Functions for ALL: Toward a Rigorous and Thorough Understanding

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What is a function?

“A function is a social gathering. No, seriously, it’s a table where each \( x \)-value corresponds to exactly one \( y \)-value.”

“An equation that can be graphed”

“A function is an equation with no repeating domains.”

“Specifically, it’s an equation where you have one input and you get just one single output for that one input.”

“It’s a relation between the inputs and the outputs.”

“Wait a minute, let me ask Siri.”

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OK, I found this on the web for ‘what is a function’:

**Web Search**

what is a function

**Function (mathematics) - Wikipedia, the free encyclopedia**
https://en.wikipedia.org
In mathematics, a function [1] is a relation between a set of inputs and a set of permissible outputs with

**What is a Function**
https://www.mathsisfun.com
What is a Function? A function relates an input to an output. It is like a machine that has an input and an

**Functions - Free Math Help**
www.freemathhelp.com
Functions What is a function? A function is a set of mathematical operations performed on one or more
The Experts Said...

“If now a unique finite \( y \) corresponding to each \( x \), and moreover in such a way that when \( x \) ranges continuously over the interval from \( a \) to \( b \), \( y = f(x) \) also varies continuously, then \( y \) is called a continuous function of \( x \) for this interval. It is not at all necessary here that \( y \) be given in terms of \( x \) by one and the same law throughout the entire interval, and it is not necessary that it be regarded as a dependence expressed using mathematical operations.” – Dirichlet, 1837

“In general the function \( f(x) \) represents a succession of values or ordinates each of which is arbitrary...” – Fourier, 1822

“For half a century we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible the honest functions which serve some purpose.” – Poincare, 1899

“Let \( E \) and \( F \) be two sets, which may or may not be distinct. A relation between a variable element \( x \) of \( E \) and a variable element \( y \) of \( F \) is called a functional relation in \( y \) if for all \( x \in E \) there exists a unique \( y \in F \) which is in the given relation with \( x \).” – Bourbaki, 1939
“A definition or theorem is not just a static statement, it is a weapon for deducing truth.”

- Charles Wells, 2016

http://abstrusegoose.com/353
Standards for Mathematical Practice

• **MP.3 Construct Viable Arguments**
  • Mathematically proficient students **understand** and **use** stated assumptions, **definitions**, and previously established results in **constructing arguments**.

• **MP.6 Attend to Precision**
  • Mathematically proficient students try to communicate precisely to others. They try to **use clear definitions** in discussion with others and in their own **reasoning**. By the time they reach high school they have learned to examine claims and **make explicit use of definitions**.
The Importance of Definitions

“Teachers can help students see that some words that are used in everyday language, such as similar, factor, area, or function are used in mathematics with different or more-precise meanings. This observation is the foundation for understanding the concept of mathematical definition.”

NCTM Principles and Standards for School Mathematics, 2000

“A function is a social gathering. No, seriously, it’s a table where each $x$-value corresponds to exactly one $y$-value.”
The Concept Image vs. The Concept Definition

**A function is ...**
- a graph,
- a table,
- a set of ordered pairs,
- a rule or equation
- a way to find the y-value

**FUNCTION.** A *function* is a correspondence between two sets, $X$ and $Y$, in which each element of $X$ is matched to one and only one element of $Y$. The set $X$ is called the *domain of the function*.

The **concept image** is “built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.”

The **concept definition** may be learned “in a rote fashion or more meaningfully ... related to a greater or lesser degree to the concept as a whole.” (Tall and Vinner, 1981)
Students struggle to use definitions to make arguments ...

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-2</th>
<th>0</th>
<th>5</th>
<th>6</th>
<th>5</th>
<th>-1</th>
<th>3</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>10</td>
<td>12</td>
</tr>
</tbody>
</table>

No, this is not a function, on a graph this table's points would fail the pencil test.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-2</th>
<th>0</th>
<th>5</th>
<th>6</th>
<th>5</th>
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<td>9</td>
<td>11</td>
<td>10</td>
<td>12</td>
</tr>
</tbody>
</table>

This is not a function because there are multiple ranges scheduled to one domain.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-2</th>
<th>0</th>
<th>5</th>
<th>6</th>
<th>5</th>
<th>-1</th>
<th>3</th>
<th>8</th>
</tr>
</thead>
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<td>0</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>10</td>
<td>12</td>
</tr>
</tbody>
</table>

NO because $x=5$ happens twice with 2 different answers making it not a function.
The essential question when investigating functions is: “Does each element of the domain correspond to exactly one element in the range?”
Function Concepts in Grades K–5

Correspondence (Grade K Application Problem)

Pretend your linking cubes are little baskets. Use your clay to make as many balls as there are baskets. Check your work by putting a ball in each basket. Do you have just enough? Score 1 point for every basket you made!

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Function Concepts in Grades K–5

Fluency—Counting by Units

1 eight is assigned to 8 ones because $1 \times 8 = 8$.
2 eights are assigned to 16 ones because $2 \times 8 = 16$. 

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Function Concepts in Grades K–5

• Grade 4: Equal Measurements

<table>
<thead>
<tr>
<th>Feet</th>
<th>Inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>36</td>
</tr>
<tr>
<td>4</td>
<td>48</td>
</tr>
<tr>
<td>5</td>
<td>60</td>
</tr>
<tr>
<td>6</td>
<td>72</td>
</tr>
<tr>
<td>7</td>
<td>84</td>
</tr>
<tr>
<td>8</td>
<td>96</td>
</tr>
<tr>
<td>9</td>
<td>108</td>
</tr>
<tr>
<td>10</td>
<td>120</td>
</tr>
</tbody>
</table>

• Grade 5: Graph Ordered Pairs and Write a Rule

1. Complete the chart. Then, plot the points on the coordinate plane below.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>(x, y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>(2, 3)</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>(4, 5)</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>(6, 7)</td>
</tr>
</tbody>
</table>

a. Use a straightedge to draw a line connecting these points.

b. Write a rule showing the relationship between the x- and y-coordinates of points on the line.

Y is 1 more than x.

c. Name 2 other points that are on this line.

(7, 8) (9, 10)
Functions in Grade 8

• **Definition:** A function is a correspondence between a set (whose elements are called inputs) and another set (whose elements are called outputs) such that each input corresponds to one and only one output.

<table>
<thead>
<tr>
<th>t (sec)</th>
<th>D (feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td>1.5</td>
<td>36</td>
</tr>
</tbody>
</table>
Functions in Grade 8

• Multiple representations emphasized

<table>
<thead>
<tr>
<th>t (sec)</th>
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<tr>
<td>0.5</td>
<td>4</td>
</tr>
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<td>1</td>
<td>16</td>
</tr>
<tr>
<td>1.5</td>
<td>36</td>
</tr>
</tbody>
</table>

$D = 16t^2$

• Some functions have algebraic rules and some do not.
Functions in Grade 8

• The graph of a function can be classified as discrete or not discrete.

The graph of a linear function takes the form \( y = mx + b \). The graph of a continuous linear function is a non-vertical line.

Connections are made to bivariate data sets, scatter plots and using a linear function to model bivariate data.

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Algebra 1: Sequences are Functions

1. Write the first three terms of the sequence. Is it arithmetic or geometric?

\[ A(n + 1) = \frac{1}{2} A(n) \text{ for } n \geq 1 \text{ and } A(1) = 4 \]

2. Identify the sequence 14, 11, 8, 5, ... as arithmetic or geometric. Explain your answer, and write an explicit formula.
Eureka Math Definition of a Function

**FUNCTION:** A function is a correspondence between two sets, $X$ and $Y$, in which each element of $X$ is matched to one and only one element of $Y$. The set $X$ is called the domain of the function.

The notation $f: X \rightarrow Y$ is used to name the function and describes both $X$ and $Y$. If $x$ is an element in the domain $X$ of a function $f: X \rightarrow Y$, then $x$ is matched to an element of $Y$ called $f(x)$. We say $f(x)$ is the value in $Y$ that denotes the output or image of $f$ corresponding to the input $x$.

The range (or image) of a function $f: X \rightarrow Y$ is the subset of $Y$, denoted $f(X)$, defined by the following property: $y$ is an element of $f(X)$ if and only if there is an $x$ in $X$ such that $f(x) = y$.

Notation matters!

$x$ is an element in the domain.

$f(x)$ is the element in the range matched with $x$.

The range of a function is a subset of set $Y$. 
The Concept Image

Assignment Exercise: Match each picture to the correct word by drawing an arrow from the word to the picture.

Domain: {four animal pictures}
Range: {elephant, camel, polar bear, zebra}

Is this a function? What is the domain? What is the range?
Building the Concept of a Function

\( f: \{\text{Students in your class}\} \to \{\text{English teachers in your school}\} \)
Let \( f \) assign each student in your class to their English teacher.

Domain: \( \{\text{Students in your class}\} \)
Range: \( \{\text{English teachers of students in your class}\} \)

What does \( f(\text{Pablo}) = \text{Mrs. Yates} \) mean?
- Pablo is a student in Mrs. Yates’ English class.

Which students satisfy \( f(x) = \text{Mr. De La Cerda} \)?
- Students in my class that have Mr. De La Cerda for English.
Building the Concept with Numbers

\( f: \{1, 2, 3, 4\} \rightarrow \{5, 6, 7, 8, 9\} \)
\( f = \{(1, 7), (2, 5), (3, 6), (4, 7)\} \)

Domain: \( \{1, 2, 3, 4\} \)
Range: \( \{5, 6, 7\} \)

What does \( f(2) = 5 \) mean?

The domain element 2 is assigned to the range element 5.

Which value(s) of \( x \) satisfies \( f(x) = 7 \)?

The domain elements 1 and 4 are assigned to 7.
Why Can We Use an Equation to Define a Function?

**ALGEBRAIC FUNCTION:** Given an algebraic expression in one variable, an *algebraic function* is a function $f: X \to \mathbb{R}$ such that for each real number $x$ in the domain $X$, $f(x)$ is the value found by substituting the number $x$ into all instances of the variable symbol in the algebraic expression and evaluating.

**The Squaring Function**

Let $f: X \to Y$ be the function such that $x \mapsto x^2$, where $X$ is the set of real numbers.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>−2</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td></td>
</tr>
<tr>
<td>$\sqrt{2}$</td>
<td></td>
</tr>
</tbody>
</table>
Why Can We Use an Equation to Define a Function?

**The Squaring Function**

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</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>1/4</td>
<td>1/16</td>
</tr>
<tr>
<td>$\sqrt{2}$</td>
<td>2</td>
</tr>
</tbody>
</table>

Is this equation true for all values of $x$?

$$f(x) = x^2$$

$f(0) = 0$

$f(3) = 9$

$f(-2) = 4$

$f\left(\frac{1}{4}\right) = \frac{1}{16}$

$f(\sqrt{2}) = 2$

New notation:

Let $f(x) = x^2$ where $x$ is any real number.
Let $f(x) = 2^x$, for integer $x$ from 1 to 5.
Domain: \{1, 2, 3, 4, 5\}
Range: \{2, 4, 8, 16, 32\}
The Graph of a Function

This code creates and plots a set of ordered pairs.

\[ \{(2, 7), (3, 9), (4, 11), (5, 13), (6, 15), (7, 17), (8, 19)\} \]

\( G \) is the graph of a function \( f(x) = 2x + 3 \) for integer values of \( x \) from 2 to 8.
The Graph of a Function

This code also creates and plots a set of ordered pairs.

```
Declare x real
Let f(x) = 2x + 3
Initialize G as {}
For all x such that 2 ≤ x ≤ 8
    Append (x, f(x)) to G
Next x
Plot G
```
The Graph of the Equation $y = f(x)$

For $f(x) = x(x - 2)(x + 2)$ with domain and range all real number, the graph is given by $\{(x, y) | x \text{ real and } y = f(x)\}$. 
Aren’t graphs of functions and graphs of equations the same thing?

**The graph of an equation**
- Nested For-Next Loop
  - For each $x$, check each $y$ to see if $(x, y)$ satisfies the equation $y = f(x)$.
  - Create a set of ordered pairs that are solutions to the equation $y = f(x)$.
- Plot the ordered pairs.

**The graph of a function**
- For-Next Loop
  - For each $x$, determine $f(x)$.
  - Create a set of ordered pairs $(x, f(x))$.
- Plot the ordered pairs.
Creating New Functions from Old Functions

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = x^2$</th>
<th>$g(x) = f(2x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
<td>36</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>16</td>
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<td>3</td>
<td>9</td>
<td>36</td>
</tr>
</tbody>
</table>

**Closing:** Discuss how the horizontal scaling by a scale factor of $k$ of the graph of a function $y = f(x)$ corresponds to changing the equation of the graph from $y = f(x)$ to $y = f(kx)$.

How does a solid understanding of functions help with transformations?
Algebra I Exit Ticket

Let $f(x) = x^2$, $g(x) = (3x)^2$, and $h(x) = \left(\frac{1}{3}x\right)^2$, where $x$ can be any real number. The graphs above are of $y = f(x)$, $y = g(x)$, and $y = h(x)$.

1. Label each graph with the appropriate equation.

2. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = g(x)$. Use coordinates of each to illustrate an example of the correspondence.

Geometric transformations are functions!
The graph of a piecewise function $f$ is shown below.

Let $p(x) = f(x - 2)$, $q(x) = \frac{1}{2}f(x - 2)$, and $r(x) = \frac{1}{2}f(x - 2) + 3$.

Graph $y = p(x)$, $y = q(x)$, and $y = r(x)$ on the same set of axes as the graph of $y = f(x)$.
Functions in Algebra II and Advanced Math

• Defining New Functions and Modeling with Functions
  • Sine and cosine functions
  • Logarithm functions
  • Rational functions

• Building New Functions
  • Extending transformations of graphs of functions to new function families
  • Inverses of functions
  • Composition of functions

• Matrix operations, vectors, and linear transformations
  • Geometric transformations
  • Representations
References


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