output* or *image* of f corresponding to the *input* x.

The *range (or image)* of a function $f: X \to Y$ is the subset of Y, denoted $f(X)$, defined by the following property: y is an element of $f(X)$ if and only if there is an x in X such that $f(x) = y$.

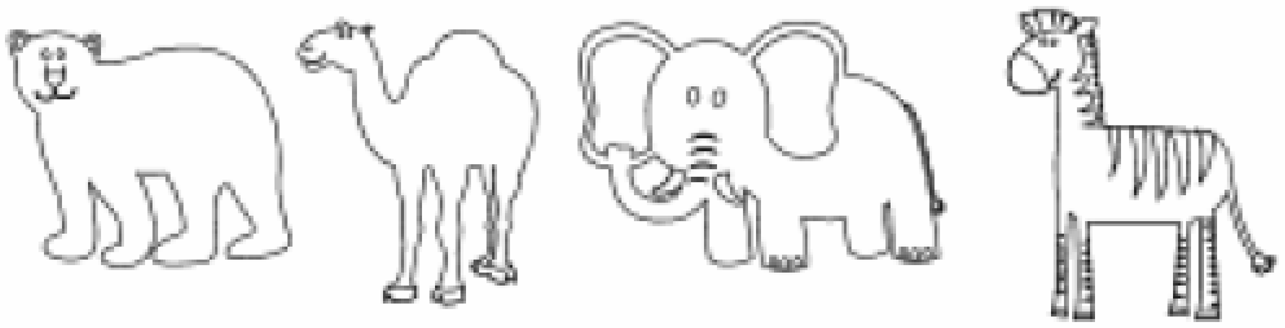
Notation matters!
x is an element in the domain.
$f(x)$ is the element in the range matched with x.
The range of a function is a subset of set Y.

The Concept Image

f: {animal pictures} → {animal names}
Assign each animal picture to its proper name
Domain: {four animal pictures}
Range: {elephant, camel, polar bear, zebra}

Opening Exercise

Match each picture to the correct word by drawing an arrow from the word to the picture.



Elephant

Camel

Polar Bear

Zebra

Is this a function? What is the domain? What is the range?

Building the Concept of a Function

$f: \{\text{Students in your class}\} \to \{\text{English teachers in your school}\}$
Let f assign each student in your class to their English teacher.

Domain: $\{\text{Students in your class}\}$
Range: $\{\text{English teachers of students in your class}\}$

What does $f(\text{Pablo}) = \text{Mrs. Yates}$ mean?

Pablo is a student in Mrs. Yates' English class.

Which students satisfy $f(x) = \text{Mr. De La Cerda}$?

Students in my class that have Mr. De La Cerda for English.

Building the Concept with Numbers

$f: \{1, 2, 3, 4\} \rightarrow \{5, 6, 7, 8, 9\}$
$f = \{(1, 7), (2, 5), (3, 6), (4, 7)\}$

Domain: $\{1, 2, 3, 4\}$
Range: $\{5, 6, 7\}$

What does $f(2) = 5$ mean?

The domain element 2 is assigned to the range element 5.

Which value(s) of x satisfies $f(x) = 7$?

The domain elements 1 and 4 are assigned to 7.

Why Can We Use an Equation to Define a Function?

ALGEBRAIC FUNCTION: Given an algebraic expression in one variable, an *algebraic function* is a function $f: X \to \mathbb{R}$ such that for each real number x in the domain X, $f(x)$ is the value found by substituting the number x into all instances of the variable symbol in the algebraic expression and evaluating.

The Squaring Function

Let $f: X \to Y$ be the function such that $x \mapsto x^2$, where X is the set of real numbers.

x	$f(x)$
0	
3	
-2	
$1/4$	
$\sqrt{2}$	

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Why Can We Use an Equation to Define a Function?

The Squaring Function

Let $f: X \to Y$ be the function such that $x \mapsto x^2$, where X is the set of real numbers.

x	$f(x)$
0	0
3	9
-2	4
$1/4$	$1/16$
$\sqrt{2}$	2

$$f(0) = 0$$

$$f(3) = 9$$

$$f(-2) = 4$$

$$f\left(\frac{1}{4}\right) = \frac{1}{16}$$

$$f(\sqrt{2}) = 2$$

Is this equation true for all values of x?
$f(x) = x^2$

New notation:
Let $f(x) = x^2$ where x is any real number.

A Function vs. the Graph of a Function

Pseudocode:

```
Declare x integer
For all x from 1 to 5
        Print 2^x
Next x
```

Declare x integer – Specifies the domain of the variable of x to be the set of integers.

For all x from 1 to 5 – Performs the instructions in the *loop body* first for x equal to 1, then 2, then 3, then 4, then 5.

Print 2^x – *loop body*

Next x – Substitutes the next element in the set for x, and runs the *loop body* instructions for that value of x. For example, if the loop just completed for $x = 3$, *Next* x tells the computer to run the *loop body* instructions for $x = 4$.

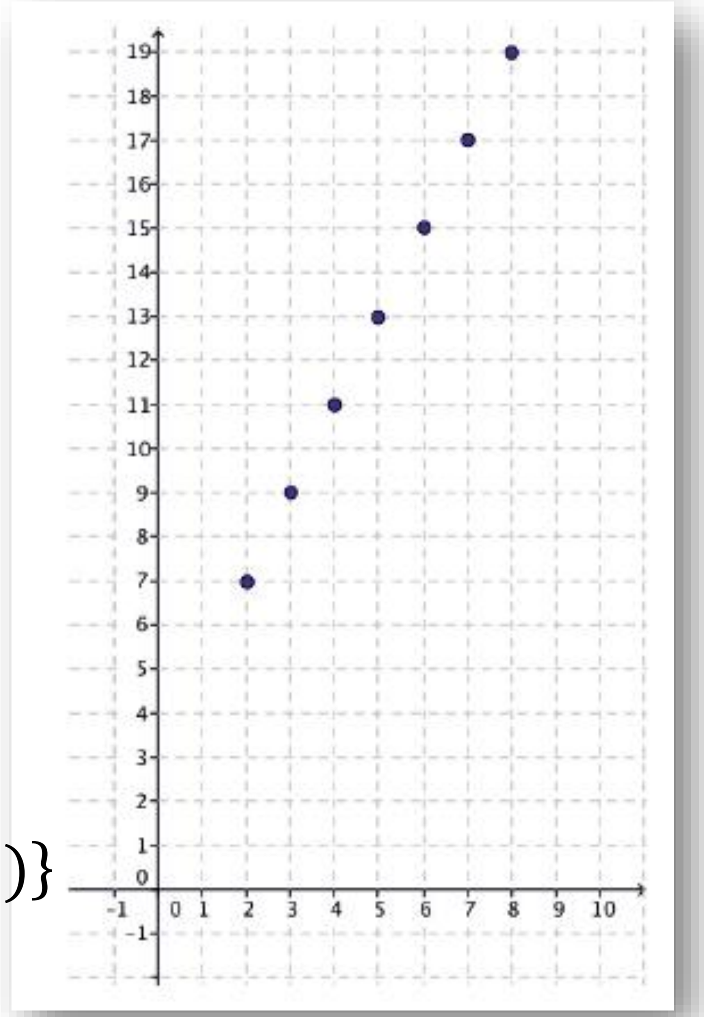
Let $f(x) = 2^x$, for integer x from 1 to 5.
Domain: {1, 2, 3, 4, 5}
Range: {2, 4, 8, 16, 32}

The Graph of a Function

```
Declare x integer
Initialize G as {}
For all x from 2 to 8
        Append (x, 2x + 3) to G
Next x
Plot G
```

This code creates and plots a set of ordered pairs.

$\{(2,7),(3,9),(4,11),(5,13),(6,15),(7,17),(8,19)\}$

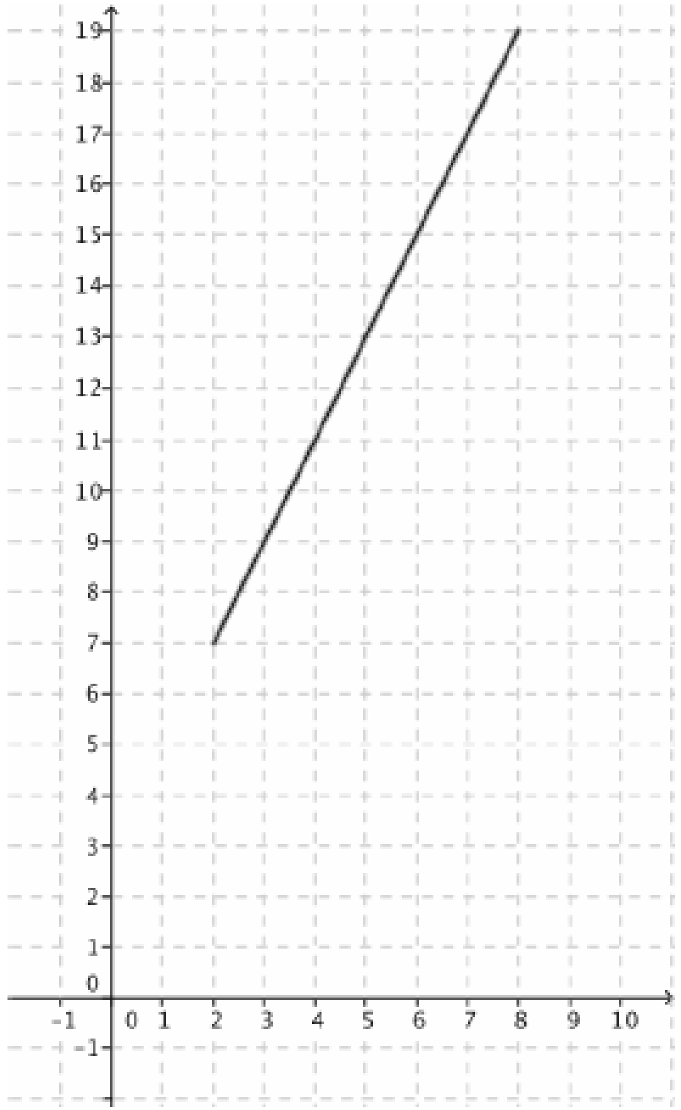


G is the graph of a function $f(x) = 2x + 3$ for integer values of x from 2 to 8.

The Graph of a Function

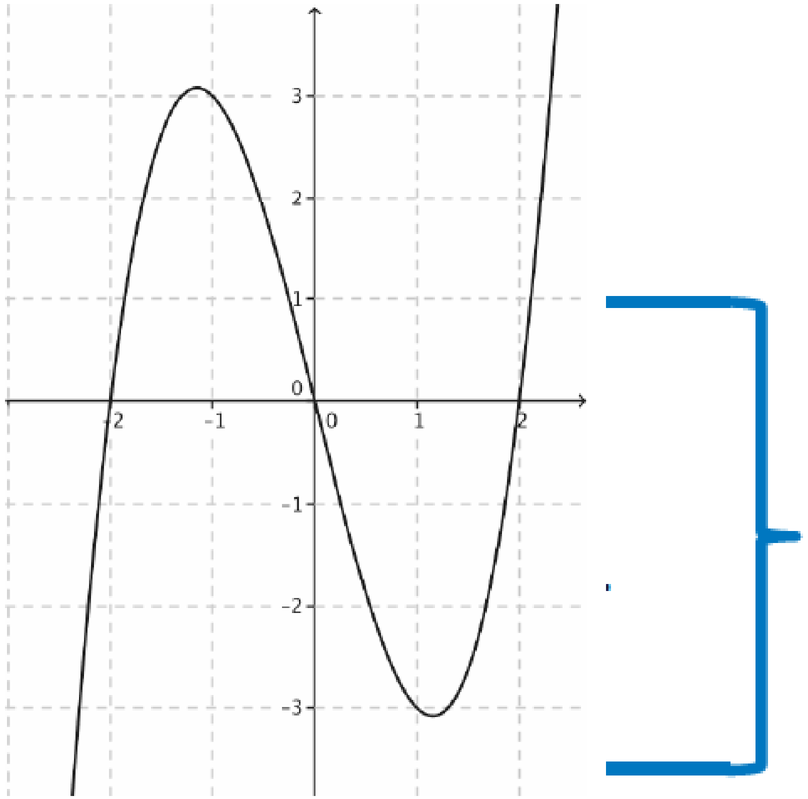
Declare x real
Let $f(x) = 2x + 3$
Initialize G as {}
For all x such that $2 \le x \le 8$
    Append $(x, f(x))$ to G
Next x
Plot G

This code also creates and plots a set of ordered pairs.



The Graph of the Equation $y = f(x)$

```
Declare x and y real
Let f(x) = x(x − 2)(x + 2)
Initialize G as {}
For all x in the real numbers
    For all y in the real numbers
        If y = f(x) then
            Append (x, y) to G
         else
            Do NOT append (x, y) t
        End if
    Next y
Next x
Plot G
```



For each x-value, the code loops through all y-values.

For $f(x) = x(x-2)(x+2)$ with domain and range all real number, the graph is given by $\{(x, y) |\ x \text{ real and } y = f(x)\}$.

Aren't graphs of functions and graphs of equations the same thing?

The graph of an equation

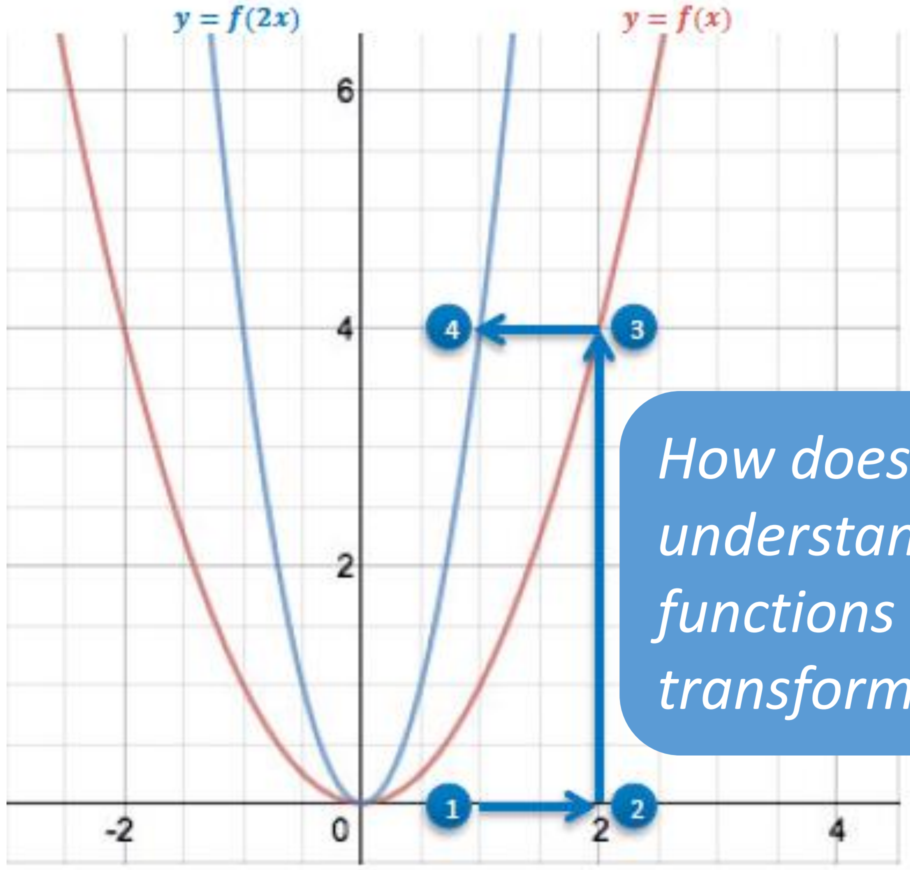
- Nested For-Next Loop
  - For each x, check each y to see if (x, y) satisfies the equation $y = f(x)$.
  - Create a set of ordered pairs that are solutions to the equation $y = f(x)$.
- Plot the ordered pairs.

The graph of a function

- For-Next Loop
  - For each x, determine $f(x)$.
  - Create a set of ordered pairs $(x, f(x))$.
- Plot the ordered pairs.

Creating New Functions from Old Functions

x	$f(x) = x^2$	$g(x) = f(2x)$
-3	9	36
-2	4	16
-1	1	4
0	0	0
1	1	4
2	4	16
3	9	36



Closing: Discuss how the horizontal scaling by a scale factor of k of the graph of a function $y = f(x)$ corresponds to changing the equation of the graph from $y = f(x)$ to $y = f(kx)$.

How does a solid understanding of functions help with transformations?

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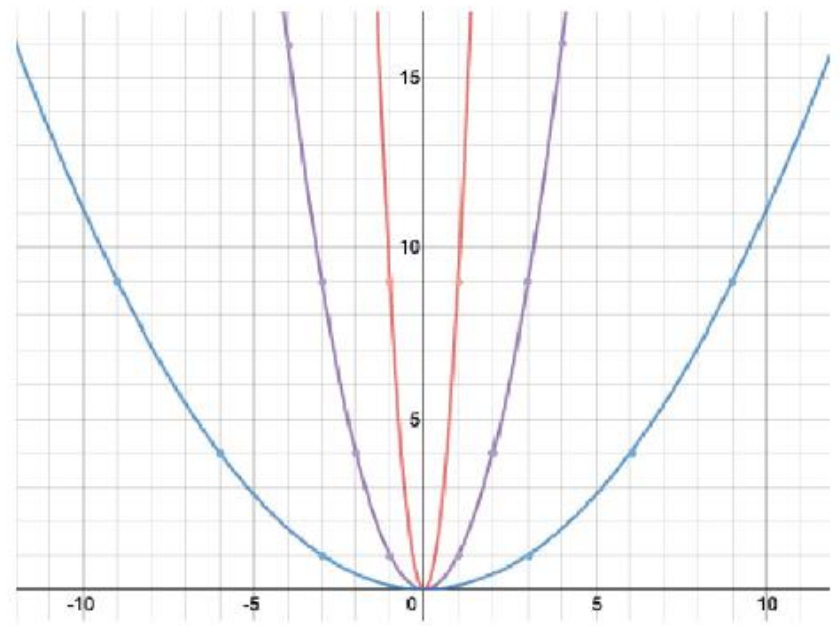
Algebra I Exit Ticket

Let $f(x) = x^2$, $g(x) = (3x)^2$, and $h(x) = \left(\frac{1}{3}x\right)^2$, where x can be any real number. The graphs above are of $y = f(x)$, $y = g(x)$, and $y = h(x)$.

1. Label each graph with the appropriate equation.

2. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = g(x)$. Use coordinates of each to illustrate an example of the correspondence.



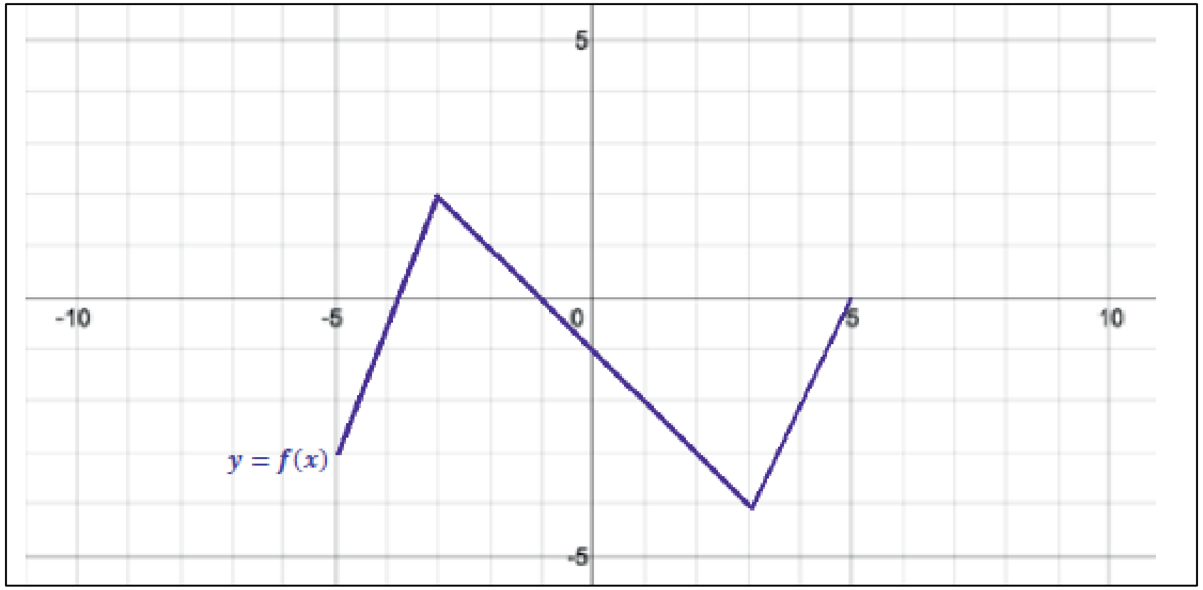
Geometric transformations are functions!

Algebra I Exit Ticket

The graph of a piecewise function f is shown below.

Let $p(x) = f(x-2)$, $q(x) = \frac{1}{2}f(x-2)$, and $r(x) = \frac{1}{2}f(x-2) + 3$.

Graph $y = p(x)$, $y = q(x)$, and $y = r(x)$ on the same set of axes as the graph of $y = f(x)$.



Functions in Algebra II and Advanced Math

- Defining New Functions and Modeling with Functions
  - Sine and cosine functions
  - Logarithm functions
  - Rational functions
- Building New Functions
  - Extending transformations of graphs of functions to new function families
  - Inverses of functions
  - Composition of functions
- Matrix operations, vectors, and linear transformations
  - Geometric transformations
  - Representations

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