A Deep Dive into Fraction Operations

"I'm going to dress up as a fraction for Halloween because I can't think of anything scarier." –Former 6th grade student

Avery Pickford
5th & 6th Grade Math Teacher
The Nueva School

avery.pickford@gmail.com
@woutgeo

Session #392 (3012 Moscone West)

Fraction Operations

- Conceptual understanding of the operation
- Flexible methods that promote conceptual understanding of procedure
- Using low-bar high-ceiling problems
- Assessment

Potentially Controversial Statement #1:

Students don't *really* understand fractions.

Potentially Controversial Statement #1:

~~Students~~ People don't *really* understand fractions.

Potentially Controversial Statement #1:

~~Students People don't *really* understand fractions.~~

College teachers: Stop blaming high school teachers for your students' lack of deep understanding of fractions.

High school teachers: Stop blaming middle school teachers for your students' lack of deep understanding of fractions.

Middle school teachers: Stop blaming elementary school teachers for your students' lack of deep understanding of fractions.

Mistakes are expected, inspected, and respected.

Saying you're finished learning fractions is like saying you're finished learning to paint with watercolors.

Practice can be useful, but exercises should be problematized.

Students should be creators of mathematics, not just consumers.

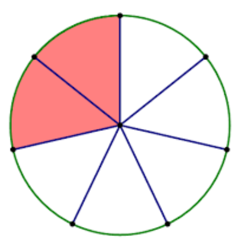
Defining a Fraction

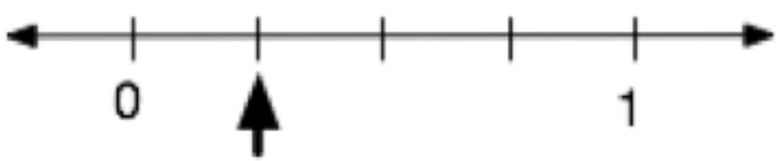
Is this a fraction?

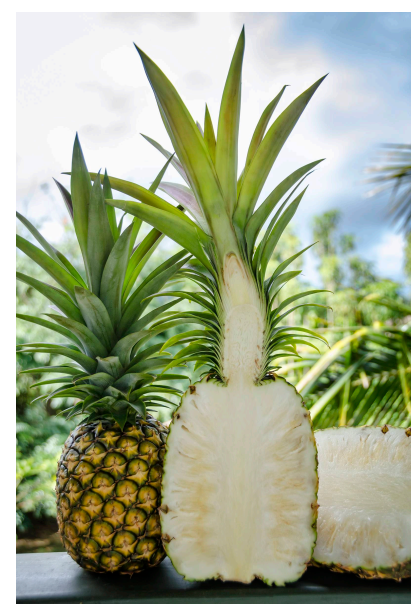
$\frac{3}{7}$ $\frac{0}{9}$ $\frac{2}{0}$ 4 π

$\frac{\pi}{3}$ $\frac{4}{2}$ $2\frac{3}{4}$ $2\frac{4}{3}$ 25%

0.25

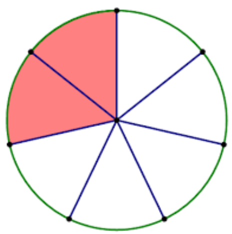






Fractions can mean many different things depending on context.

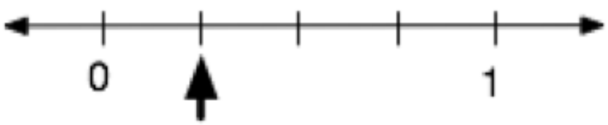
Parts of a whole



Parts of a group

Ratio

The name for a point



Division

Conceptual Understanding of Addition

What does it mean to add?

What does it mean to add 3 oranges to 2 apples?

How is this related to fractions?

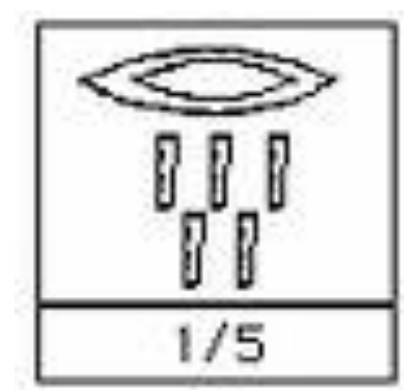
What might the 3 and the 4 represent in the fraction $\frac{3}{4}$?

Adding Fractions:
Flexible methods

$4\frac{3}{8} + 7\frac{1}{6}$

Egyptian Fractions

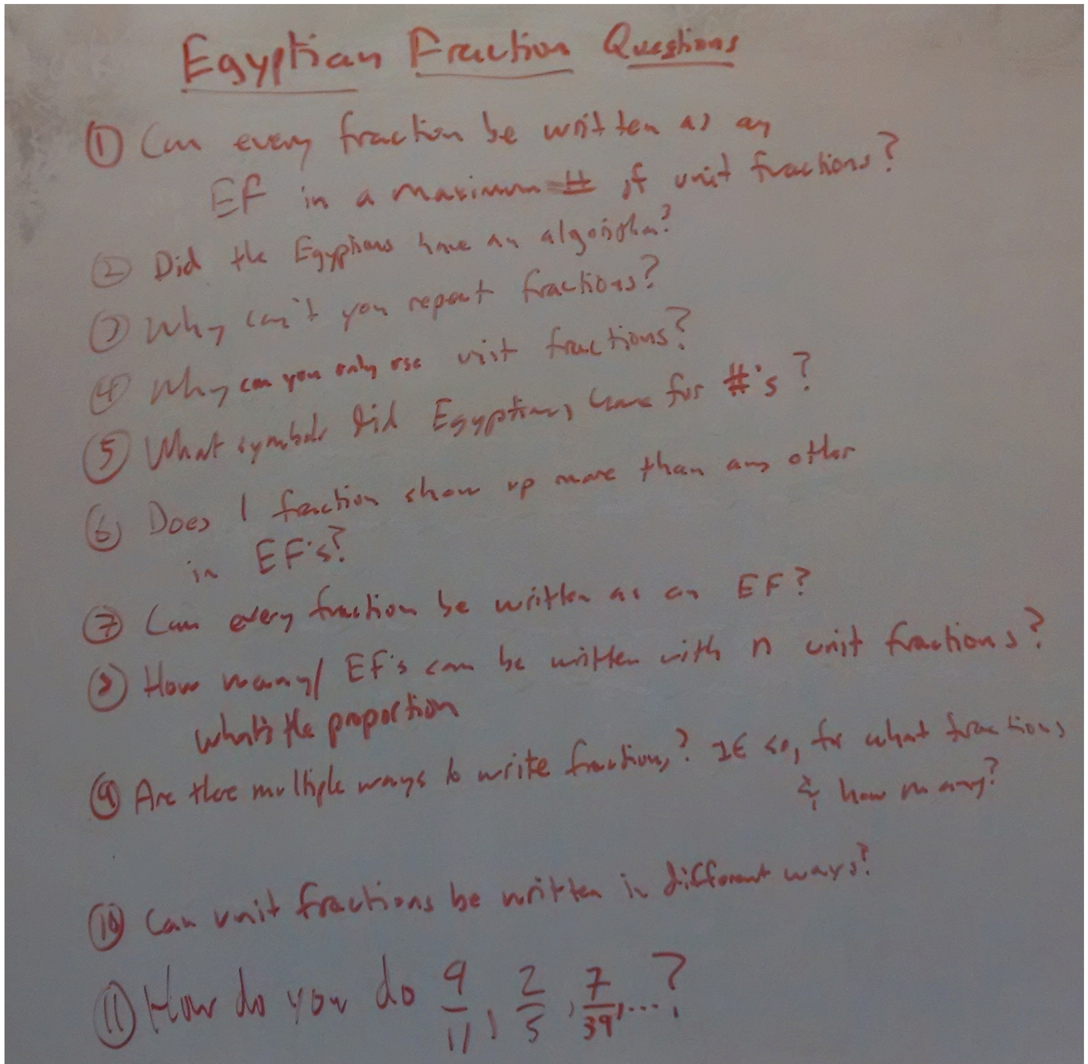
- Only used unit fractions (**1 in the numerator)**
- Could describe amounts using **addition problems** (similar to how we describe mixed numbers)
- Every fraction in your addition problem **must be different**
- As an example, Egyptians would write 3/8 as 1/4 + 1/8.



Egyptian Fractions: Low-bar high-ceiling problem

- For some, it's trial and error practice adding fractions
- For some, it's coming up with interesting things to notice/wonder
- For some, it's proving the things they notice/wonder

5th Grade Questions



Some Possible Questions/Proofs

- Prove that every rational number be written as an Egyptian Fraction?
- Fractions be written as Egyptian fractions in more than one way. Always, sometimes, or never?
- Find an algorithm for finding an Egyptian fraction for fractions of the form 3 over an even number, i.e. 3/2n

Erdős–Straus Conjecture: Another low-bar high-ceiling problem

The fraction 4/n can be written as an Egyptian fraction with three unit fractions.

Adding Fractions:

Your reminder that they're probably not experts yet.

1. In the final regular season game of the Warrior's (basketball) historic season, Stephen Curry made 8 out of 12 shots in the first half and 6 out of 9 shots in the second half of yesterday's game (true story). What fraction of his shots did he make in the game?

2. Find 1+1/2+1/4+1/8+… (so on and so on forever)

3. $$1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{4}}}}}.$$

Conceptual Understanding of Subtraction

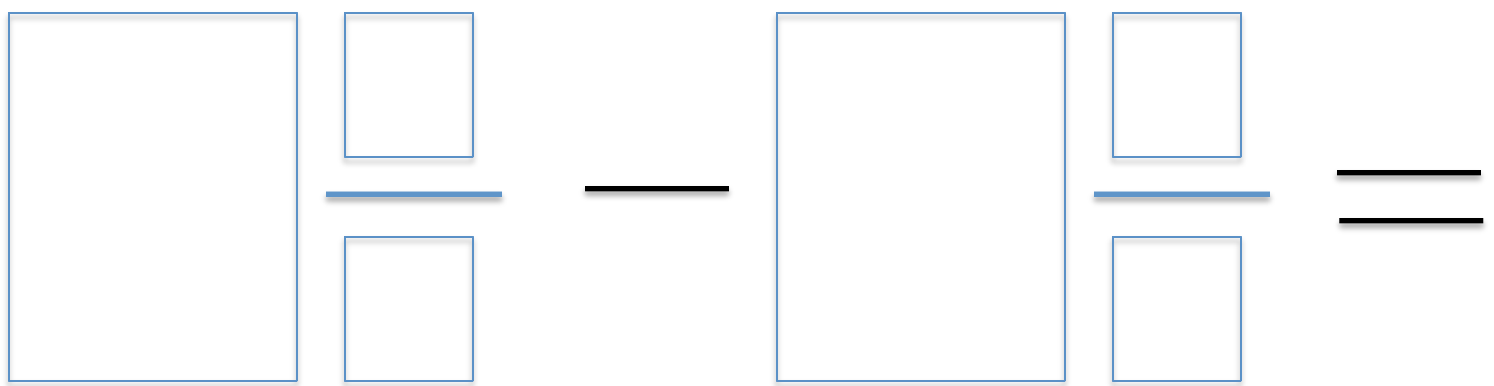
What does it mean to subtract?

How is subtracting fractions similar/different from adding fractions?

Subtracting Mixed Numbers: Flexible methods

- Converting to improper fractions
- Regrouping
- Using negative intermediate results

Number Lottery



Solitaire

Using all the numbers 1 through 9, create a true equation.

$$\Box\,\frac{\Box}{\Box} - \Box\,\frac{\Box}{\Box} = \Box\,\frac{\Box}{\Box}$$

Inspired by Kent Haines at http://www.kenthaines.com/blog/2016/2/19/integer-solitaire

Subtracting Fractions: Where's the cognitive demand?

- For some, it's with the procedure of subtracting fractions
- For some, it's coming up with strategies to optimize answer or noticing/wondering
- For some, it's calculating probabilities or proving things they've noticed/wondered

Subtracting Fractions:

Your reminder that they're probably not experts yet.

1. Will the expression $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ be more than $\frac{1}{2}$ or less than $\frac{1}{2}$? Explain your reasoning.

2. $1 - \frac{1}{10} - \frac{1}{100} - \frac{1}{1000} - \cdots - \frac{1}{10^{10}} =$

3. Create and describe a procedure for finding the fraction exactly between two given fractions.

Multiplying Fractions: Models for multiplication

Example: 4x3

Repeated addition

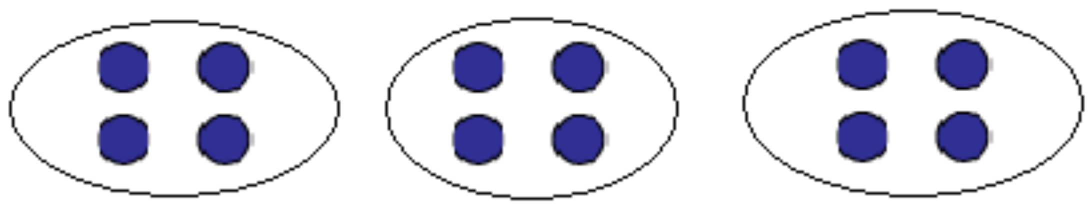
4 + 4 + 4

Multiplying Fractions: A conceptual understanding

Example: 4x3

Repeated addition (4+4+4)

Groups of



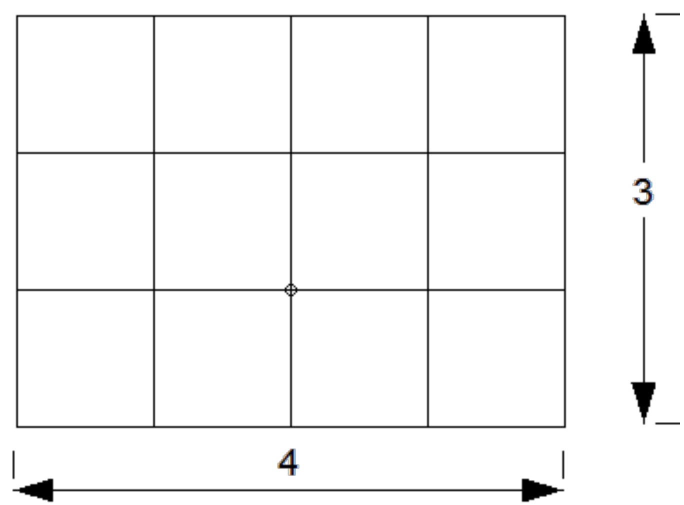
Multiplying Fractions: A conceptual understanding

Example: 4x3

Repeated addition (4+4+4)

Groups of (3 groups of 4)

Area (area of a 4 by 3 rectangle)



Multiplying Fractions: A conceptual understanding

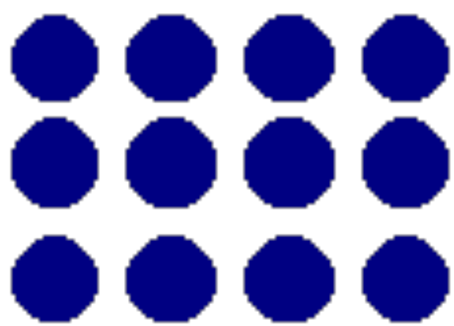
Example: 4x3

Repeated addition (4+4+4)

Groups of (3 groups of 4)

Area (area of a 4 by 3 rectangle)

Array



Multiplying Fractions: A conceptual understanding

Example: 4x3

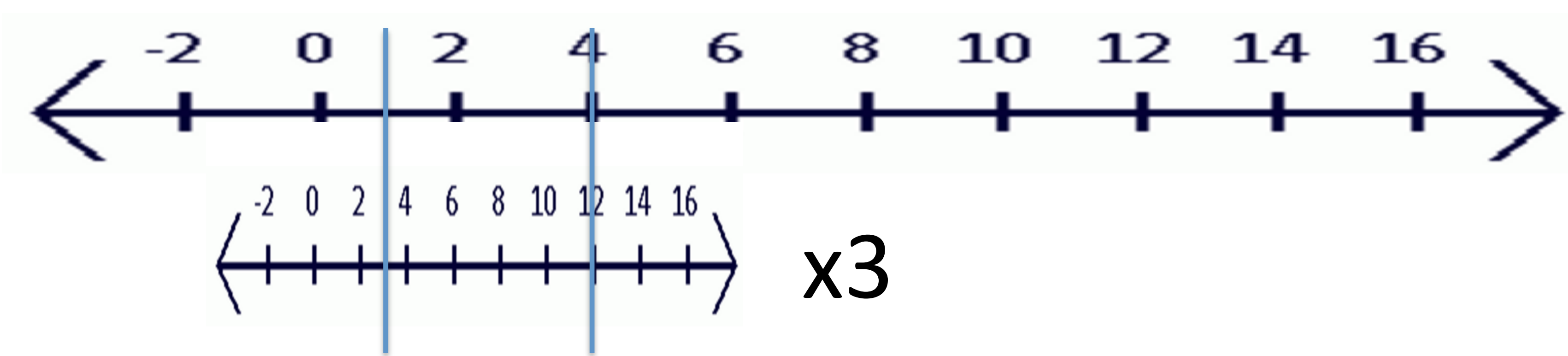
Repeated addition (4+4+4)

Groups of (3 groups of 4)

Area (area of a 4 by 3 rectangle)

Array (number of dots in a 4 by 3 array)

Scaling of number line



x3

Multiplying Fractions: Flexible methods

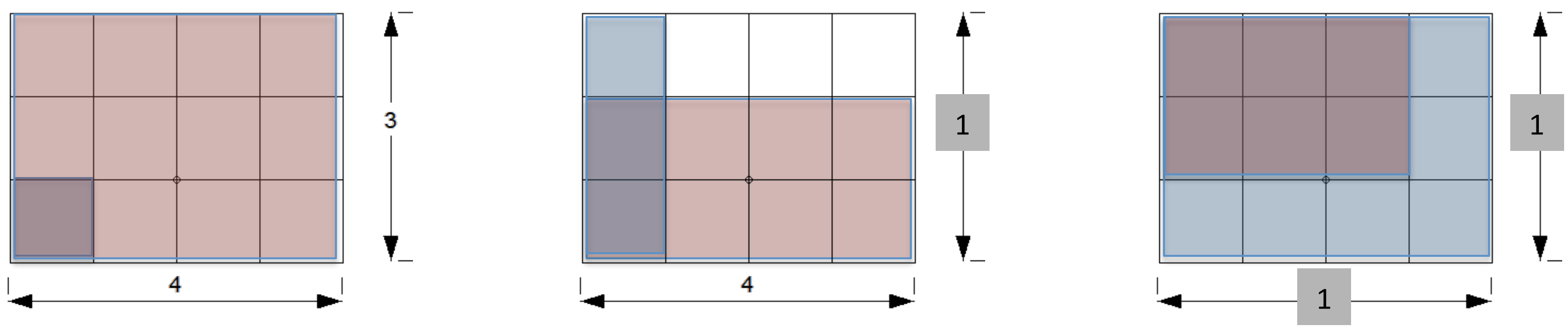
- Which of the previous models can we (should we) use for different problems?

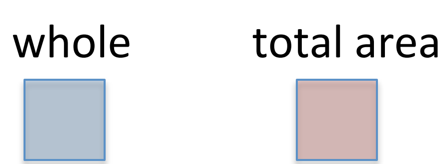
Potentially Controversial Statement #2:

We should NOT make the math as easy as possible for students.

Multiplying Fractions: Using the area model

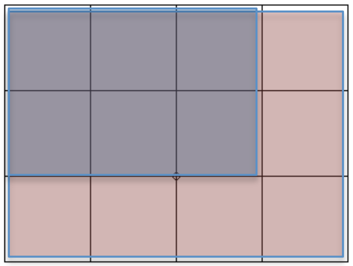
$4 \bullet 3$ $\qquad$ $4 \bullet \frac{2}{3}$ $\qquad$ $\frac{3}{4} \bullet \frac{2}{3}$



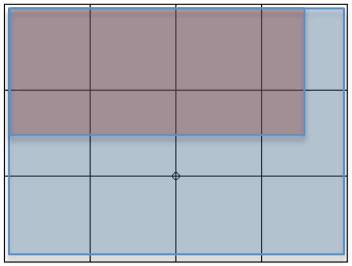


Multiplying Fractions: Using the area model

$$1\frac{1}{3} \bullet 1\frac{1}{2}$$



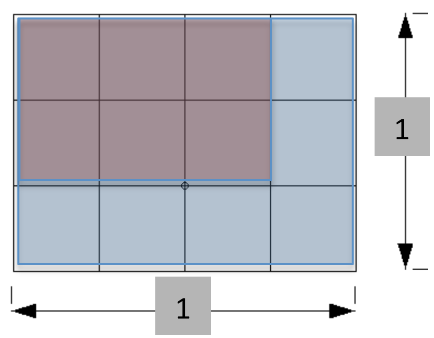
$$\frac{3\,{}^{1}\!/_{2}}{4} \bullet \frac{1\,{}^{1}\!/_{2}}{3}$$



Multiplying Fractions: Why I like this.

- Connects to standard procedure

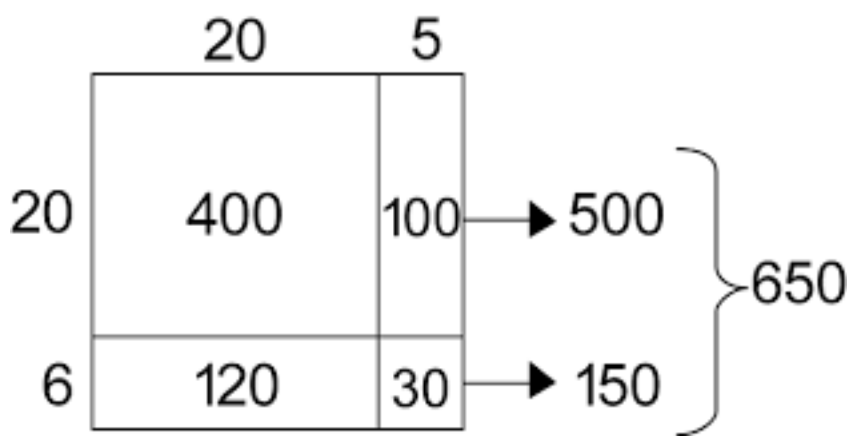
$$\frac{3}{4} \bullet \frac{2}{3} = \frac{6}{12}$$



Multiplying Fractions: Why I like this.

- Useful with whole numbers

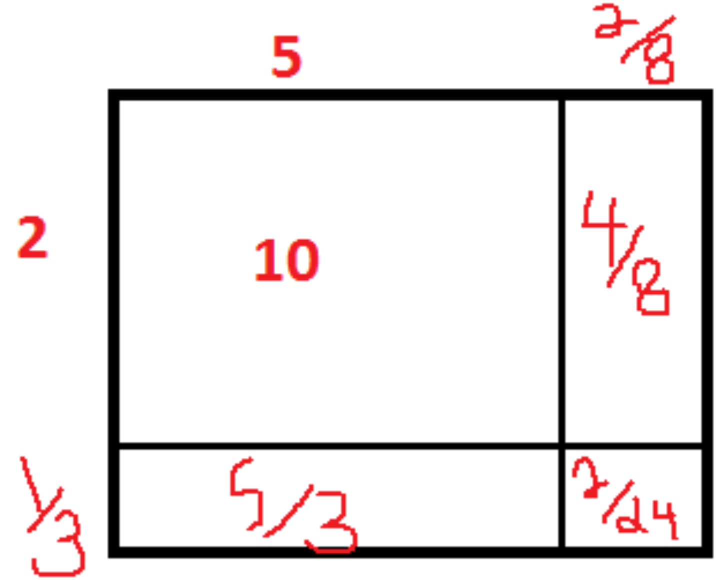
$$26 \bullet 25 = 650$$



Multiplying Fractions: Why I like this.

- Intermediate step of efficiency

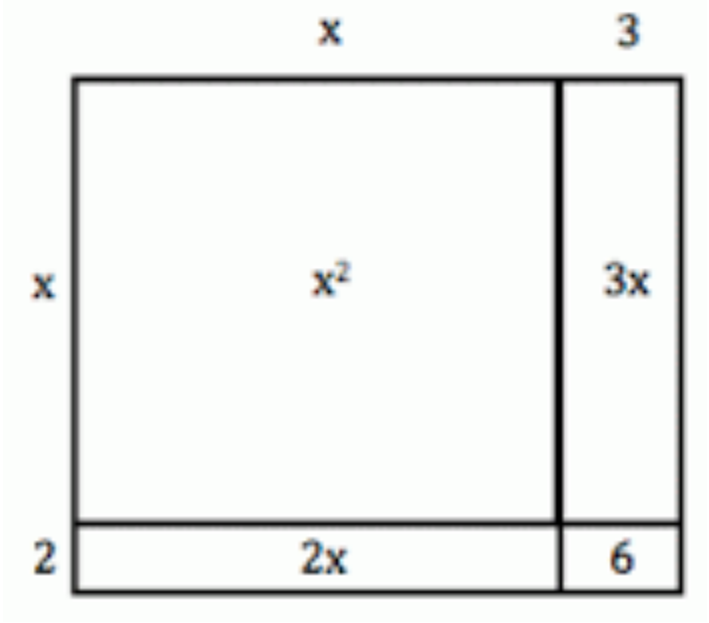
$$2\frac{1}{3} \bullet 5\frac{2}{8}$$



Multiplying Fractions: Why I like this.

- Useful in the future

$$(x+2)(x+3) = x^2 + 5x + 6$$



The Orange Juice Problem

You have a pitcher of orange juice and a pitcher of lemonade. You take a tablespoon of the lemonade and mix it thoroughly into the orange juice. You then take a tablespoon of the orange juice (with the mixed in lemonade) and pour it back into the lemonade pitcher. Which is greater and why, the amount of lemonade in the orange juice mixture or the amount of orange juice in the lemonade mixture?

Multiplying Fractions: Where's the cognitive demand?

- For some, it's making a physical model
- For some, it's exploring whether the size of the pitchers matters
- For some, it's figuring out how much lemonade would be in the orange juice if you repeated this 10 times

Multiplying Fractions:
Your reminder that they're probably not experts yet.

1. Use the "groups of" concept of multiplication to solve the problem $2\frac{1}{3} \cdot 5\frac{2}{8}$.

2. Find $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{98}{99} \cdot \frac{99}{100}$.

Potentially Controversial Statement #3:

Dividing fractions is the most difficult concept in all of K-12 mathematics.

Dividing:
A conceptual understanding

$$12 \div 3$$

Repeated addition/subtraction (chunking)

How many 3's must I add to get to 12?

$$\underbrace{3 + 3 + 3 + 3}_{4} = 12$$

Dividing:
A conceptual understanding

$$12 \div 3$$

Repeated addition/subtraction

Split into a certain number of groups (partitioning)

12 objects are evenly split into 3 groups. How many objects are in 1 whole group?

1	1	1	1
1	1	1	1
1	1	1	1

← 1 whole =
4 objects

Dividing:
A conceptual understanding

$$12 \div 3$$

Repeated addition/subtraction

Split into a certain number of groups

Split into certain sized groups

12 objects are evenly split into groups of 3. How many groups are there?

1 1 1	1 1 1
1 1 1	1 1 1

↑ 1 group

4 groups total

Dividing:
A conceptual understanding

$$12 \div 3$$

Repeated addition/subtraction

Split into a certain number of groups

Split into certain sized groups

Fractions

What is the reduced value of $\frac{12}{3}$?

Dividing:
A conceptual understanding

$$12 \div 3$$

Repeated addition/subtraction

Split into a certain number of groups

Split into certain sized groups

Fractions

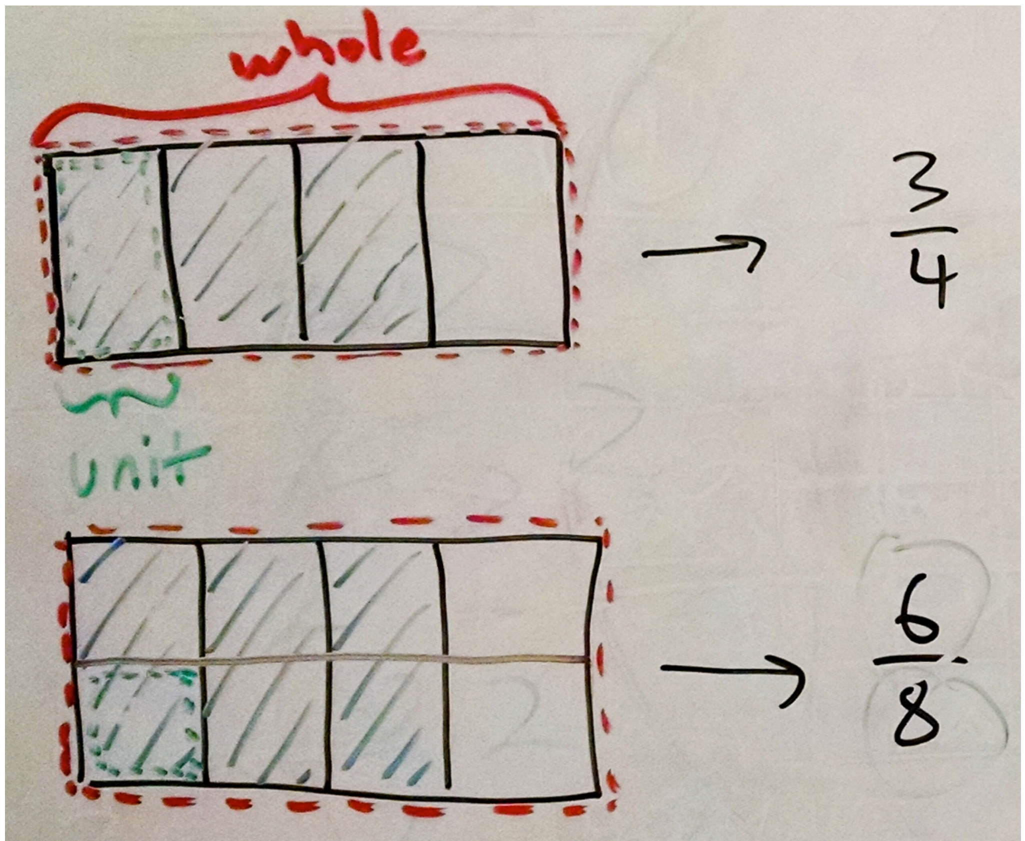
Inverse of multiplication

3 times what equals 4?

$$3 \bullet \_\_ = 12$$

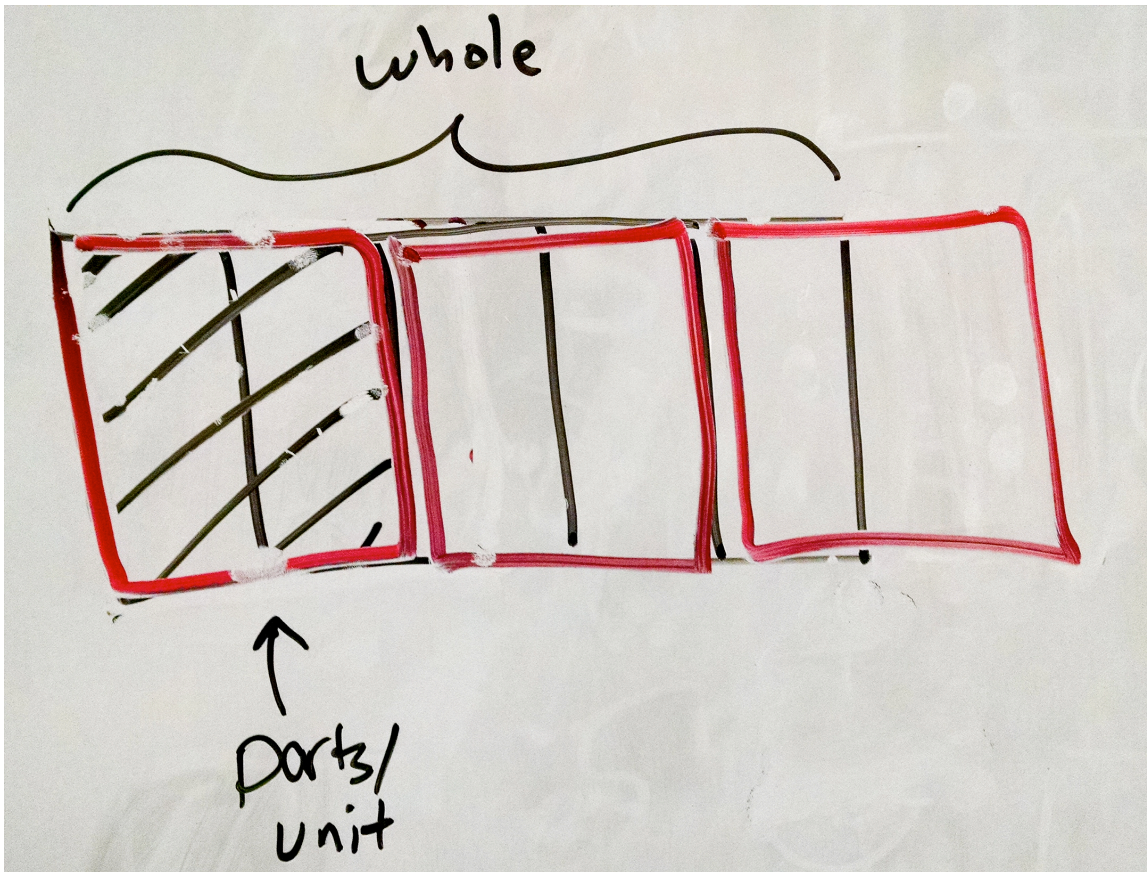
Dividing Fractions: Important Scaffolding

- Finding the fraction given the whole and the unit



Dividing Fractions: Important Scaffolding

- Finding the fraction given the whole and the unit



$$\frac{1}{2\frac{1}{2}}$$

Dividing Fractions: Important Scaffolding

Finding equivalent fractions by multiplying by "fancy ones."

Dividing Fractions: Important Scaffolding

The concept of the multiplicative inverse (both how to find it and why it is useful)

Rational tangles: http://www.geometer.org/mathcircles/tangle.pdf

Dividing Fractions: Flexible methods

Can you describe different division

Dividing Fractions: Flexible methods

Name: ______________________

Concepts of Division: dividend ÷ divisor = quotient

	1) $12 \div 3$	**2)** $12 \div \frac{1}{3}$	**3)** $12 \div \frac{3}{4}$	**4)** $\frac{3}{4} \div \frac{3}{8}$	**5)** $\frac{15}{17} \div \frac{3}{17}$	**6)** $8\frac{3}{4} \div 1\frac{3}{4}$
A. Repeated addition or subtraction (chunking)	Q: How many 3's must I add to get to 12? $\underbrace{3+3+3+3}_{4} = 12$	Q:	Q:	Q:	Q:	Q:
B. Split into a certain number of groups (partitioning)	Q: 12 objects are evenly split into 3 groups. How many objects are in 1 whole group? 1 1 1 1 1 1 1 1 1 1 1 1 ← 1 whole = 4 objects	Q:	Q:	Q:	Q:	Q:
C. Split into certain sized groups	Q: 12 objects are split into groups of 3. How many groups are there? 1 1 1 | 1 1 1 1 1 1 | 1 1 1 ↑ 1 group 4 groups total	Q:	Q:	Q:	Q:	Q:
D. Fractions	Q: What is the reduced value of $\frac{12}{3}$? $12 \times {}^{1}/_{3}$ $3 \times {}^{1}/_{3}$ $= \frac{4}{1}$ $= 4$	Q:	Q:	Q:	Q:	Q:
E. Multiplication	Q: 3 times what equals 12? $3 \cdot 4 = 12$	Q:	Q:	Q:	Q:	Q:

Dividing Fractions: Flexible methods

	7) $3 \div 12$	8) $\frac{1}{3} \div 12$	9) $\frac{3}{4} \div 12$	10) $\frac{3}{8} \div \frac{3}{4}$	11) $\frac{3}{4} \div 7\frac{1}{2}$	12) $1\frac{3}{4} \div 8\frac{3}{4}$
A Repeated addition or subtraction (chunking)	Q:	Q:	Q:	Q:	Q:	Q:
B Split into a certain number of groups (partitioning)	Q:	Q:	Q:	Q:	Q:	Q:
C Split into certain sized groups (quotative)	Q:	Q:	Q:	Q:	Q:	Q:
D Fractions	Q:	Q:	Q:	Q:	Q:	Q:
E Multiplication	Q:	Q:	Q:	Q:	Q:	Q:

Chocolate Bar Problem

4 tables each have some bars of chocolate on them. Everyone in the class will first secretly pick a table they would like to sit at. Once everyone is at their table, the chocolate is distributed evenly.

Which table would you like to sit at?

Table 1	Table 2	Table 3	Table 4
$2\frac{2}{3}$	$\frac{4}{5}$	$1\frac{2}{3}$	$1\frac{4}{7}$

Dividing Fractions: Where's the cognitive demand?

- For some, it's with the division
- For some, it's coming up with a strategy
- For some, it's exploring this problem with different numbers of people/amounts

Dividing Fractions:

Your reminder that they're probably not experts yet.

1. Reduce the fraction $\dfrac{1}{\frac{2}{\;}\!\big/\frac{3}{4}}$.

- 3/5 of the people in a cafe are seated in 2/5 of the chairs. The rest of the people in the room decide to stand. If there are 27 empty chairs, how many people are standing?

A Deep Dive into Fraction Operations

Avery Pickford

avery.pickford@gmail.com

@woutgeo

bit.ly/NCTM2016fractions

Want More?

I'll be leading/co-leading 2 week-long courses at Stanford University as part of their Center to Support Excellence in Teaching. Get more info and sign-up at:

https://cset.stanford.edu/pd/courses/math