Everything Old Is New Again: Connecting Calculus To Algebra

Andrew Freda
(afreda@deerfield.edu)

1) Limits

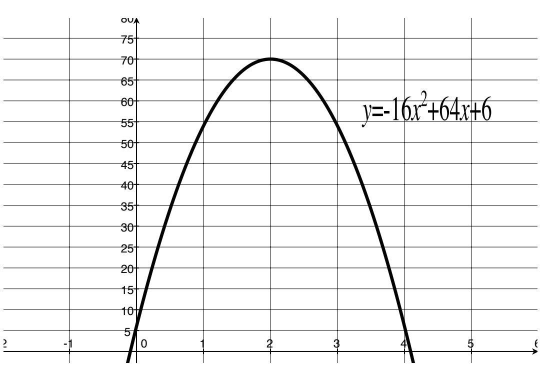
a) Newton's Idea of a Limit

"Perhaps it may be objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But by the same argument it may be alleged, that a body arriving at a certain place, and there stopping, has no ultimate velocity; because the velocity, before the body comes to the place, is not its ultimate velocity; when it has arrived, is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives; that is, that velocity with which the body arrives at its last place, and with which the motion ceases. And in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities, not before they vanish, nor afterwards, but with which they vanish." [Principia].

Projectile Motion problem for an Algebra class (note the connection between Newton's idea and question part "f")

If you throw a rock directly upward against gravity, its position (height above its starting point at any given second if we ignore wind, air-resistance, humidity, etc.), can be found by using:
$position = \left(\frac{1}{2}\right)(gravity)t^2 + \left(velocity_{original}\right)t + \left(position_{original}\right)$ where gravity is negative (because it works against the original movement of the rock). If a rock is thrown with a velocity of $64\frac{ft}{\sec}$ from a height of 6 feet, then its position is given by $position = \left(\frac{1}{2}\right)(-32)t^2 + (64)t + (6)$ or $position = -16t^2 + 64t + 6$ or $position = -16(t-2)^2 + 70$. Note that the fastest baseball ever pitched had an initial velocity of 154 feet per second.

a) What is the height off the ground of the rock after 2 seconds?
b) What is the height off the ground of the rock after 3 seconds?
c) When does the rock reach its maximum height?
d) When does the rock get "caught" by the thrower (reach a height of 6 feet on its way down)?
e) Please make a sketch with the x-axis as time in seconds and the y-axis as height above ground in feet. Use the values from your previous answers for your sketch.
f) What is the speed of the rock at time = 0 seconds? At time = 2 seconds? At time = 4 seconds?



b) Delta-Epsilon Definition of a Limit

All Algebra I students are asked to work with absolute value on the number line and they should understand the meaning of "difference" and "distance" in an algebra setting. That is, $(x - 5) \neq (5 - x)$ because the "difference" between two numbers is a "signed value" which must reflect whether we are taking a larger value from a smaller or a smaller value from a larger. However, $|x - 5| = |5 - x|$ because the "distance" between two numbers must be non-negative and reflects the separation between two values. If we remind students that they know this, then we can ask them to understand and explain the relation between the informal definition of a limit (which relies on verbs of motion and words in English) and a more formal definition for Limit (which uses mathematical symbols). In English we say "If f(x) gets closer than any given value, or equals, a given place "L" on the vertical number line, then it must be true that "x" gets closer than any given value but does not reach a place "a" on the horizontal number line."

Using symbols from Calculus if we say that $\lim_{x\to a} f(x) = L$ **then** for each given $\varepsilon > 0$ (no matter how small) there is a corresponding $\delta > 0$ such that $|f(x) - L| < \varepsilon$, provided that $0 < |x - a| < \delta$

c) Infinity Minus Infinity Limit Problems – three cases

We ask student to move square roots from the denominator to the numerator ("rationalizing the denominator") for a variety of reasons before calculus, but there are wonderful limit problems which require little more than their Algebra skills to see for themselves that $\infty - \infty$ can equal infinity, zero, or even 2.5. The exciting part of these limit problems is that they provide great motivation for students to see the importance of algebraic skills. With these skills, they can understand by themselves the richness contained by the idea expressed by $\infty - \infty$, but without these skills, they must take someone else's word for it.

First Case of $\infty - \infty$

$$\lim_{x\to\infty}\frac{\sqrt{x^6+5}-x^3}{1} = \lim_{x\to\infty}\left(\frac{\sqrt{x^6+5}-x^3}{1}\bullet\frac{\sqrt{x^6+5}+x^3}{\sqrt{x^6+5}+x^3}\right) = \lim_{x\to\infty}\left(\frac{x^6+5-x^6}{\sqrt{x^6+5}+x^3}\right) = \lim_{x\to\infty}\left(\frac{5}{\sqrt{x^6+5}+x^3}\right) = 0$$

Second Case of $\infty - \infty$

$$\lim_{x\to\infty}\frac{\sqrt{x^8+5}-x^3}{1}$$

$$\lim_{x\to\infty}\left(\frac{\sqrt{x^8+5}-x^3}{1}\bullet\frac{\sqrt{x^8+5}+x^3}{\sqrt{x^8+5}+x^3}\right) = \lim_{x\to\infty}\left(\frac{x^8+5-x^6}{\sqrt{x^8+5}+x^3}\right) = \lim_{x\to\infty}\left(\frac{\frac{x^8}{x^4}+\frac{5}{x^4}-\frac{x^6}{x^4}}{\sqrt{\frac{x^8}{x^8}+\frac{5}{x^8}}+\frac{x^3}{x^4}}\right) = \lim_{x\to\infty}\left(\frac{x^2\bullet\left(x^2-1\right)+\frac{5}{x^4}}{\sqrt{1+\frac{5}{x^8}}+\frac{1}{x}}\right) = \infty$$

Third Case of $\infty - \infty$

$$\lim_{x\to\infty}\frac{\sqrt{x^6+5x^3}-x^3}{1}$$

$$\lim_{x\to\infty}\left(\frac{\sqrt{x^6+5x^3}-x^3}{1}\bullet\frac{\sqrt{x^6+5x^3}+x^3}{\sqrt{x^6+5x^3}+x^3}\right) = \lim_{x\to\infty}\left(\frac{x^6+5x^3-x^6}{\sqrt{x^6+5x^3}+x^3}\right) = \lim_{x\to\infty}\left(\frac{\frac{5x^3}{x^3}}{\sqrt{\frac{x^6}{x^6}+\frac{5x^3}{x^6}}+\frac{x^3}{x^3}}\right) = \lim_{x\to\infty}\left(\frac{5}{\sqrt{1+\frac{5}{x^3}}+1}\right) = \frac{5}{2}$$

2) Continuity

Algebra students have a strong understanding that all polynomials can be sketched without lifting one's pen off the page. Students are so prejudiced in favor of polynomials, that they often disregard piece-wise functions as too artificial to "really count" as functions and consider Rational functions to be continuous, except where they aren't continuous (which is, of course, quite correct). This prejudice can be used to help students picture "well-behaved" functions which approach a given point in the plane "nicely" from both the left and right eventually meeting at that given point, unlike the lack of continuity that we can see in some Piece-Wise or Rational functions.

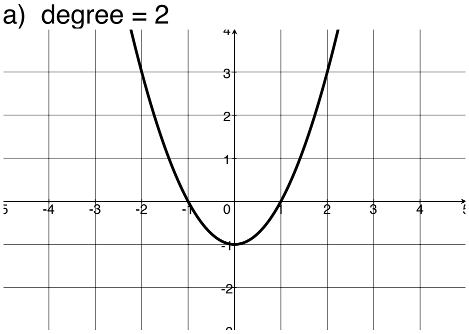
One way to start the study of continuity in Calculus not with the formal definition or with examples of discontinuity, but with polynomials of various degree followed by Rational functions. Students will quickly agree that any polynomial can be sketched without "lifting pen from paper," and this will lead to any Rational function being sketched without "lifting pen from paper, except where you have to "lift pen from paper." The knowledge that students have from Algebra II of graphing polynomials and the issue that division by zero creates with one specific Rational function (where the function is not continuous), is a great place to start the conversation on what we mean by a continuous function and what is meant by $\lim_{x\to a} f(x) = f(a)$.

Please sketch the following– label x and y intercepts, vertical asymptotes, if they exist, and give the general shape

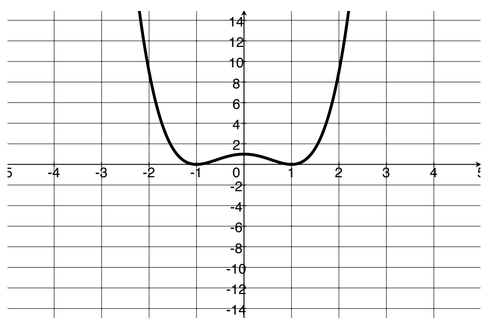
1. Please sketch $a)\ y=\dfrac{(1)}{(x-4)}$ $\quad b)\ y=\dfrac{(x-4)}{(x-4)}$ $\quad c)\ y=\dfrac{(x-4)(x-1)}{(x-4)}$ $\quad d)\ y=\dfrac{(x-4)}{(x-4)(x-1)}$

2. Please find an equation for the following sketches

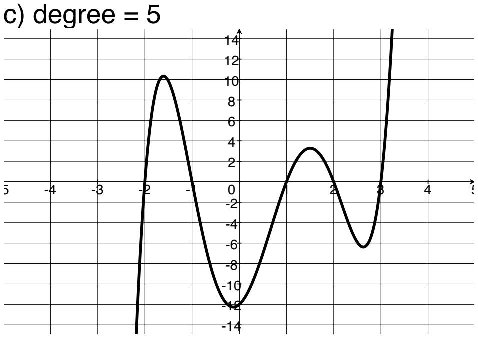
a) degree = 2



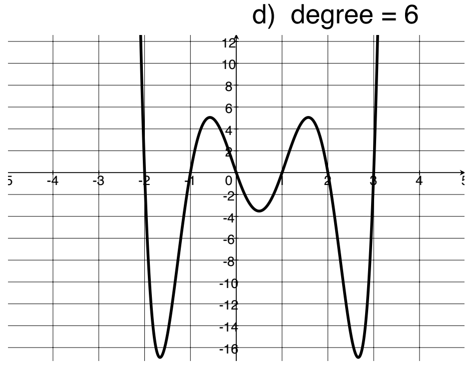
b) degree = 4



c) degree = 5

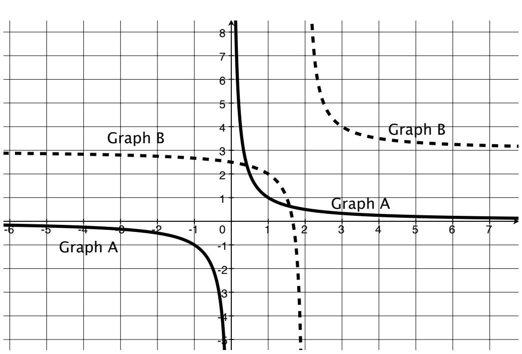


d) degree = 6

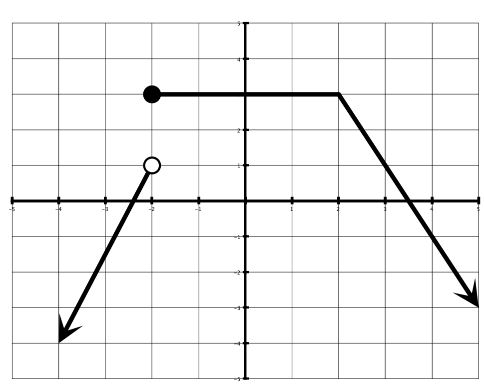


4. Please find an equation for each sketch.

a) Graph A is a "toolkit" graph and Graph B has the same shape but has been moved right and up



b) This graph is made of two rays and segment

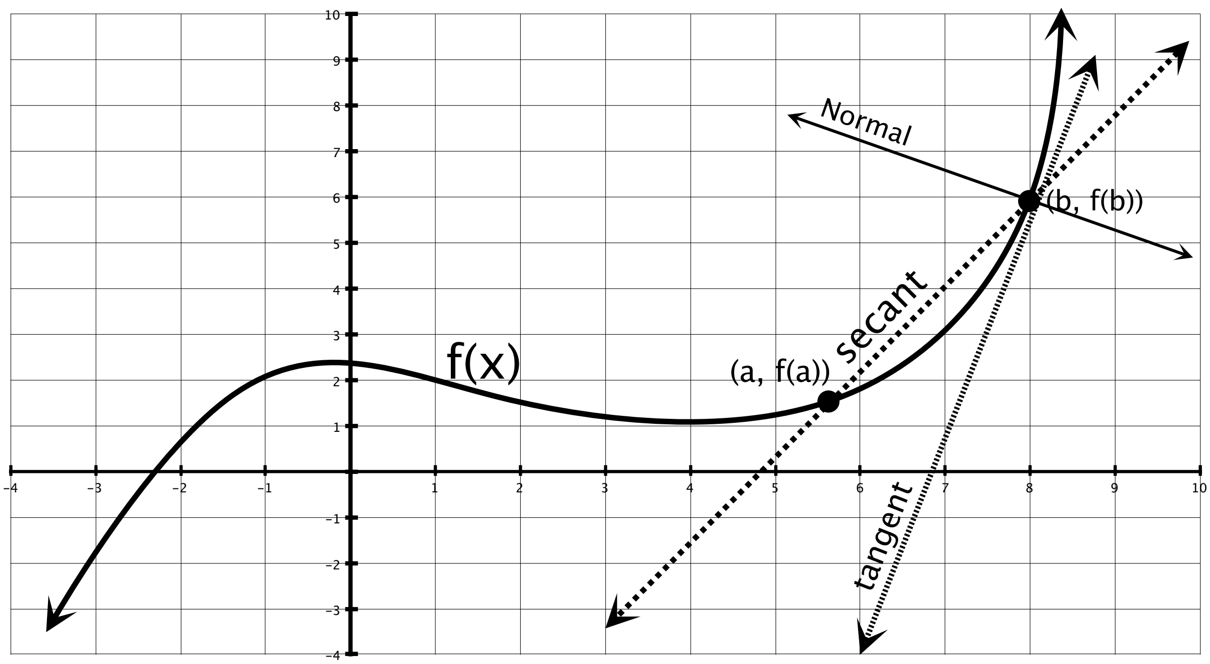


3) Derivatives

I tell students that **if** they want to break my heart, **then** they they may write the formal definition of a derivative as: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. I ask instead that they write $\lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-(x)}$ because I believe that it helps them understand the derivative more fully if they remind themselves constantly that they are taking the limit of an old friend: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-(x)}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{x_2\to x_1}\frac{y_2-y_1}{x_2-x_1}$

a) Below is a worksheet that I use to help students understand the algebraic expressions for tangents, secants, and normal lines.

Please match each expression or equation below with something from the graph or explain why there is no match



a) $\frac{f(b)-f(a)}{b-a}$ b) $\frac{f(a)-f(b)}{a-b}$ c) $\frac{f(a)-f(b)}{b-a}$

d) $\frac{1}{\left(\frac{f(a)-f(b)}{b-a}\right)}$ e) $\lim_{b\to a}\frac{f(b)-f(a)}{b-a}$ f) $\lim_{a\to b}\frac{f(b)-f(a)}{b-a}$

g) $(y-f(b))=\left(\frac{f(b)-f(a)}{b-a}\right)(x-b)$ h) $(y-f(b))=\left(\frac{1}{\left(\frac{f(a)-f(b)}{b-a}\right)}\right)(x-b)$

i) $(y-f(b))=\left(\lim_{a\to b}\frac{f(b)-f(a)}{b-a}\right)(x-b)$ j) $(y-f(a))=\left(\lim_{a\to b}\frac{f(b)-f(a)}{b-a}\right)(x-a)$

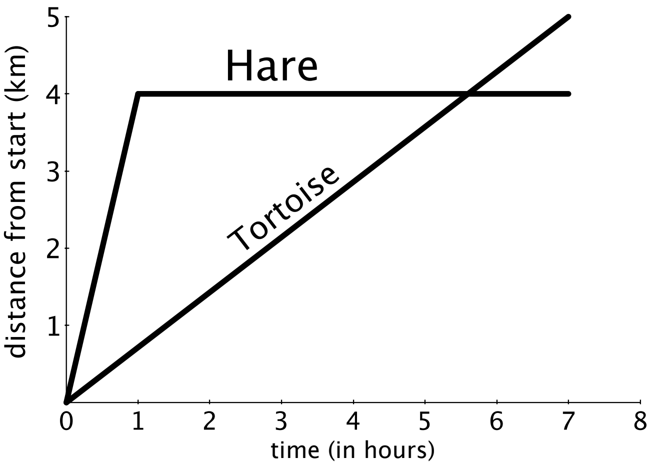
k) $(y-f(a))=\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$ l) $(y-f(b))=\left(\lim_{a\to b}\frac{1}{\left(\frac{f(a)-f(b)}{b-a}\right)}\right)(x-b)$

b) Mean Value Theorem for Derivatives:

Before I introduce my students to "If a function f(x) is continuous on a closed interval [a,b] and is differentiable on the open interval (a,b), then there exists a number "c" in (a,b) such that $f(b) - f(a) = f'(c)(b - a)$," I make them think about average velocity, an idea that they studied in their algebra classes.

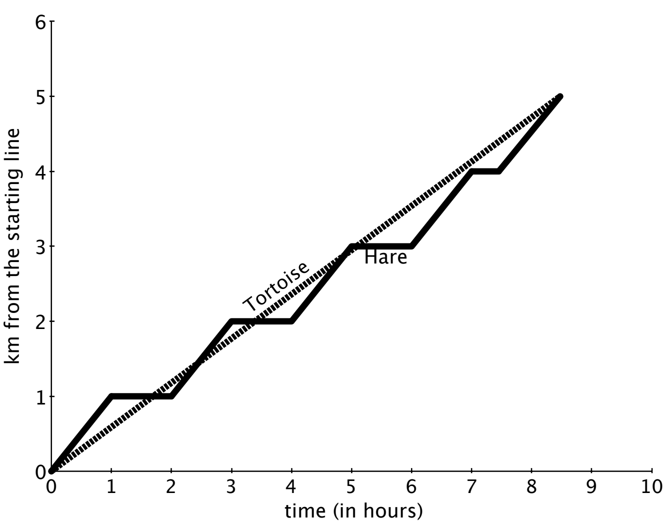
The Tortoise and the Hare have a series of 5k races, please describe what happens in each race

Race #1



a) What is the Tortoise's average velocity for the race?
b) What is the Tortoise's velocity at t = 3?
c) What is the Hare's average velocity for the race?
d) What is the Hare's velocity at t = 3?
e) Do they ever have the same velocity?
f) Do they ever have the same average velocity?

Race #2



a) What is the Tortoise's average velocity for the race?
b) What is the Tortoise's velocity at t = 3?
c) What is the Hare's average velocity for the race?
d) What is the Hare's velocity at t = 1?
e) Do they ever have the same velocity?
f) Do they ever have the same average velocity?

Race #3

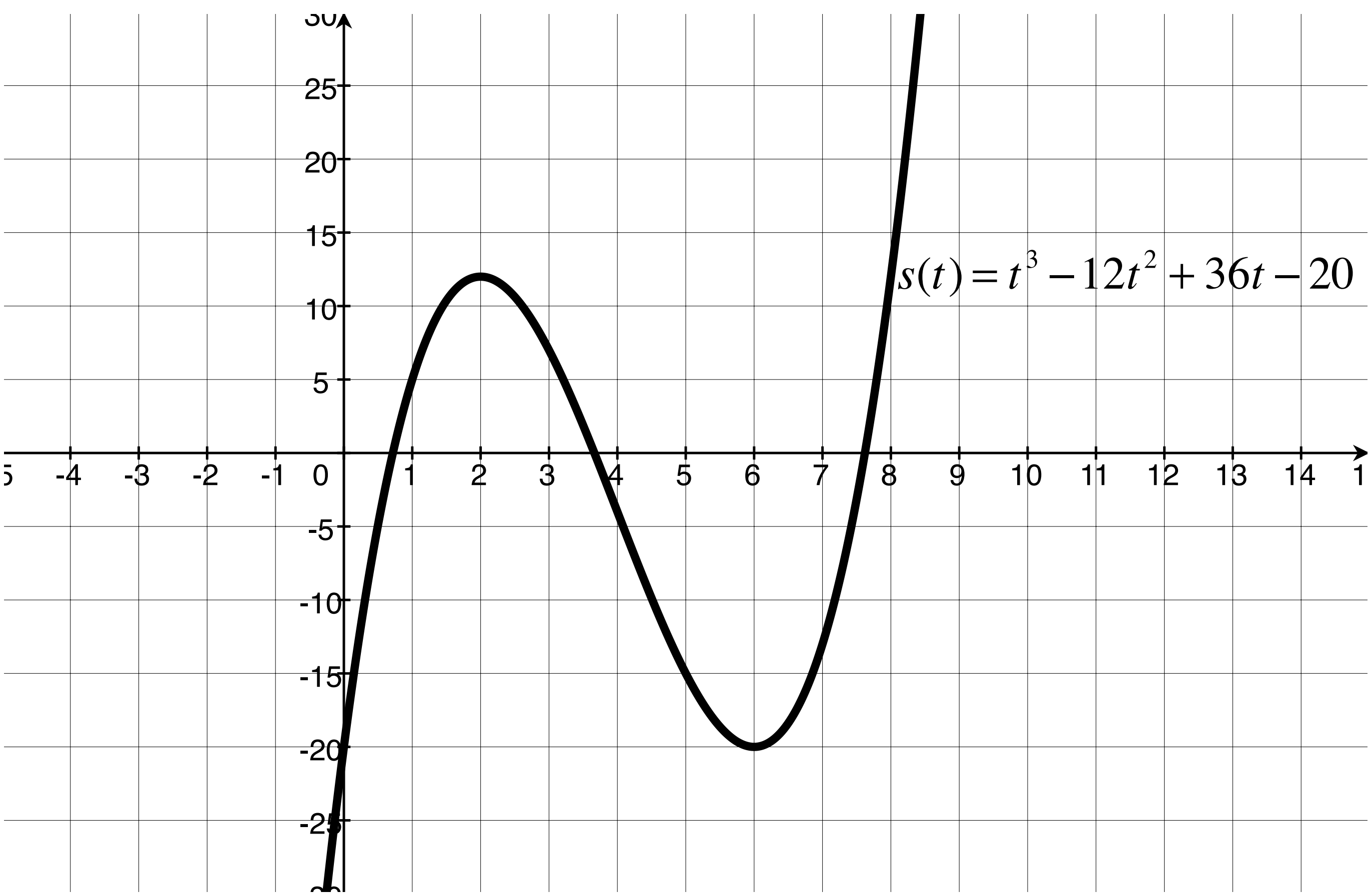
a) What is the Tortoise's average velocity for the race?
b) What is the Tortoise's velocity at t = 3?
c) What is the Hare's average velocity for the race?
d) When is the Hare's velocity greatest?
e) Do they ever have the same velocity?
f) Do they ever have the same average velocity?

Students are then ready for a more general discussion of position, velocity, and acceleration but the meaning of these key terms should be understandable to them if they remember what they learned in their Algebra class from the Tortoise and the Hare. This is also a good time to remind students of Newton's idea of what is meant by a limit.

The position function of a point moving on a coordinate line is given by:

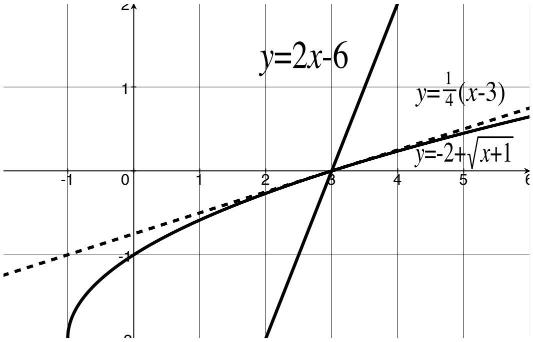
$s(t) = t^3 - 12t^2 + 36t - 20$

a) Please find the velocity function
b) Please find the acceleration function
c) Please find the average velocity for the interval [1,2]
d) Please find the velocity at the instant when t = 1
e) Please find the velocity at the instant when t = 2
f) Please find all times in the interval [0,8] when velocity at an instant equals zero
g) Please explain the visual significance of your answer for part " f "
h) Please find the acceleration at t = 2. What is the meaning of the sign of acceleration?
i) Please explain the meaning of the x-intercepts (.716, 3.663, 7.620) and the y-intercept
j) Please find when the point is farthest to the right during the interval [0,8]
k) Please find when the point is farthest to the left during the interval [0,8]
l) Please sketch the derivative of position on the graph above
m) Please explain what the point is doing when the y-values of the derivative are below the x-axis



c) L'Hospital's Rule and Local Linearity

Let us look at a limit problem which can be nicely resolved by substituting tangent lines at a point for the given curves. Students are asked to find $\lim_{x\to 3}\left(\frac{2x-6}{-2+\sqrt{x+1}}\right)$ and they quickly see that the "substitution" technique gives $\frac{0}{0}$ which means "not enough information."



A first way to look at L'Hospital's Rule is to see that we must find the limit of the quotient of functions where the "x" value that we evaluate is the x-intercept for both functions

Students could then manipulate the expression algebraically in order to be able to use the "substitution" technique:

$$\lim_{x\to 3}\frac{2x-6}{-2+\sqrt{x+1}}=\lim_{x\to 3}\frac{2x-6}{-2+\sqrt{x+1}}\bullet\frac{-2-\sqrt{x+1}}{-2-\sqrt{x+1}}=\lim_{x\to 3}\frac{-4x-2x\sqrt{x+1}+6\sqrt{x+1}+12}{4-(x+1)}=\lim_{x\to 3}\frac{(3-x)\left[4+2\sqrt{x+1}\right]}{(3-x)}=8$$

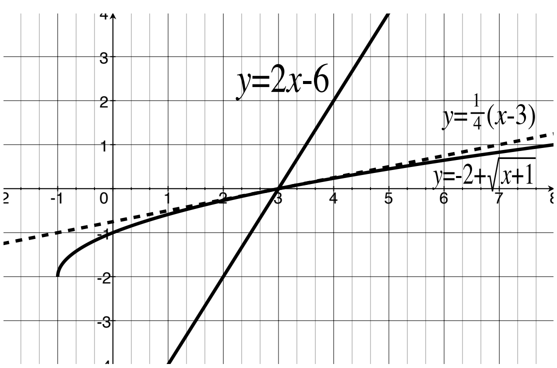
<u>or</u> they could use L'Hospital's rule:

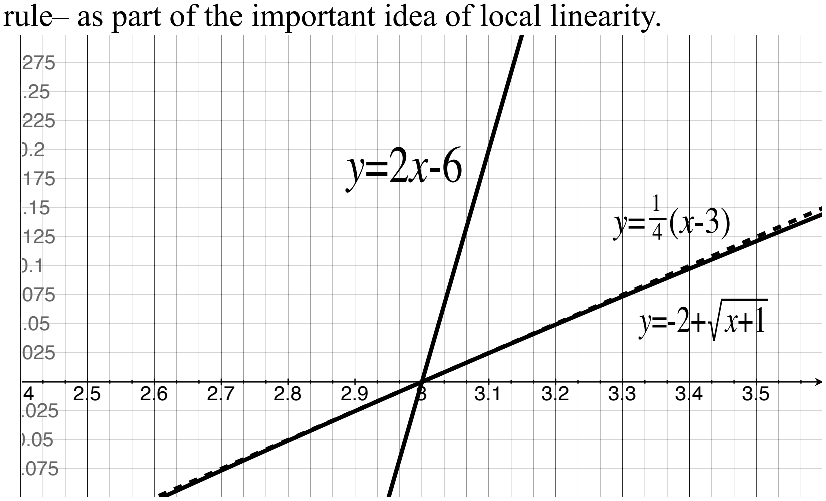
let $f(x)=2x-6$ *and* $g(x)=-2+\sqrt{x+1}$ $\qquad \lim_{x\to 3}\frac{f(x)}{g(x)}=\lim_{x\to 3}\frac{f'(x)}{g'(x)}=\lim_{x\to 3}\left[\frac{2}{\frac{1}{2}(x+1)^{-\frac{1}{2}}}\right]=8$

We can see that local linearity is the basis for L'Hospital's rule. That is, because we can't compare the given curves, we use the tangent lines to both curves at (3,0). The ratio of the two functions at any common point is the ratio of the slopes of their tangent lines at that point.

$$\lim_{x\to 3}\frac{2x-6}{-2+\sqrt{x+1}}=\lim_{x\to 3}\frac{\text{tangent line to f(x) at (3,0)}}{\text{tangent line to g(x) at (3,0)}}=\frac{f'(3)(x-3)+0}{g'(3)(x-3)+0}=\frac{(2)(x-3)+0}{\left(\frac{1}{2}(3+1)^{-\frac{1}{2}}\right)(x-3)+0}=\frac{(2)}{\left(\frac{1}{2}(3+1)^{-\frac{1}{2}}\right)}=8$$

If we "zoom in" on the sketch, we see that f(x) is eight-times as steep as g(x) in the x = 3 "neighborhood" and students can understand a powerful technique–L'Hospital's rule– as part of the important idea of local linearity.





The function and its tangent line may be the same (as in the case of y = 2x - 6) or they may share only the point of tangency (as in the case of $y=-2+\sqrt{x+1}$) but we can see that as we approach the point of tangency, the difference between the function and its tangent line is either zero or gets smaller so that at the limit, we can substitute the tangent line for the function

Summary

I want my students to understand that $\lim_{x\to 0}\left(\frac{x^4}{x}\right)=0$ and I want to them to imagine "zooming in" on this function in the "neighborhood" of x = 0. I want my students to understand that $f(x)=\left(\frac{x^4}{x}\right)$ is not continuous for all numbers in the interval which includes x = 0 even though it has a limit for all Real numbers. I want my students to understand "local linearity" and how that term applies to $f(x)=|x|$ (which can be said to be made of line segments, but is not locally linear in the neighborhood which includes x = 0) or $f(x)=\sin(x)$ (which has no straight segments anywhere but is locally linear everywhere). I tell my students that a function is differentiable where it is locally linear and this powerful idea that they first met in Algebra I will lead to Taylor Polynomials in Calculus II and Tangent Planes in Calculus III.

Some Big Ideas of Calculus I with Connections to Algebra **(Integrals are not discussed in this talk)**

Definitions

Limit:

if we say that $\lim_{x\to a} f(x)=L$ **then** for each given ε >0 (no matter how small) there is a corresponding δ >0 such that $|f(x)-L|<\varepsilon$, provided that $0<|x-a|<\delta$

If we "approach" a given x value very closely from both greater and lesser values, then the f(x) value must get closer than any given number to the Limit (a Real number).

Continuity:

$\lim_{x\to a} f(x)=f(a)$

If for some given interval we can draw the graph of a function without lifting our pen, then we call it continuous for that point which is in that interval

Derivative:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-(x)}=\lim_{b\to a}\frac{f(b)-f(a)}{(b)-(a)}$$

The slope of a segment between any two points on a function gets closer than any given number to the slope of the tangent line to the function at an endpoint of that segment

Theorems

Intermediate Value Theorem:

If a function f(x) is continuous on a closed interval [a,b] and if f(a) ≠ f(b),
Then f(x) takes on every value between f(a) and f(b) in the interval [a,b].

If a function is continuous, then it can't "skip" values for f(x) as as the function moves (or does not move) up and down as we read from left to right

Mean Value Theorem for Derivatives:

If a function f(x) is continuous on a closed interval [a,b] and is differentiable on the open interval (a,b), then there exists a number "c" in (a,b) such that f(b) – f(a) = f ' (c) (b – a)

If a function is differentiable, then the "average slope" must equal the "slope at an instant" at least once