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Building Proficiency in Mathematical Modeling

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What is Mathematical Modeling?

At NCSSM, we consider a mathematical experience in which students make choices about how to use mathematics to create representations of a real-world process to be a form of mathematical modeling.

Modeling is the process of creating representations (models) that help us understand a phenomenon while using *mathematical* concepts and the principles and language of mathematical symbolism.

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What is Mathematical Modeling?

“Mathematical Modeling is when you use mathematics to understand a situation in the real world, and then perhaps use it to take action or even to predict the future, and where both the real world situation and the ensuing mathematics are taken seriously.”

“What is Mathematical Modeling”.
H. Pollak, Teachers College, Columbia University.

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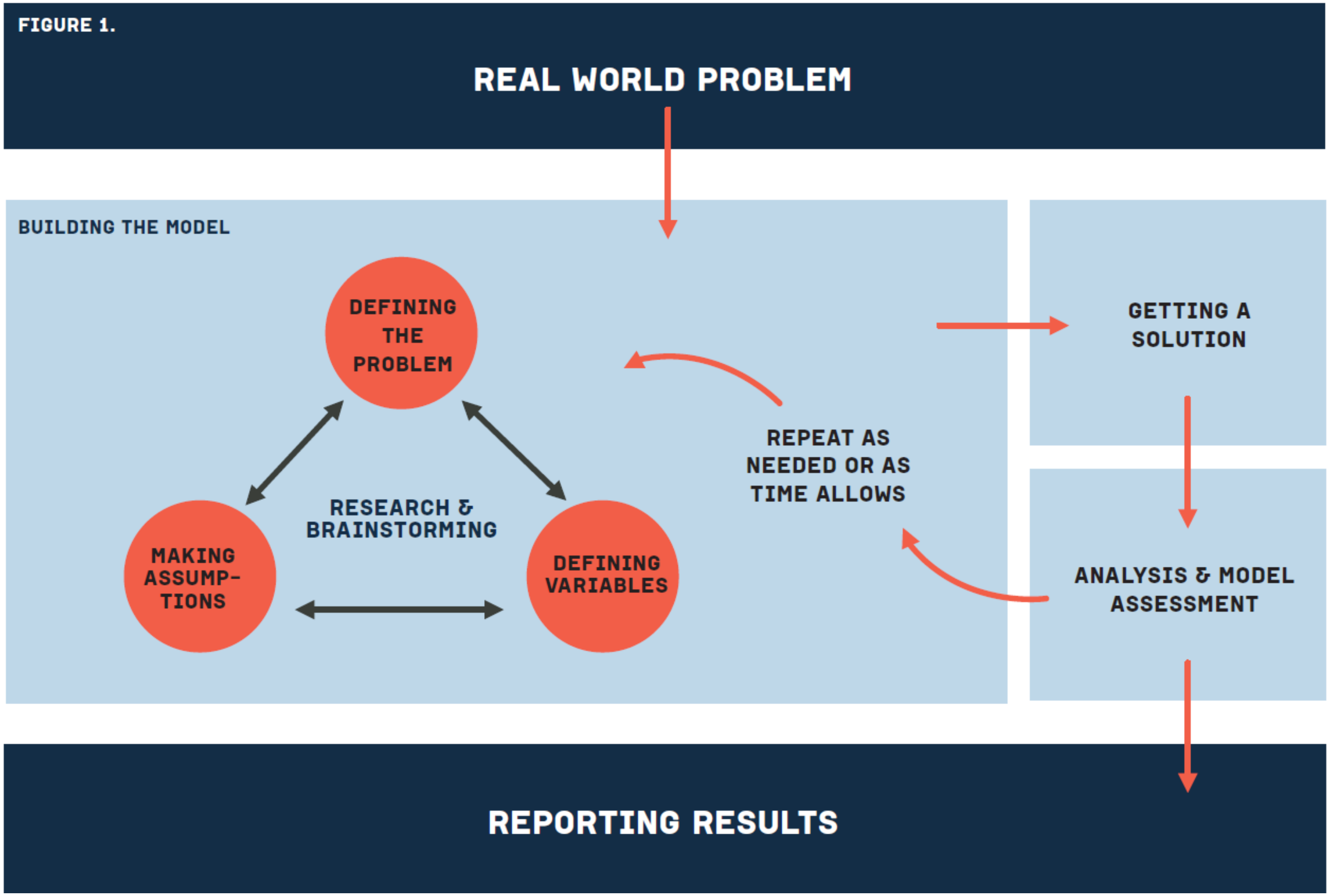
Is it Modeling?

Are the students *remembering*?

or

Are the students ***thinking***?

The Modeling Cycle



From SIAM *Modeling Getting Started and Getting Solutions Handbook*
https://m3challenge.siam.org/resources/modeling-handbook

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A Note about Modeling

For students to develop their mathematical modeling skills, it is important for them to be free to make decisions and try new ideas at each stage in the modeling process.

Students, not their teachers, need to make important decisions about where to focus their attention, how to proceed, and, later in the process, how to evaluate or assess their models.

Students' decisions and their own creativity drive the modeling process.

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An Obstacle to Success

"Frustration is often a first reaction on the part of students. For many, their past mathematics classroom experiences have led them to believe that when given a problem, they are supposed to be able to immediately search for, identify and apply the correct procedure. Thus, when they are unable to identify a particular procedure right away, they feel the problem is unfair, or that the teacher has poorly prepared the students for the task. . . .

Further, many students and teachers find it difficult to tolerate the inefficient approaches and wrong directions that typically surface early in the modeling episodes."

Zawojewski, Judith S., Richard A. Lesh, and Lyn D. English. "A Models and Modeling Perspective on the Role of Small Group Learning Activities." In *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching,* edited by Richard Lesh and Helen M. Doerr, pp. 337–58. Mahwah, N.J.: Lawrence Erlbaum Associates, Inc., 2003.

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Ease in to Modeling

Incorporating smaller problems that require students to think and struggle is beneficial in its own right.

This also helps prime the students for more extensive modeling in the future.

One way this can be done is by giving students problems to work on before showing them a new technique.

Another approach can be to take away some details of a problem so that they must make decisions about what information they need.

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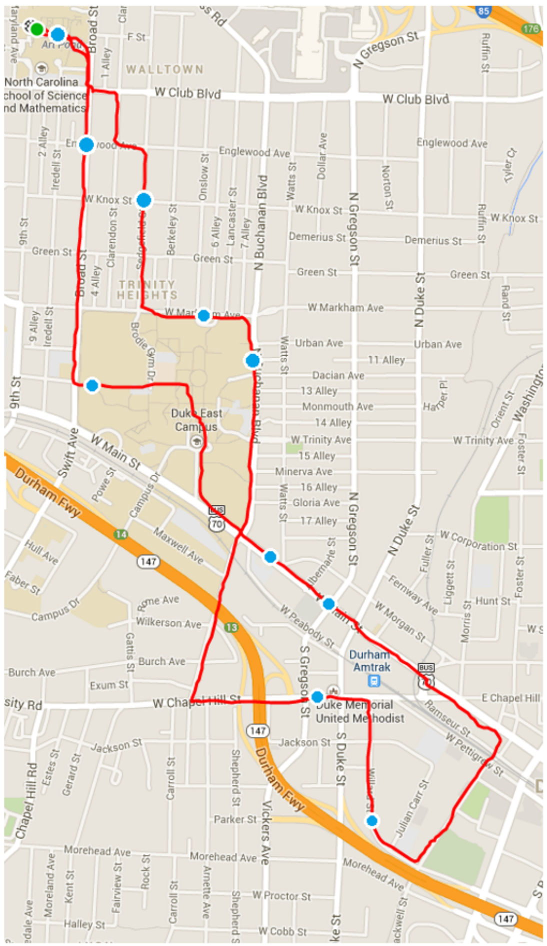
Running Data

During a run through Durham, the following data were collected:



Time (h:mm:ss)	Distance Travelled (miles)	Elevation (ft)
0:00:34	0.1	414
0:02:47	0.4	384
0:07:21	1.0	396
0:12:36	1.7	409
0:14:11	1.9	383
0:21:49	2.9	344
0:25:49	3.3	416
0:34:17	4.4	410
0:36:22	4.6	414
0:39:42	5.0	366
0:44:40	5.5	408

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Running Data

Instead of asking a Calculus class "What is the average speed of the runner between $t = 0$ and $t = 7{:}21$?",

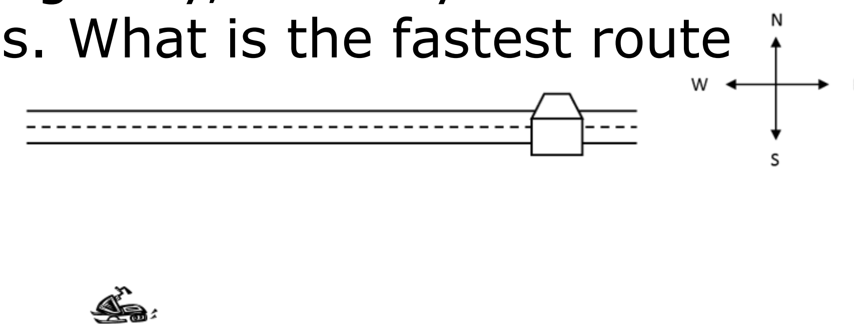
I ask "What questions could we explore with this data?"

Some questions generated were:

- *How did the speed change on inclines/declines?*
- *What was the runner's pace?*
- *How did speed vary over the run?*
- *How fast was the runner going at some particular time?*

Minimizing Travel Time

Your snowmobile is out of gas, and you are 3 miles due south of a major highway. The nearest service station is on the highway 6 miles east, and it closes at midnight. You can walk 4 miles per hour along the highway, but only 3 miles per hour through the snowy fields. What is the fastest route to take?

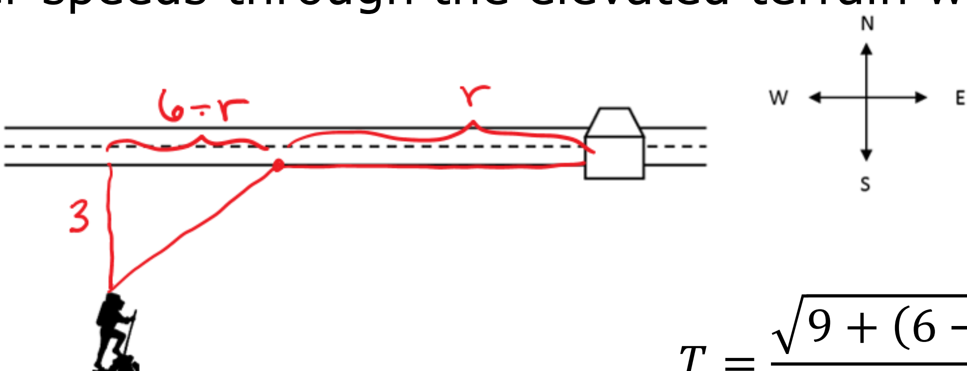


When given after similar problems, or after doing many optimization problems, the students are remembering…but if given before talking about what approach to take the students have more thinking to solve the problem and see why creating a model for time would be useful.

But we can also make it more open ended…

Minimizing Travel Time – A variation

It is well known that hilly/rocky terrain slows one's walking speed (common estimates specify that the average walking speed is about 3 miles in an hour on level ground with an extra hour added in for each 2000 feet of elevation). Suppose a hiker is walking along elevated terrain toward a level road. The hiker is 3 miles due south of the road and traveling toward a rest area that is 6 miles east along the road. Assume the hiker can travel 3 mph on the road. Explore how different slower speeds through the elevated terrain will affect her optimal path.



$$T = \frac{\sqrt{9 + (6-r)^2}}{k} + \frac{r}{3}, \qquad 0 \le r \le 6$$

where k = speed through elevated terrain

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Energy Expenditure in Salmon

Salmon swim against the current to return to their birthplace to spawn. Researchers have studied the way they swim upstream and have discovered by observation that usually the velocity of the salmon is about 1.5 times the rate of the current of the stream. The salmon chose this rate because instinct leads them to swim at a velocity that uses the minimum energy possible. Researchers also realize that any salmon who swims near the rate of the current will make little progress and will expend much energy on the trip.

We will make several assumptions. First, the longer the salmon swims the more energy is expended. If the salmon swims at a rate "close" to the rate of the stream, they will expend more energy. Based on the two clues and the two assumptions, we can develop a function to describe the energy of the salmon as it swims upstream.

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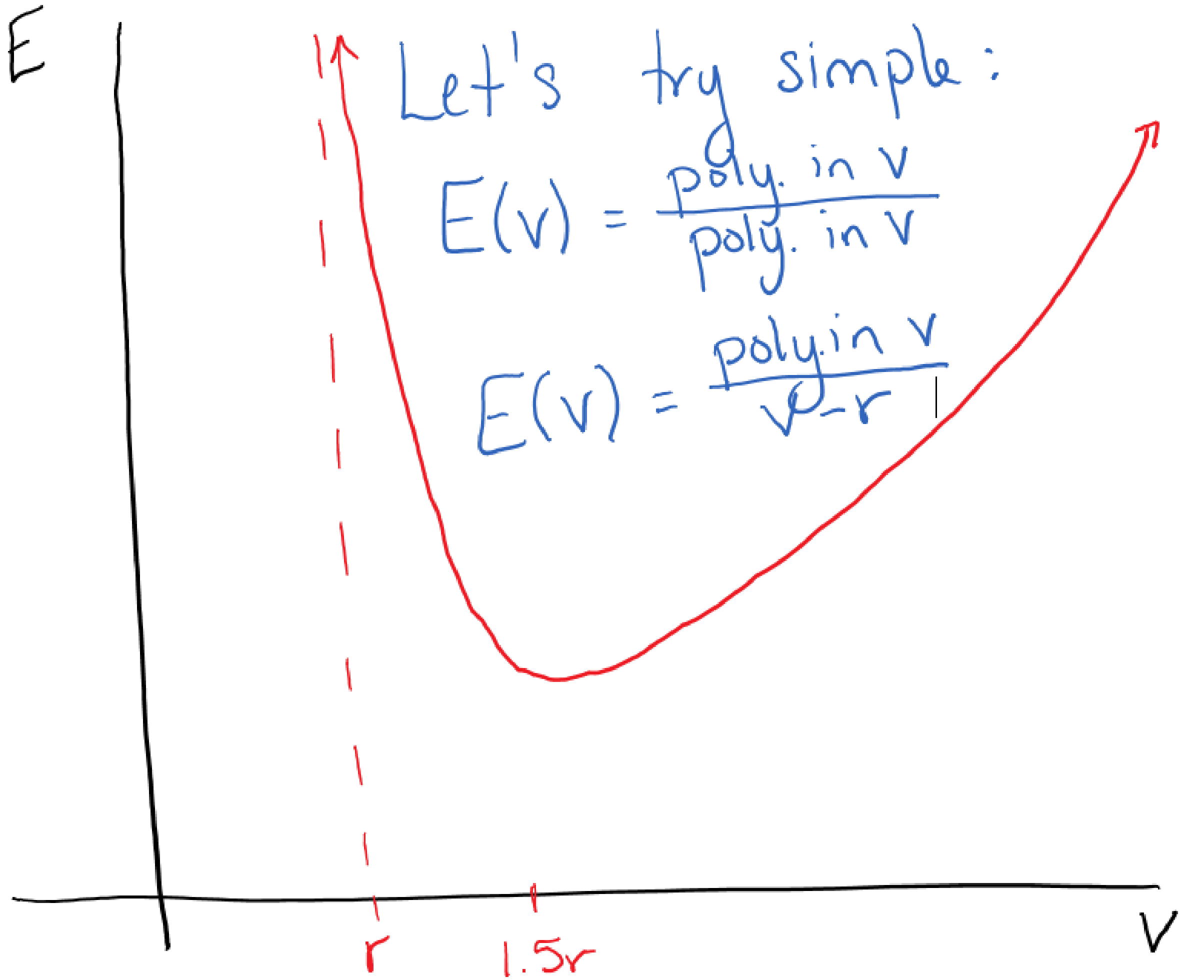
Energy Expenditure in Salmon



Let's try simple:

$$E(v) = \frac{\text{poly. in } v}{\text{poly. in } v}$$

$$E(v) = \frac{\text{poly. in } v}{v-r}$$

Energy Expenditure in Salmon

Simplest with $v - r$ in the denominator: $E(v) = \frac{1}{v-r}$

This doesn't work due to incorrect long term behavior…

Next effort: $E(v) = \frac{v}{v-r}$

This doesn't work since this function will not have a local minimum.

Then we can explore $E(v) = \frac{v^p}{v-r}$ for various values of p.

Through exploration or using calculus we find that $p = 3$ gives the minimum at the correct location.

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A Teacher’s Role in Modeling

A teacher’s first job in the modeling process, is offering appropriate problems.

The social and biological sciences offer many nice problems that do not require a technical background. The modeling process will go more smoothly if you begin by engaging the students in the problem in some way such as with discussion, videos, or simulation.

The students’ experiences or lack of experience with open ended modeling problems will affect the problem choices.

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A Teacher's Role in Modeling

Throughout the modeling activity, the teacher is listening to their ideas and asking questions that might cause them to rethink their approach.

Students may need suggestions about a new direction to try, encouragement to continue pursuing their current approach, or nudging to abandon an unproductive strategy.

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Mantids & Modeling

Some modeling activities are tightly focused, giving students limited ability to “go their own way” as they create a solution to a problem, but decisions must be made.

These work well for students with limited experience in modeling.

In the Mantid problem, modeling comes in the form of choosing appropriate functions to represent data, as well as making decisions about how to combine and transform functions.

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Reaction Distance vs. Satiation

Satiation (cg)	11	18	23	31	35	40	46	53	59	66	70	72	75	86	90
Reaction Distance (mm)	65	52	44	42	34	23	23	8	4	0	0	0	0	0	0

We will make a scatter plot of the data to determine what type of function will best fit the data.

http://betterlesson.com/community/lesson/547204/the-math-of-mantids-a-math-lesson-involving-functions-graphing-and-bugs

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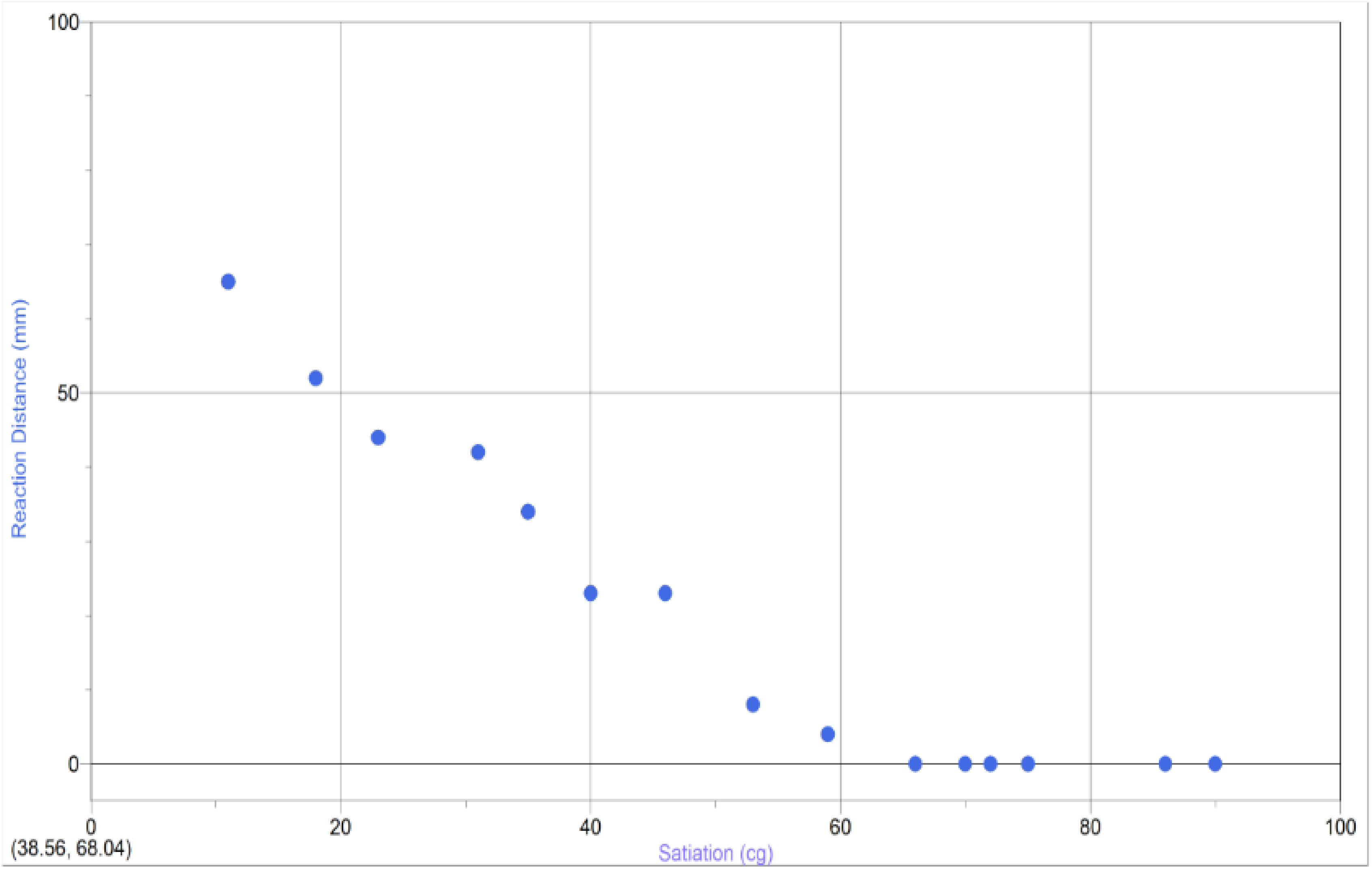
Reaction Distance vs. Satiation



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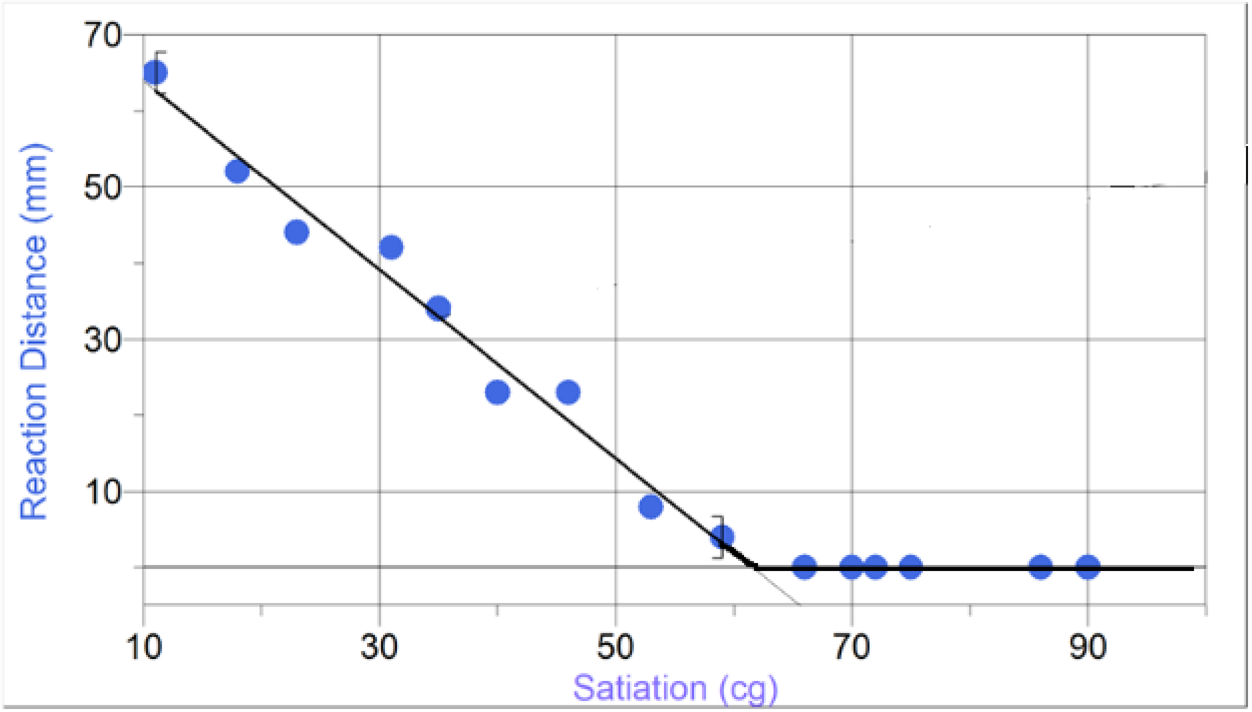
Finding a Model

The function below models the relationship between Satiation (S) and the Reaction Distance (R) a mantid will travel for food.

$$R(S) = \begin{cases} -1.24S + 76.26 & \text{if } 0 \leq S < 61.5 \\ 0 & \text{if } S \geq 61.5 \end{cases}$$



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Interpreting the Model

The (non-zero) slope for our model is -1.24.

This tells us that according to our model, for every additional centigram of food in the mantid's stomach, it is willing to travel approximately 1.24 millimeters *less* to get to food.

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Satiation vs. Time

Time (hr)	0	1	2	3	4	5	6	8	10
Satiation (cg)	94	90	85	82	88	83	70	66	68
Time (hr)	12	16	19	20	24	28	36	48	72
Satiation (cg)	50	46	51	41	32	29	14	17	8

We will again make a scatter plot of the data to determine what type of function will fit best.

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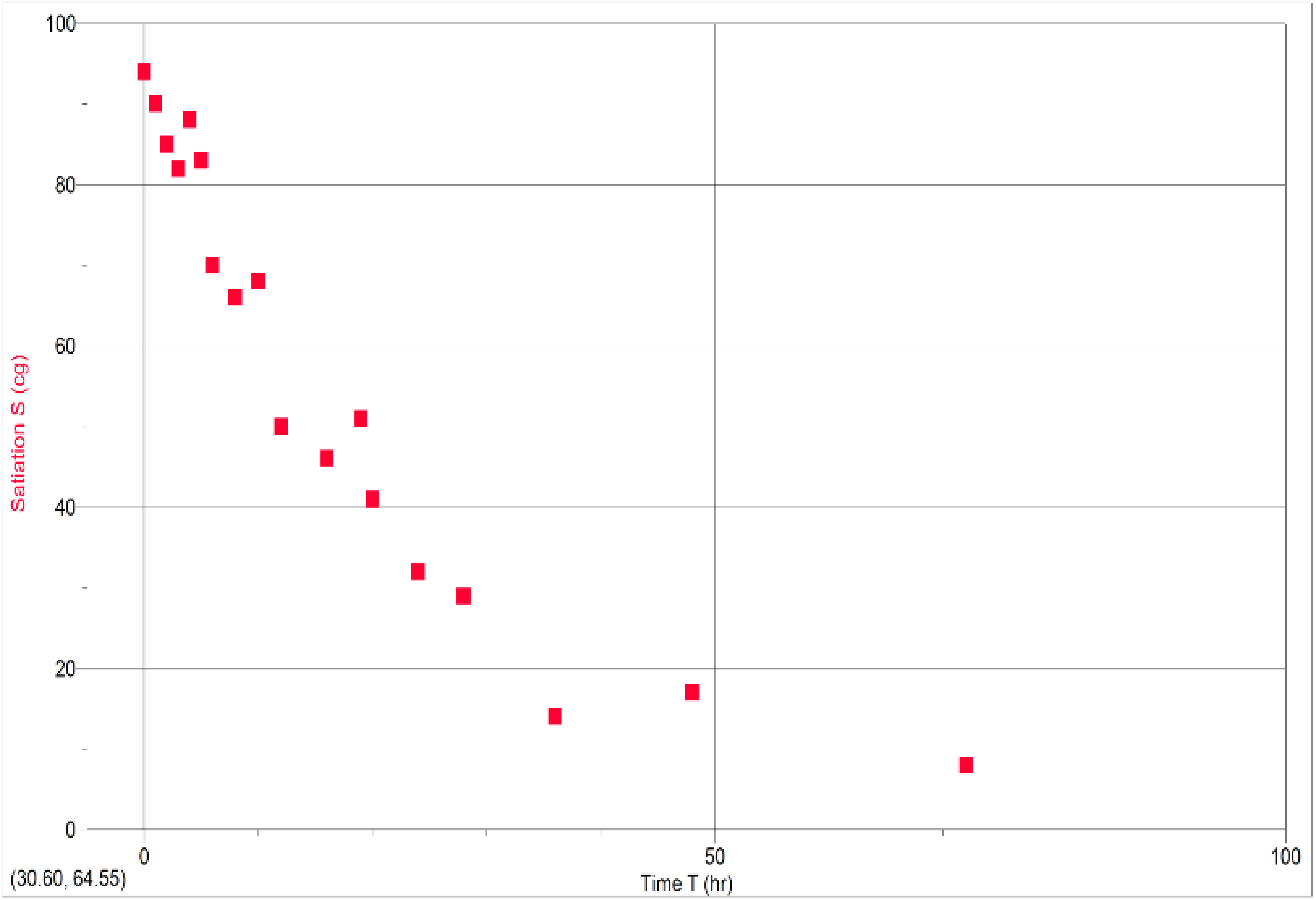
Satiation vs. Time



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Finding a Model

The biologists assume that the mantid will digest a fixed percentage of the food in its stomach each hour. That information, together with the graph, tells us that an *exponential function* should be a good fit.

Through investigation, we find the following function models this behavior well:

$\boldsymbol{S = 0.96^T \cdot 94}$**,** where T is the number of hours since the mantid has filled its stomach and S is its satiation in cg.

Notice that **0.96** is the percentage of food in the mantid's stomach and **94** represents the initial satiation.

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Combining the Models

Notice the relationship between our two functions:

$$R(S) = \begin{cases} -1.24S + 76.26 & if\ 0 \leq S < 61.5 \\ 0 & if\ S \geq 61.5 \end{cases}$$

$$S(T) = 0.96^T \cdot 94$$

The output for the second is the same quantity as the input for the first.

We can write a function for Reaction Distance in terms of time as a composition of functions.

The non-zero part is:

$$R\big(S(T)\big) = -1.24 \cdot (0.96^T \cdot 94) + 76.26$$

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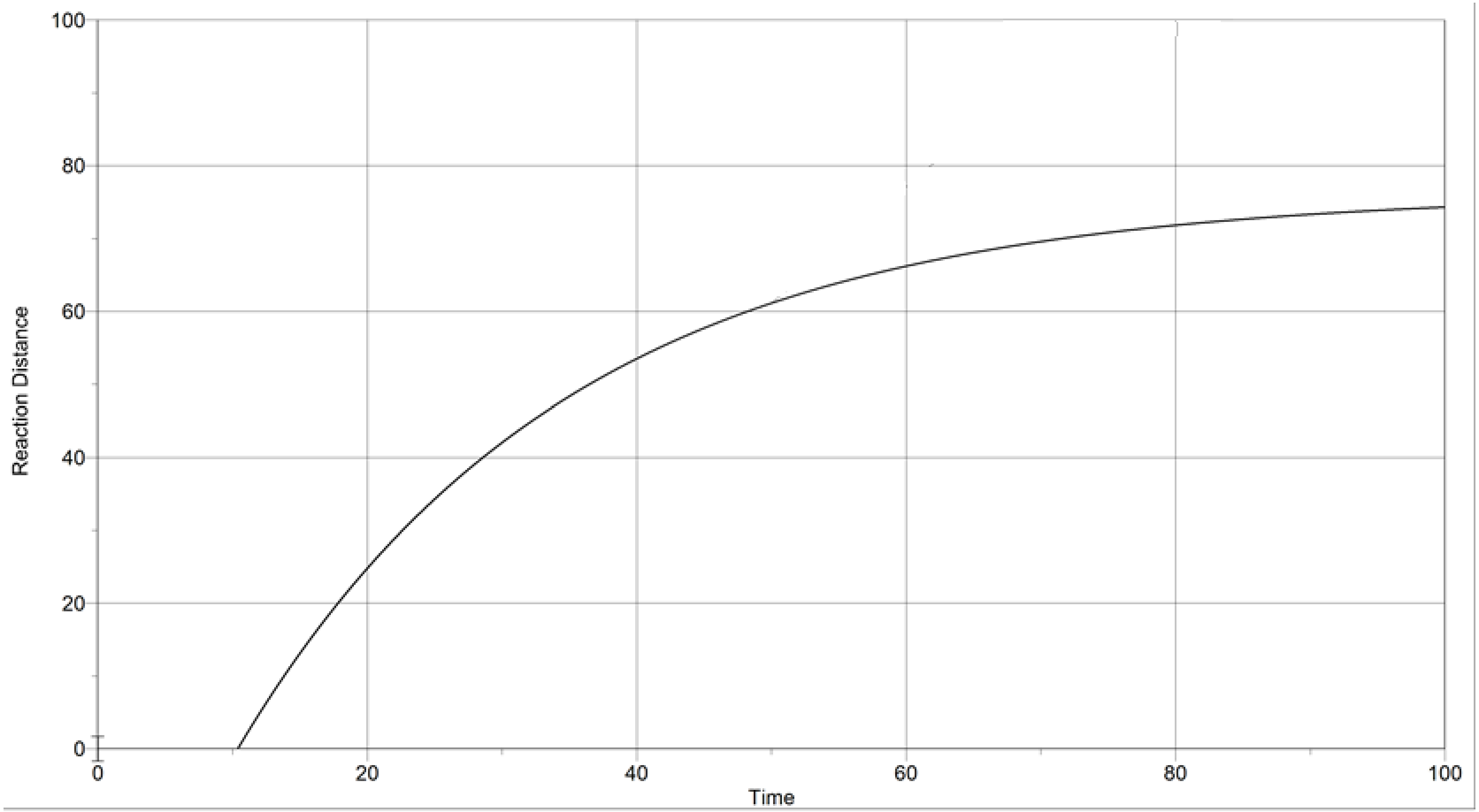
Reaction Distance vs. Time



It may be tricky for students to recognize that since S decreases over time, in the R vs. T graph, the zero portion will be first…

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Driving for Gas

Every driver recognizes the fluctuations in gas prices that happen almost on a weekly basis. Is it worth the drive across town for less expensive gas?

If you know the locations and the prices at all gasoline stations, at which station should you buy your gas?

Create an App that would allow users to choose the best gas station to visit.

This problem is a variation on a problem developed for class use by Landy Godbold (Westminster School in Atlanta).
Image from xkcd.com

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Driving for Gas

Advanced students should be able to refine the problem on their own to create a more clearly defined question to answer. For beginning students, the teacher will need to formulate the problem in a more tractable and clean form.

For example:

You drive to school every day. On the route you take from home to school, there are several gas stations. Unfortunately, the prices on your route are always high. A friend tells you she buys her gas at a station several miles off your normal route where the prices are cheaper. Would it be more economical for you to drive the extra distance for the less expensive gas than to purchase gas along your route?

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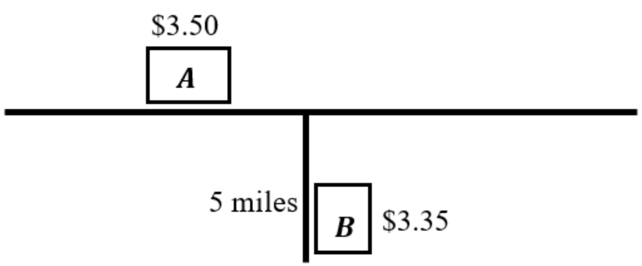
Driving for Gas

We encourage our students to create a small, specific version of the problem and try to answer the question about that specific model. This special case will help the students clarify the problem, determine the important variables, and suggest approaches that could be useful for the general case. One example of such a problem is:

Suppose there is a station on your normal route that sells gas for \$3.50 a gallon. A station 5 miles off your route sells gas for \$3.35 a gallon. Should you travel the extra distance to buy gas at that station?



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Driving for Gas

Why wouldn't we drive the five miles to buy cheaper gas? If we changed the problem so that the cheap gas is 50 miles away, we can quickly see that we would use up the money we saved in purchasing the less expensive gas by driving to the station. This tells us that the number of miles per gallon (mpg) of our car gets driving in town would be an important factor, and suggests that the answer to the question might be different for two students who drive different cars.

The student driving an SUV getting 15 mpg might be less likely to drive the 10 miles (5 miles out to the station and 5 miles back to your route) than a student driving a small car getting 40 mpg. We now have a more refined problem on which we can work.

Driving for Gas

- Let P^* represent the price per gallon at the station along our route and P the price per gallon at the station we are considering.
- Let D represent the distance in miles from the normal route that must be driven to get to the gas station.
- Let M represent the miles per gallon of the car you drive along the route to the gas station.
- Let T represent the number of gallons of gas we purchase when we buy gas.

One possible metric we can use to compare gas stations is:

$$\frac{T \cdot P}{T - \frac{2D}{M}} < P^*$$

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Driving for Gas

The previous model compares “cost per useable gallon of gas”.

Other variations:

- Allow for varying parameters such as car mpg
- We could consider buying miles of driving. Then we would be comparing “driving miles per dollar”.
- We could incorporate a cost for the time used up.

IF YOU SPEND NINE MINUTES OF YOUR TIME TO SAVE A DOLLAR, YOU'RE WORKING FOR LESS THAN MINIMUM WAGE.

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Choosing Problems

If you have a good problem and want to tailor it to a group of students consider whether it is best to:

Scaffold – give more direction to help beginning modelers avoid the frustration

Scale – help the students make simplifying assumptions that scale the problem down, making the problem easier to approach

Save – if neither of those will work, consider saving the problem for later in the students' modeling education

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A Teacher’s Role in Modeling

Having students share their results either written or orally can be an effective way to more formally assess the work of modeling.

Asking the students to focus their writing/reflection on the *process* of modeling is useful for helping students become more proficient on future modeling experiences.

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A Teacher’s Role in Modeling

Writing about and/or presenting models and results provides its own set of challenges for high school students.

The students will need guidance as to how to present their findings in a way that will be understandable to someone who has not been a part of the idea development.

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A Teacher’s Role in Modeling

When projects are turned in, the teacher needs to read the project from the student’s point of view. Given their approach, what did they do well and what could they have done better? Can you repeat the process they used? Did they see the strengths and weaknesses of their model? Did they test their model in some way?

Providing instructive feedback will help to improve the quality of their work and their writing.

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Learning More about Modeling

Check out NCTM's ***Annual Perspectives on Mathematics Education 2016*** for more information and insight on mathematical modeling.

Our chapter "*Moving From Remembering to Thinking: The Power of Mathematical Modeling*" contains more examples of how NCSSM has used modeling in our courses.

http://www.nctm.org/Store/Products/Annual-Perspectives-in-Math-Ed-2016--Mathematical-Modeling/