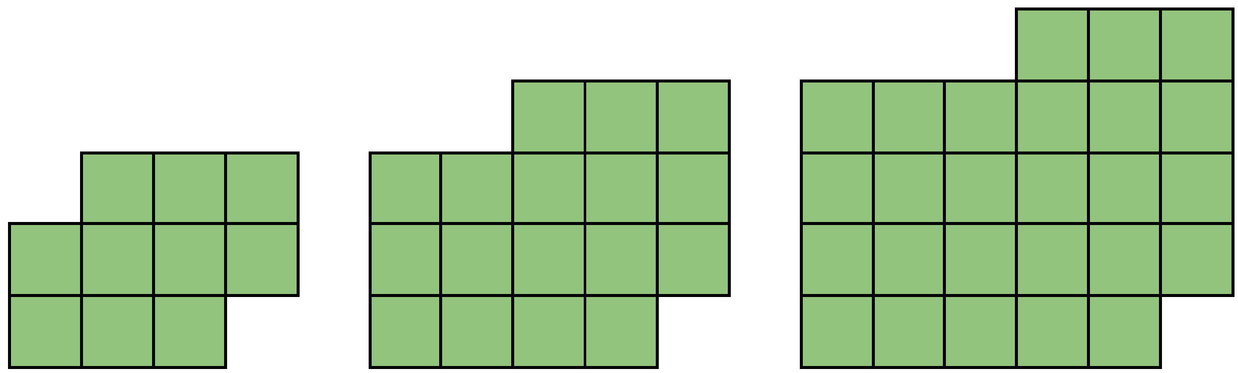
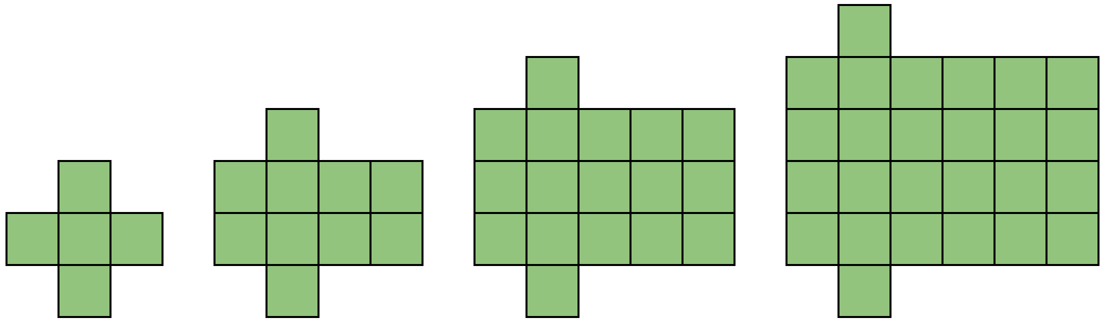


2 3 4

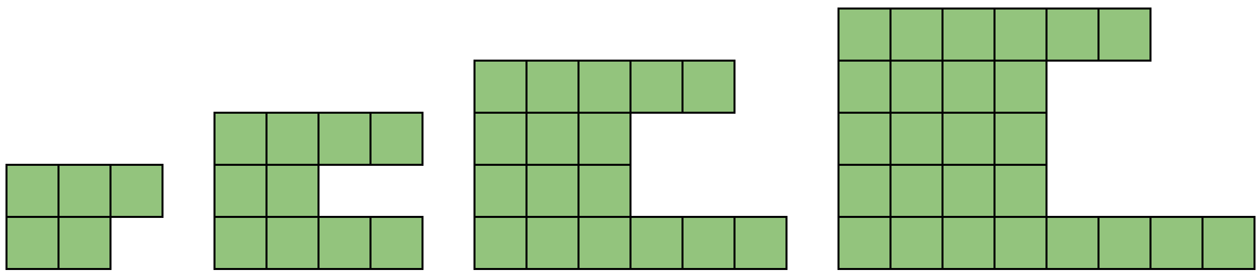
What do you notice?

Pose some questions.

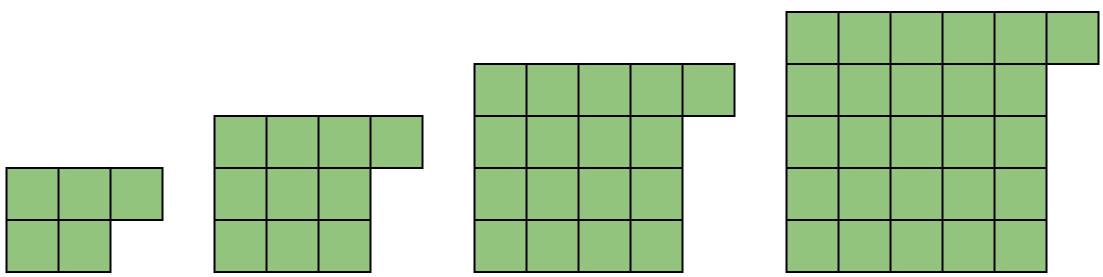
Note: This visual growth sequence is inspired by **visualpatterns.org**, curated by Fawn Nguyen, a middle school math teacher in California.



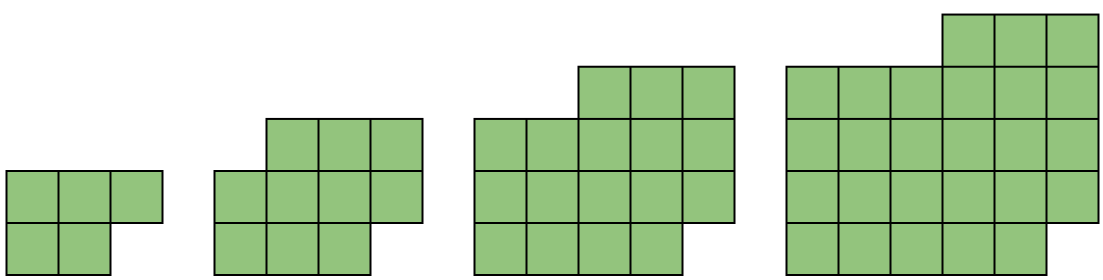
1 2 3 4



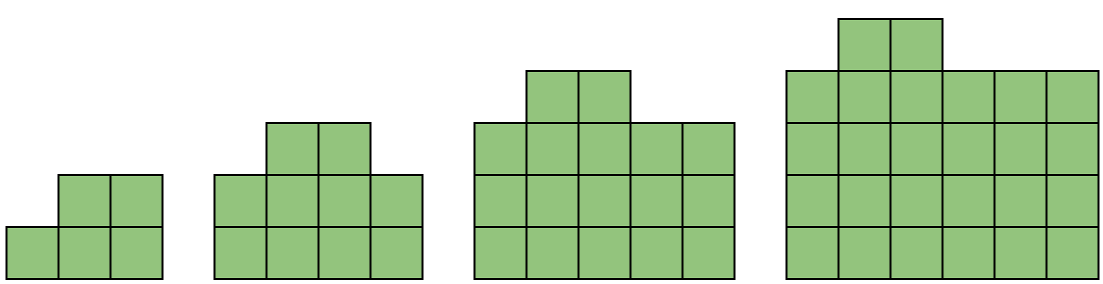
1 2 3 4



1 2 3 4

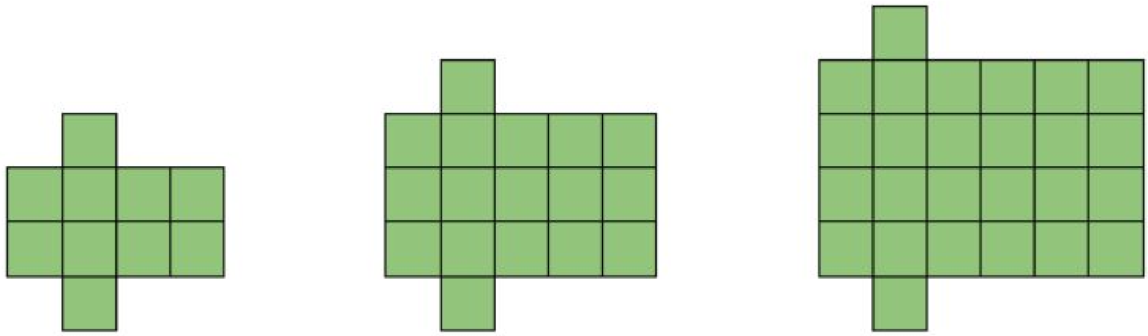


1 2 3 4



1 2 3 4

Visual Pattern #19



2 3 4

1. Sketch and label the fifth and sixth figure.
2. Sketch and label the first figure.
3. How many square tiles are needed to construct each of these figures?
4. Describe with a written explanation how you would sketch or construct the 100^{th} figure.
5. Using the picture directly, describe with words two different ways you could determine the number of tiles needed to make the n^{th} tile in the sequence.
6. If you did not already do so, write a rule or formula that matches each of the ways you described in #4. Define your variables explicitly.
7. Share your results with a partner. How do you justify your expression? How do you verify results?
8. Sketch figure 0.
9. Would the 10^{th} figure have an even number of squares or an odd number? Why? How do you know?
10. Would the n^{th} figure have an even number of squares or an odd number? Why?
11. Is the pattern growing proportionally? Why or why not? If not, could you create a pattern that grows proportionally from the picture where n=2?

*Thanks to Dr. Usha Kotelawala for the questions and Fawn Nguyen for visualpatterns.org!

UNIT 8

Developing Algebraic Reasoning through Visual Patterns

Introduction: More Than Solving for *x*

Overview

- What is Algebra?
- **Lesson I:** Introduction to Patterns and a Scaffolded Approach to Arch Problem
- **Lesson II:** Open-ended Approach to Pattern #141
- Sample Progression of Visual Patterns
- Additional Recommended Resources

What is Algebra?

Before we talk about ways to teach algebra, let's reflect on what algebra is, what it could be, and how it connects to our students and their self-concept.

Here are a few particularly rich definitions:

> "Algebra is the fundamental language of mathematics. It enables us to create a mathematical model of a situation, provides the mathematical structure necessary to use the model to solve problems, and links numerical and graphical representations of data. Algebra is the vehicle for condensing large amounts of data into efficient mathematical statements."
>
> –from *Math Matters: Understanding the Math You Teach* by Suzanne H. Chapin and Art Johnson

> "Algebraic thinking or algebraic reasoning involves forming generalizations from experiences with number and computation, formalizing these ideas with the use of a meaningful symbol system, and exploring the concepts of patterns and functions."
>
> –from *Elementary and Middle School Mathematics: Teaching Developmentally* by John A. Van de Walle

> "Algebra is a tool for making sense of the world–for making predictions and for making inferences about things you cannot measure or count."
>
> –from "Some Thoughts on Algebra for the Evolving Workforce" by Romberg and Spence (as cited by Manly and Ginsburg, 2010)

Algebra is a way of thinking and reasoning that allows us to create models, study relationships, and solve problems.

WHY DO STUDENTS STRUGGLE WITH ALGEBRA?

When you ask adult education students to define algebra, they tend to give a vague list of disconnected words which somehow relate to procedures for manipulating symbols and equations. They will say things like “Algebra is x and y”, or “variables”, or “negative numbers”, or “finding the unknown” without any clear idea of what those things mean or how they are related. These associations are a result of much of algebra instruction, which focuses on developing the procedures to manipulate equations, which is only one facet of the richness of algebra. Compare these impressions with the definitions above.

If you push them further to talk about algebra, many of our students also say things like “Algebra is heartbreak,” or “That’s when I left school,” or “It has nothing to do with real life,” or “Algebra is something that makes me want to give up.” Both categories of answers give us insight into the root of their struggle. As Fosnot and Jacob (2010) stated: *“It is human to seek and build relations. The mind cannot process the multitude of stimuli in our surroundings and make meaning of them without developing a network of relations,”* (as cited in Van de Walle, 2013). When it comes to learning algebra, most adult education students need more than a disparate set of procedures without links to other kinds of math or to anything they can connect with. For most of them it can be like trying to learn shorthand before you learn how to write.

In addition to not having a framework or structure for developing that “network of relations,” students often come to us with incomplete or incorrect prior knowledge of some of the core concepts of algebraic thinking. Variables, patterns, generalizations, equality/balance, and symbolic representations are some of the big ideas in algebraic thinking. Students’ prior understandings of these concepts are the soil in which we plant new ideas, and we need to aim our instruction at these fundamental misconceptions. For example, consider the fact that we use variables in a variety of different ways.

Before reading the next sentence, pause and try to come up with at least three different uses for variables.

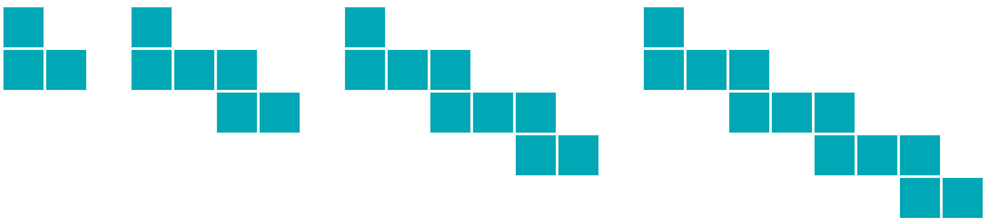
One use is in formulas, as in the circumference of a circle can be found using $C = \pi d$. We also use variables to stand for specific unknowns. Students are most familiar with this use—for example $2x + 17 = 63$. We use variables to express generalizations about mathematical relationships and functions, as in $y = 4x + 8$. These are all quite different, and we need to help students differentiate between them by being explicit and by giving students opportunities to reflect on each.

Another common misconception that adult students have about algebra it that is has no connection to the math they already know. This cuts students off from the mathematical foundations they have already built, both in terms of content and in terms of confidence.

What is a Visual Pattern and How Can it Help?

A visual pattern is a sequence of figures that can be used to bridge the gap between what students know and the algebraic reasoning they need to develop.

So how does it work? Let's start with the first four figures in an example of a visual pattern:



MATH CONTENT YOU CAN TEACH THOUGH VISUAL PATTERNS

- recognizing patterns
- predictions
- organizing data (tables and graphs)
- creating/constructing expressions
- creating/constructing equations
- understanding multiple uses of variables
- constants
- linear equations (like the arch problem)
- matching function equation to a situation
- recursive and explicit rules
- rate of change/slope
- connecting parts of equations to concrete pictures
- starting amount/y-intercept
- graphing (coordinate plane, ordered pairs)
- equivalent functions/expressions
- combining like terms
- evaluating functions
- identifying graph of function (linear and quadratic)
- simplifying expressions
- input/output tables
- independent/dependent variables
- coefficients
- the difference between an expression and an equation
- quadratic equations (like pattern #141 in this unit)
- comparing linear, quadratic, cubic functions
- second differences in quadratic functions
- algebraic notation/function notation
- polynomials
- solving for a specific value of a variable (i.e. Given the number of squares, can you figure out the figure number?)
- order of operations
- skip counting
- area and perimeter
- exponents
- perfect squares
- diagramming/sketching as a problem-solving tool

The list above does not cover all of the math content you can address through visual patterns, but it gives you a sense of the scope and range of what is there. Focusing on an exploration of visual patterns is a nonintimidating way for students to make connections between different algebraic problems and concepts and build up a structure and coherence for understanding how those concepts fit together. Visual patterns are also a great way to introduce algebraic concepts by drawing out a need for them. To put it another way, I'll adapt the words of Dan Meyer (blog.mrmeyer.com): "If algebra is the aspirin, how do you create the headache?" For example, visual patterns allow students to develop statements generalizing patterns they have identified. By having to talk about those generalizations with others, who might have different generalizations, students will be begging to write them as expressions—once you've shown them how. Instead of seeing algebra as something arbitrary and removed, students will begin to see variables and expressions as tools for expressing generalizations efficiently.

You can focus your line of questioning about the pattern to draw out different aspects of algebraic reasoning. For example, you could return to a visual pattern you have explored with one series of questions and then ask another, more complex series of questions. New math content can also be drawn out, depending on the questions you choose. For example, later in this unit you will read a lesson detailing how to use a visual pattern (the arch problem) to create a linear function equation. When returning to the pattern, add a question about how many toothpicks it would take to create a given figure and the task changes to looking for a quadratic function.

Below, you will find some effective questions to use with visual patterns to draw out algebraic thinking and introduce algebraic notation. Please note that you would not want to ask them all at once.

- *How many different patterns do you see?*
- *How do you draw the next two figures?*
- *How do you draw the 10th figure?*
- *How would you draw the 25th figure?*
- *How would you tell someone how to draw any figure?*
- *How could we create a table organizing what we know about the figure numbers and the number of squares in each figure?*
- *If you had a box of 25 squares, what would the figure look like? Would you have any left over?*
- *How would you figure out how many squares are in the 99th figure?*

- *Describe how you would figure out how many squares there are in any figure number.*
- *How many squares would it take to build the nth figure?*
- (Given a set of expressions) *Which expression(s) for the nth figure would work? How does the expression connect to the picture?*
- *Using the picture, describe with words two different ways you could determine the number of tiles needed to make the nth figure in the sequence. Then write a rule or formula that matches each of the ways you described. Define your variables explicitly.*
- *Which figure would have exactly 138 squares?*
- *What is the perimeter of the 10th figure? The 99th figure? The nth figure?*

In the following pages, you will find the following supports for weaving visual patterns into the fabric of your classroom:

LESSON I: Introduction to Visual Patterns & A Scaffolded Approach to the Arch Problem

This lesson plan begins with some activities to introduce patterns in general. It then models a scaffolded approach to visual patterns with a selection of questions from the list above to build towards the development of linear equation(s) that describe the relationship between the number of squares and the figure number. The lesson goes a little deeper in making explicit how each question can be used to develop specific algebraic content.

For additional support refer to the video: **Introducing Patterns and Developing Algebraic Reasoning Through Visual Patterns: A Scaffolded Approach.** Visit the CUNY HSE Curriculum Framework web site at http://literacy.cuny.edu/hseframework.

LESSON II: An Open-Ended Exploration of Pattern #141

This lesson plan models an approach to using visual patterns that is more open than Lesson I. As with the arch problem, the activity and questions can be used with almost any visual pattern. Using the open-ended activity described here will not only help students develop their algebraic thinking, but it will give them opportunities to revise and improve their precision in mathematical communication. Since students are creating their own problems and choosing which ones to work on, this activity works especially well with a mixed-level class. Everyone is working from the same visual patterns, but at their own pace.

Pattern #141 is so named because it is the 141st visual pattern on visualpatterns.org. See *Recommended Resources for Visual Patterns* to learn more about this great resource.

Some Suggested Patterns

This is a sample progression of visual patterns that you can do with students over the course of a semester. The patterns in the progression get more complicated, moving from linear to quadratic to exponential growth. If you use this sample progression with your students, you might use the arch problem from Lesson I to start, or perhaps between the cross and the factory problems. You might use Visual Pattern #141 between the lobster claw and super columns problems.

Additional Recommended Resources

For teachers interested in learning more about visual patterns and how to bring them into your classroom, we offer some high utility resources for support.

A FINAL WORD ON HOW TO USE VISUAL PATTERNS WITH YOUR STUDENTS

Work on visual patterns regularly in your class. Sometimes that will mean dedicating a whole lesson to them, and sometimes it will mean giving students a visual pattern as a warm-up exercise. You can use the scaffolded approach and the open-ended approach with any visual pattern.

The lessons in this unit offer a model for a scaffolded approach and for a more open approach to working with visual patterns. They are a suggestion of how you might begin and where you might end up. Use the suggested resources and learn more about all the different kinds of activities that have worked for other teachers. Experiment and see what works best with your students. Use the list of “Math Content You Can Teach Through Visual Patterns” as a guide and create lessons targeting that content through visual patterns.

Lesson Plan I: Introduction to Visual Patterns

A Scaffolded Approach to the Arch Problem

OBJECTIVES

By the end of this lesson, students will have an experience with:

- ✔ Looking for and discussing patterns.
- ✔ Using patterns to make predictions and generalizations.
- ✔ Collecting data in a table.
- ✔ Developing strategies to move from concrete to abstract models.
- ✔ Finding recursive and explicit rules.
- ✔ Creating a written description to define a linear function relationship.
- ✔ Creating a linear function equation that matches a situation.
- ✔ Understanding the use of a variable in the context of a function with two unknowns, as opposed to solving for a specific value of *x*.
- ✔ Seeing a connection between pattern exploration and algebra.

COMMON CORE STANDARDS OF MATHEMATICAL PRACTICE

MP1; MP2; MP3; MP4; MP5; MP6; MP7; MP8

NOTE TO THE TEACHER

This lesson may take up to three classes depending on the length of your classes.

To see the classroom videos, **Introducing Patterns and Developing Algebraic Reasoning Through Visual Patterns: A Scaffolded Approach**, visit the CUNY HSE Curriculum Framework web site at http://literacy.cuny.edu/hseframework.

ACTIVITY 1 Launch: Algebra and Pattern Brainstorm

MATERIALS: Board/Newsprint

STEPS:

1. Ask students to take two minutes and write down anything that comes to their mind when they hear the word *Algebra*.
2. Bring the class together and ask them to call out their ideas and write them down as students share. You should preface this part of the activity by saying they should not worry about repeating anything or censoring themselves. Students should just say whatever occurs to them. You should also point out that you want to get everyone's ideas, so if they call something out that you miss, they should keep saying it until they see it up on the board. Put

their answers in two columns. If students say things that have to do with algebra topics, write that on the left. If they say things that have to do with disposition, mindset, emotions, self-concept, record those things on the right. In the end, you'll get something that looks like this:

• letters	• a different kind of math
• variables	• makes me want to give up
• expressions	• when I left school
• equations	• confusing
• x and y	• college level math
• slope	• frustrating
• solving for x	• the power when you say "a-ha!"
• formulas	• not connected to the real world
• finding what's missing	• logical
• finding the meaning of letters	

3. Next ask students to take two minutes and talk to the person next to them about what comes to their mind when they hear the word *patterns*—what are they and where do we find them?

4. Have students share their ideas and record them on the board. Encourage students to keep sharing and you'll end up with a collection of diverse patterns. Your board may look something like this:

- The things that happen every day (like daily routines)
- Repeated behaviors ("I always date the same kinds of people")
- Like if someone in your family is bald, their children may carry that gene and develop baldness
- Plaid, paisley, quilts (design patterns)
- Music and dance
- Seasons, days of the week

5. Suggest to students that it is interesting that one word applies to so many seemingly different situations. What do dancing, repeated behaviors, genetic dispositions, and fashion designs have in common? What is a pattern?

You may get words/comments like: sequences; the way something is supposed to happen; something that repeats itself; something based on past experiences.

6. A good way to help students think about this is to talk about the weather. Ask the class if anyone knows what the weather is going to be like over the next few days. Ask, *How can we possibly know what the weather is going to be like tomorrow? Do meteorologists have time machines?*

   The concept you want to draw out of the discussion is that patterns are about making observations and collecting information and using that to make predictions. The simplified version behind weather reports is that humanity has been observing the weather patterns in nature for thousands of years and we have learned that when certain things happen, certain other things tend to follow. And we have lots of ways to gather information and we use that information to make predictions, based on what has come before. The idea of making observations and making predictions is an important one, so go back to their brainstorm and see how each pattern they cited in the Pattern brainstorm fits that definition.

ACTIVITY 2 1, 2, 3, 4, 1, 2, 3, 4...

MATERIALS:

- **1, 2, 3, 4 Pattern (handout)**
- **1, 2, 3, 4 Pattern—The First 30 Numbers (optional handout)**

STEPS:

1. Tell the class that we are going to look at patterns as a way to help us understand something about algebra. To begin, write **1, 2, 3, 4** on the board and ask them to predict the next number in the pattern. Most of them will say "5". Tell them you understand why they would think that, but that you are going to give them a little more information. Write **1, 2, 3** after the **1, 2, 3, 4** you already have written and ask them what the next number will be. Ask them to keep predicting until you have the following sequence on the board:
   1, 2, 3, 4, 1, 2, 3, 4, 1, 2...

2. Give them the 1, 2, 3, 4 PATTERN handout and ask them to work on the first question on their own for one minute. After a minute have them share their thinking with a partner and work on the problems together.

3 As students are working, walk around and look at their strategies. Here are some examples of what you might see:

B

Used 4 Times Table

4th #	4
8th #	4
12th #	4
16th #	4
20th #	4
40th #	4
80th #	4
92nd #	4

C

10th → 2
20th → 4
30th → 2
40th → 4
50th → 2
60th → 4
70th → 2
80th → 4
90th → 2
100th → 4

A. Find someone who continues the pattern and writes out the sequence.

B. Look for any students who recognize that every multiple of 4 will be a 4.

"If I know every fourth number is a 4, than I know the 16th, 40th and 92nd numbers will all be 4s. Then I know the 19th will be a 3, the 41st would be a 1 and the 91st would be a 3."

C. See if anyone uses the given section of the pattern and just counts moving a pencil from number to number until they get to the 19th, 41st and 91st.

Look for anyone who continues the pattern to a point, but then starts to find other patterns.

"I started just writing out the pattern and then I noticed that every 10 numbers it goes back and forth between a 2 and a 4."

4 When you go over the handout, have someone who continued the pattern and wrote out the sequence until the 19th term to share their strategy.

5. Before you talk strategies, take a poll and write all of the answers students got. There may not be agreement on the answers. Having more than one answer on the board allows you to ask students to explain how they got their answers with the goal of convincing someone else of their answer. When you start to discuss the 41st number in the pattern, first ask if we could use the method demonstrated in Step #4 and continued writing out the pattern until we got to the 41st number. Some of them may want to say there are other ways, which you will look at, but make sure you make the point that the first strategy would still work. If someone used multiples of 4, ask them to explain how they used their method to figure out the 41st figure.

6. Begin your discussion of the 91st number by taking a poll of answers and writing them on the board. Ask if the previous methods could be used to figure out the 91st number. (They can.) Then ask if someone has another way.

7. Optional—If no one continued the sequence and found other patterns, you can give students an opportunity to play around with that idea. Tell students we are going to take another look at a table for this pattern to see if we can use the information to make observations that might help us with our predictions. Give out the 1, 2, 3, 4 Pattern—The First 30 Numbers handout.

Ask students to take a few minutes and look for any patterns they can find in the numbers that might help make predictions.

Some patterns students might identify:

> *"If you add 20 to any number in the sequence you will get the same number. For example, the 9th number in the sequence is 1 and the 29th number is also 1. The 3rd number is 3 and the 23rd number is also three, etc."*

> *"Every 10th number in the sequence is either a 2 or a 4."*

> *"The 13th number in the sequence is a 1. If you add 4 (and keep adding 4) to the number in the sequence, you'll keeping getting 1. So the 17th number is also 1 and the 21st number and the 25th number, etc."*

ACTIVITY 3 Exploring Visual Patterns— A Scaffolded Approach

MATERIALS:

- The Arch Problem, part 1 (handout)
- The Arch Problem, part 2 (handout)
- The Arch Problem, part 3 (handout)
- Different color markers

STEPS:

1. Write, **Algebra is the generalization of arithmetic** on the board. Ask students what *arithmetic* is. Then ask them what a *generalization* is. Ask students what they think the sentence means.

2. Tell students: *In order to help them learn algebra, I want to start working from what you already know. You are going to look at a visual pattern and collect information on how it is changing. Then we are going to work together to use algebra to generalize the patterns we find.*

   2 — This line of discussion is adapted from *A Collection of Math Lessons, Grades 6–8* by Marilyn Burns and Cathy Humphreys.

3. Hand out The Arch Problem, part 1, and give students two minutes to look at the pattern and write a few observations down. Then have them talk in pairs about what they see in the Arch Problem. While they are talking, draw the first three figures on the board, large enough to be seen in the back of the room.

4. Once your students have had some time to make some observations, bring the class back together and ask them to share the changes they notice in each figure.

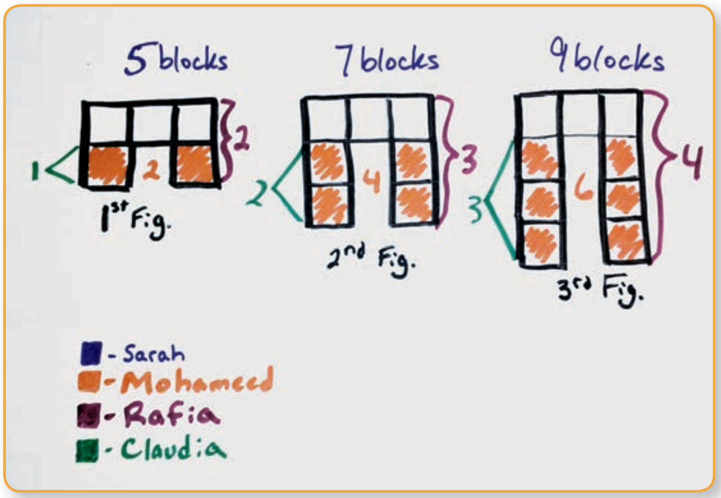
   Some things they might say:

   "The number of squares goes up by 2."

   "The number of squares underneath the top row goes up by 2."

   "As the figure number changes, the figure gets taller by 1."

5. As each student shares one of their observations, stop and ask them to explain so everyone can see it. If, for example, a student says, *"The number of squares goes up by two,"* ask her to point out specifically where she sees that in the drawing. Label the observation on the picture you drew on the board with a color and write the student's name below, in the same color. After you have recorded a few student observations, the picture might look something like this:



Identifying each student's contribution is helpful because as you go through the next activity, it allows everyone to talk about certain patterns by name—i.e., "Rafia's pattern"—instead of having to describe the whole pattern each time someone wants to reference it. Also, since everyone's heads are full of patterns they want to share, it's important to make sure everyone is listening to each specific pattern being shared.

6 Once all of the observations have been shared, tell the class they are going to use their collective observations to make predictions. Have your class get into groups of 2-3 students and hand out Arch Problem, part 2.

7 Walk around the room as the groups are working. Listen for interesting discussions, disagreements and struggles to raise during the whole class discussion. As you walk around, keep in mind the things you want to come out of the discussion for each question (detailed below) and look for connections.

Some things you should listen for/ask about:

a. When students sketch the next two figures, do their figures follow all of the patterns identified and shared in step 4?

b. How do students start their sketch of the next two figures? Some might draw three squares across and then draw the "legs" coming down. Some might draw one square in the middle and then draw the legs coming down. Some might draw the first figure and then add some squares to the bottom of each "leg." Try to take note of who is doing what.

c. Instead of describing what the tenth figure will look like, some groups might write, "The 10th figure has 23 squares in it." If you come across this, ask them to read the question again.

7a

When beginning to work with visual patterns, there are often 1-2 students who struggle with drawing the next two figures. It depends on which observation(s) they are using. Say for example they are only focusing on the number of blocks. There are 11 squares in the 4th figure. Eleven squares can be arranged in a lot of different ways. You can help these students by referring them to all of the patterns identified in steps 3 and 4. It not only has to have 11 squares, but it has to have a height of 5, etc.

7b

The first two ways are very common and almost always both come up. The third way is less common, and needn't be forced, but is a special treat when a student see the figures in that way.

Ask them, *"What will those 23 squares look like? How will they be arranged?"*

d. Some students might assume that if the 5th figure has 13 squares, the 10th figure will have double that amount, or 26 squares. If that happens, you might ask if there are any patterns they notice in the table. They probably already noticed that the number of squares goes up by 2. Ask them to continue the table and see if they get the same answer. Then walk around some more—give the group a chance to discuss on its own which answer they think is correct—the 23 or the 26. Check back in a few minutes and see where their conversation is. If it has stalled and there is no consensus, you can them if the 4th figure has twice as many squares as the 2nd figure, but it is much better if they can get there on their own. If they do, make sure to ask how they decided.

e. Which group has a clear set of steps for figuring out how many squares are in the 99th figure? Ask them to talk you through their steps. Look for one or two that are clear and ask the group if they would be willing to share their method during the debrief. Ask them to spend a few minutes talking about how they will demonstrate their method so it is clear to the other folks in the class.

8 Once the groups are mostly done with the first 5 questions (don't worry if no one has gotten to the bonus question), bring the class back together for a whole-class discussion.

Guide for Facilitating the Whole Class Debrief and Discussion

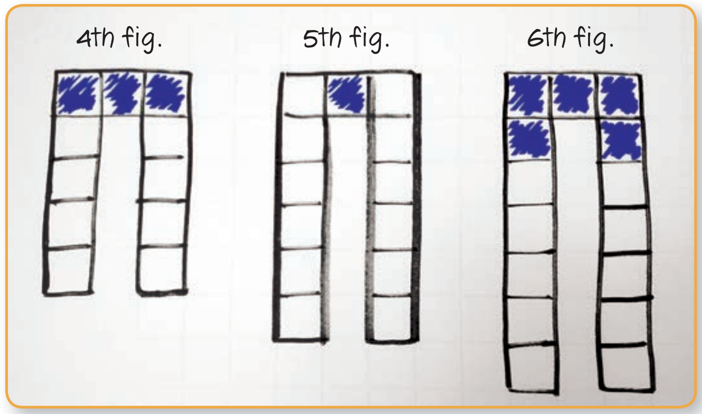
The goal in this scaffolded approach to visual patterns is to start concrete and get more and more abstract. If available, give students the option to use square-inch tiles to lay out the next two figures.

"Sketch the next two figures."

When going over this question, I tell students, *You are the brain and I am the hand.* From their desks I have them give me explicit instructions for how they want me to draw the 4th figure. There are a few different ways they might be looking at the figures and you want to draw out as many different ways of seeing the pattern as possible. What is important here is to get them to be clear about how they want you to start.

The first student might have you start by drawing the three squares across. Draw it the way they describe, next to the first three figures you have on the board. Color in the top three squares and then describe their way of seeing it—*So Roberta had me draw three across the top and then draw the legs underneath. Did anyone see it a different way?*

Try to find a student who drew the legs first and then added the square in the center of the top row (or, put another way, a student who drew the square in the center of the top row and then drew the legs coming down). Draw it and then color in the center square. Then ask if anyone saw it in a different way. If they did, you can have them describe how to draw the 6th figure.



When you are done, you'll have something like this *(see image at right)*:

"Complete the table."

Draw the table on the board and make sure you leave space below so you can continue the chart later in the debrief. Students will likely mention that they see a pattern in the number of squares: they go up by 2. If they don't bring it up, ask if anyone sees any patterns. Either way, write it on your table.

This "going up by two (+2)" is the iterative (or recursive) rule. The iterative rule is a rule that you can use to find the value of a term (number of squares in this case) by using the value of the previous term. For example, if the 10th figure has 23 squares, I know the 11th figure has 2 more, or 25 squares.

Ask the class how the table was helpful to their work. If it doesn't come up, mention that the table can help us organize the information without having to draw out all the figures. It also helps us identify more patterns.

"In a few sentences, describe what the tenth figure would look like."

Remember, you want a written description for this one. By all means, talk about the different ways students came up with 23 (and discuss any disagreement about that answer). But then get some descriptions on the board. Try to write down exactly what the student says. Then, draw a picture based on the description only—if you can playfully "misinterpret" any details and come up with a different figure, please do. Give students the opportunity to revise their description until your figure matches the picture they had in mind.

You should end up with descriptions similar to these:

> *"Draw three squares across. Then draw ten more squares down under the left square and another ten squares down from the right square."*

> *"Draw a column of 11 squares. Draw another square to the right of the top of the column. Draw another column of 11 squares down on the other side of the center square."*

You want at least one picture of the 10th figure on the board.

It's ok for the words to feel awkward. The eventual takeaway is that algebraic notation has a purpose. It is a tool that allows us to express our way of thinking clearly and concisely. Of course, you should not mention this yet.

An **explicit rule** is a rule that allows you to find the value of any term in the pattern without needing to know the value of the term before it. If you look at visual patterns in the context of functions, the explicit rule is the function rule. (And the **iterative/recursive rule** is the rate of change).

"Explain how you would figure out the number of squares in the 99th figure."

TEACHER NOTE: Students can certainly use the iterative/recursive rule to answer this question and there is usually someone who continues the chart all the way to the 99th figure. Which is great. But this question also encourages students to look for an explicit rule. When we put students in a situation where they are telling themselves, "there has to be another way", they will often start looking for patterns. We should make this explicit whenever it happens.

For this question, ask one of the groups that you identified during the group work problem-solving to come up and share their approach. After the group presentation(s), if it hasn't come up, ask the class how the 201 squares in the 99th figure would be arranged.

There are a few ways they might answer this one. They might recognize that if you draw three across the top, the number of squares down each column is equal to the figure number. They might also say, "We have 201 squares total. If we put three across the top, that leaves us with 198. Since both columns are equal, if we divide up the 198, we get 99 in each column."

For the other way of seeing the figure, they might recognize that after you put the one square in the center, the number of squares in each column is one more than the figure number. They may also use the 201 and say, "Well, after we put the 1 in the middle, we are left with 200 squares, so that is 100 on each side."

If they are not sure, you can refer to the two (or three) different ways of drawing the 10th figure and give them a few minutes to think about your question.

Teacher: How would *(student's name)* start drawing the 99th figure?

Student: She would draw three across the top.

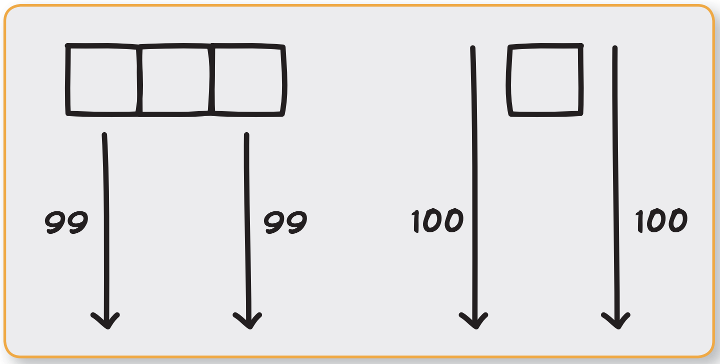
Teacher: How many squares would there be in each column going down?

Student: 99.

Teacher: What about *(student's name)*. How would she start drawing the 99th figure?

Student: She would draw the center square. She would then draw 100 squares down each side.

By the end, you want to create a visual sketch model that students can use as a shorthand.



"In a few sentences, describe how you would determine how many squares there are in any figure in the pattern."

Give the groups five minutes to work on the bonus question. Some of them who either didn't have time or were too intimidated to answer it before will have a little more confidence. If any groups finished it during the group problem-solving, ask them to try to use the debriefing notes from the board to try and find a second way.

After five minutes, bring the class back together. Remind them that when we look at information, make observations and find patterns, those patterns allow us make predictions.

The bonus question is asking us to predict how many squares there would be in any figure number, so let's look at how many squares there are in the figure numbers we already know.

Point to the visual sketch model you drew for the 99th figure. Ask how many squares are in the 99th figure. They'll either say *"99 + 99 + 3"* or *"2 × 99 + 3"*. Whichever form they use, record it on the board and use the same form for the rest.

Ask, *How did we arrange the 10th figure?*

And record their answer— either "10 + 10 + 3" or "2(10) + 3"

Continue all the way to the 1st figure.

Then, ask students how they could figure out the number of squares in the 47th figure, and record their answer: "47 + 47 + 3" or "2(47) + 3." Do it a few more times if it feels necessary.

Then ask your class: *What is changing? What is staying the same?*

Teacher: What is changing?

Students: The figure number.

Teacher: What is staying the same?

Students: We are always doubling the figure number and we are always adding three.

You want to start with the view of the figure that starts with the three squares across the top. When we get to generalizing the rule in a few steps, it is easier to work with the rule that has the simplest form of figure number in it. I recommend the first one, but the others can certainly be used as extensions for faster students. The first can be written as 99 + 99 + 3 and generalized as $2n + 3$. The other one is (99 + 1) + (99 + 1) +1 and generalized as $2(n + 1) + 1$. The rule that involves the 1st figure can be written as (99 – 1) + (99 – 1) + 5 and generalized as $2(n – 1) + 5$.

This is an opportunity to introduce students to writing 2 × 99 + 3 as 2(99) + 3.

If you prefer them to use the $2n + 3$ format, encourage them to do so by asking for another way to write adding a number to itself.

The first time you do visual patterns, focus on building only one of the ways of seeing the figure into an algebraic generalization. For future visual patterns, looking at more than one can be a great leaping off point for talking about equivalent equations.

Ask everyone to write in their notes, the steps they would take to find the number of squares in any figure number. Walk around and see what folks are writing and decide a few to share. Take a volunteer and record their steps on the board. Ask if their steps would always work. Have the class test it with the figure numbers we've already worked with, until everyone agrees it would work. Then ask if anyone has a different way.

Once you have at least one clear set of steps on the board that works, under the 47 + 47 + 3 or the 2(47) + 3, write an n. Say, *I don't know what figure number this is, but whatever figure number it is, what will I do to figure out the number of squares in it?*

Record their response—either "*n + n + 3*" or *2(n)+3*.

At the end, this piece of the board should look something like the image on the right:

$2(99)+3$
$2(10)+3$
$2(5)+3$
$2(4)+3$
$2(3)+3$
$2(2)+3$
$2(1)+3$
$2(47)+3$
$2(n)+3$

9

Using a variable to stand for the independent change in a function—as we are using it here—is only one way to use a variable.

If you want to go a little deeper into this use of a variable, you might ask some follow-up questions: What does *n* represent? How many different numbers can *n* be?

9 **Ask if anyone knows the word for when we use letters like "*n*" to represent numbers.** Chances are, someone will throw out the word "variable." Ask them what it means when something varies and remind them of your question about what was changing and what was staying the same. It is the element that was changing—in this case, the figure number—that we represent with a variable. Have students add the following definition of a variable to their notes: *"When working with visual patterns, the variable is the part of an explicit rule that changes for each figure."*

10 **Give out The Arch Problem, part 3.** It could be either the final problem-solving activity of class or it could be given as a homefun assignment.

NOTE TO TEACHER: It's too much to get into this the first time you look at a visual pattern, but this question, "*Which figure will have ____ squares*?" can be used to build towards solving one-variable equations—i.e. solving for "*n*" in the equation $2n + 3 = 175$. But hold off on making that explicit until after they've come up with their own methods and shared them. Give students a chance to work on it as it is written and it can become the foundation of students solving for a specific unknown, except instead of us having to tell students how to do it, they can tell us.

Debriefing this question is a good moment to compare an equation using a variable that has one specific value to a function, which is about the functional relationship, where the variable can be any number. Ask how this use of the variable is different from the way we used it when we came up with the rule, **$2n + 3$**.

- **"Which figure will have 175 squares in it?"**

There are a few strategies you should look for to have students share.

Some students might use guess and check. You should go over this method first.

They already know the 99th figure has 201 squares, so they know it is smaller than the 99th figure. Say, they start with the 80th figure. They will test each guess by using the rule—doubling the figure number and adding three—until they get to the 86th figure and realize the $2(86) + 3 = 175$.

Some might refer to the visual sketch of the figure and say something like, "Imagine you have 175 squares. You put three across the top, leaving you with 172 squares. You divide the 172 squares by 2 and divide them evenly to each side of the figure. That would give us a figure with three squares across and 86 squares down each side. So it is the 86th figure."

- **"Which figure will have 44 squares in it?"**

This is a bonus question because it is a trick question. There is no figure in this visual pattern that will have 44 squares in it. Many students will come to that conclusion, but the goal is for them to be able to explain why and how they know that is the case.

You need to be able to subtract 3 from the number and end up with an even number. Put another way, if you put the three squares across the top, you need an even number of squares to distribute between the two columns. 44 minus 3 is 41, which cannot be divided evenly into two columns.

11 **Re-write "Algebra is the generalization of arithmetic" on the board.** Ask students to do a pair share and discuss how the sentences connects to the work they've been doing on the arch problem.

This math log was adapted from *A Collection of Math Lessons: From Grades 6 through 8* by Marilyn Burns and Cathy Humphreys

Check-Out/Exit Ticket

- **Give students a minute or two to look over the whole board. Tell them you are going to give them some time to reflect on what they learned in class.**
- **On a separate piece of paper—you'll be collecting it—have students take notes on the following items:**
  - The date of the lesson.
  - What do you think would be a good title for today's class?
  - What happened in class today?
  - What did you learn about today?
  - What do patterns have to do with algebra?

Introduction to Visual Patterns

Examine the following pattern:

1, 2, 3, 4, 1, 2, 3, 4, 1, 2...

1. What will the 19th number in the sequence be?

2. What will the 41st number in the sequence be?

3. What will the 91st number in the sequence be?

Adapted from *Math Matters* by Suzanne H. Chapin and Art Johnson

Introduction to Visual Patterns

"1, 2, 3, 4 Pattern—The First 30 Numbers"

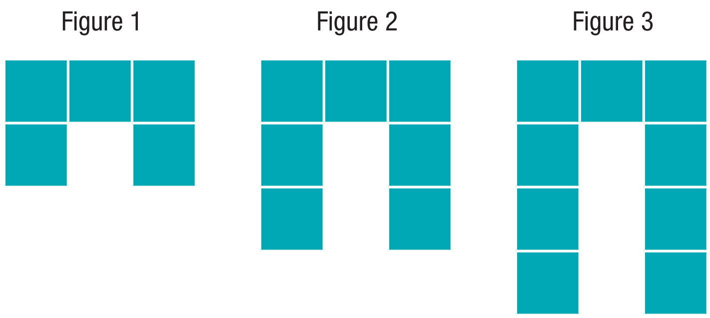
What patterns do you notice?

Number in Sequence	Number
1st	**1**
2nd	**2**
3rd	**3**
4th	**4**
5th	**1**
6th	**2**
7th	**3**
8th	**4**
9th	**1**
10th	**2**
11th	**3**
12th	**4**
13th	**1**
14th	**2**
15th	**3**

Number in Sequence	Number
16th	**4**
17th	**1**
18th	**2**
19th	**3**
20th	**4**
21st	**1**
22nd	**2**
23rd	**3**
24th	**4**
25th	**1**
26th	**2**
27th	**3**
28th	**4**
29th	**1**
30th	**2**

The Arch Problem, part 1

Look at the figures below. What do you notice?



Complete the following sentence:

As the figure number changes, ______________________________________ also changes.

The Arch Problem, part 2

1 Sketch the next two figures.

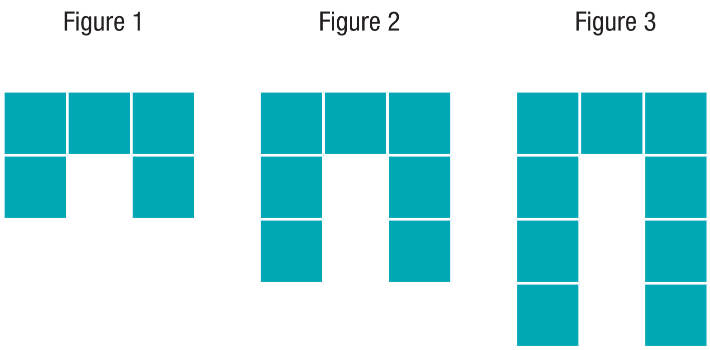


Figure 4:

Figure 5:

Figure Number	Number of Squares
1	
2	
3	
4	
5	

2 Complete the table to the right.

3 In a few sentences, describe what the tenth figure would look like.

4 Explain how you would figure out the number of squares in the 99th figure.

5 BONUS QUESTION: In a few sentences, describe how you would determine how many squares there are in any figure in the pattern.

The Arch Problem, part 3

Which figure will have 175 squares in it?

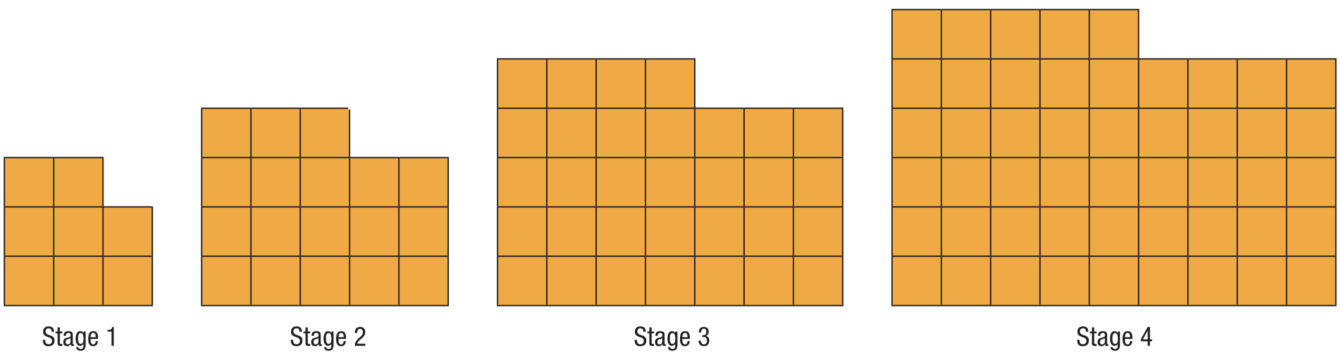
BONUS QUESTION:

Which figure will have 44 squares in it?

Lesson Plan II:
An Open Approach to Visual Patterns

In the scaffolded approach to visual patterns, the teacher identifies the change that students should observe. In many visual patterns, this change will be in the number of squares that make up the figure in each stage of the sequence.

In the pattern below, for example, you could count the number of squares in each figure, create a table to collect the data and eventually come up with a rule for finding the number of squares in any figure.



This is just one way to look at change in a visual pattern. You could also ask students to find height of the 10th figure or the width of the 20th figure. How many vertical lines would it take to draw the 15th figure? If you built one of these figures out of toothpicks, how many would you need? What is the perimeter of the 30th figure? When we look at visual patterns with new eyes, we see that are many different kinds of change that can eventually be represented algebraically.

In this lesson plan, we use an open, less scaffolded approach to encourage students to find different kinds of change in visual pattern sequences and then investigate that change. The goal is to generate interest, creativity, and flexibility in student thinking.

We want to balance scaffolding and openness. As students become comfortable working with visual patterns in a scaffolded approach, we can begin to remove the scaffolding and release responsibility to students to make decisions about what they want to investigate and how they will organize their work.

The main activity here—looking at a visual pattern and writing questions—can also be used as warm-up activity before using a more scaffolded approach similar to what is described in the first visual pattern lesson above. This can be useful to generate different ways of seeing a visual pattern before settling in to understand one kind of change together as a class.

OBJECTIVES

- Students will reflect on their own persistence and practice self-awareness while doing math.
- Students will write their own questions in response to a visual pattern.
- Students will consider audience when writing questions and have an opportunity to revise their questions based on peer feedback.
- Students will annotate posters of each other's work.

COMMON CORE STANDARDS OF MATHEMATICAL PRACTICE

MP1; MP2; MP3; MP4; MP5; MP6; MP7; MP8

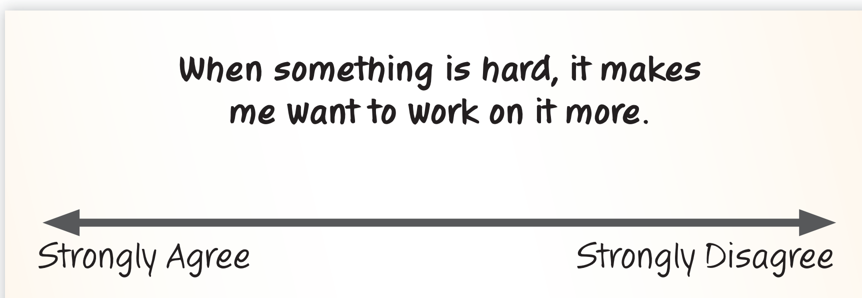
ACTIVITY 1 Launch: Persistence Spectrum

MATERIALS:

• A spectrum drawn on newsprint or the board • stickies

STEPS:

1. Before class, draw a large horizontal double-headed arrow as a scale (spectrum) on newsprint or the board. Above the arrow (leave enough room for students to place stickies with their name), write the following statement: "When something is hard, it makes me want to work on it more." Underneath the arrow, write **Strongly Agree** and **Strongly Disagree**, as shown below:



2. When students come into the room, ask them to write their name on a stickie big enough for everyone to see and then place their name along the scale. Encourage students to choose a spot along the scale that accurately indicates their feelings about the statement. They don't have to choose strongly agree or strongly disagree. They can place their name anywhere along the spectrum. As a model, place your name along the scale as well. When I did this lesson, I put my name on the left, but not all the way over. I explained that it depended on the situation. I enjoy challenges sometimes, but only if I feel that I have a fighting chance. I sometimes get discouraged and want to give up. This is a nice opportunity to have individual conversations with students how they think about this statement.

You may want to propose some situations when this statement is important (raising children, managing a household, dealing with a supervisor at work, coming back to school as an adult).

3 **Pair/Share**: Students should turn to a partner and talk about why they chose that spot on the spectrum. You might ask these pairs to share examples of things they do even though they are hard.

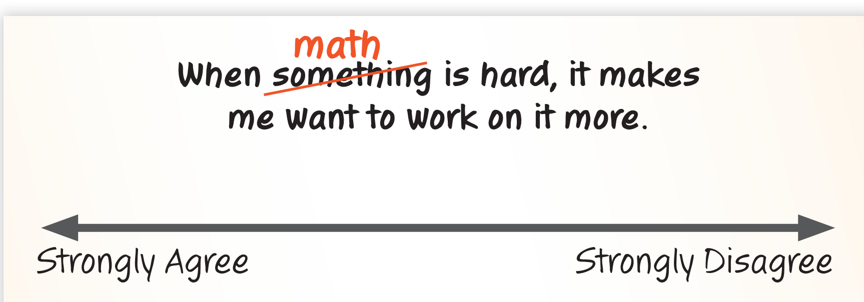
4 Ask for volunteers to share something from their conversation. Ask students to give examples of situations when they agree with the statement and situations where they disagree. Possible discussion questions:

- *What are some things that you do even though they are hard?*
- *Are there times when you feel like giving up?*
- *How do encourage yourself to keep on going when things are difficult?*
- *What strategies do you use to encourage yourself?*
- *What advice would you have for other students who are preparing for the HSE exam?*

5 Ask the group to look at the spectrum and pose the question, *What do you notice about how we placed our names?* Possible responses to the discussion include:

- *Most of us agree with the statement.*
- *A few of us are unsure about how we feel when something is difficult.*
- *The majority of the class likes a challenge.*

6 If students haven't spoken about math yet, ask how they feel about challenges in math class. Go back to the spectrum and make the following change:



Ask the class if anyone would like to move their stickie based on this change.

7 You might say at this point that you have a challenge for them today that you think will be interesting and you hope that by the end of class, some students will move their names towards the left on the spectrum. We don't expect huge changes in one day. We're looking for slow and steady improvements.

ACTIVITY 2 Writing and Sharing Questions About Visual Pattern #141

VOCABULARY

pattern
figure
stage
series

MATERIALS:

- **Newsprint with drawings of the four stages of visual pattern #141 large enough for the class to see from their seats (easel paper with a grid is helpful)**
- **Handout: Visual Pattern #141**
- **large strips of paper (newsprint cut horizontally into 4" wide strips)**
- **tape**
- **markers**

TEACHER NOTE: This is one of the more challenging puzzles from visualpatterns.org because the number of squares in each stage grows quadratically. We recommend trying a few linear growth sequences with your class before this one. You can, however, use most elements of this lesson plan with any of the visual patterns collected below.

STEPS:

1 Before starting this activity, break your class into small groups of 2-3 students. Depending on the culture of your class, you can let you students choose groups or you can decide on the groups yourself. Either way, we recommend sharing advice like this before you begin:

You'll be working in small groups for a lot of the class today. I know for myself that it doesn't feel great when I'm doing math with someone and they race ahead of me. If you're working with someone who takes their time, try to be patient and work together with them, without rushing ahead.

It's important that we all are able to have time to think and come up with ideas without being rushed. Try not to just give each other answers. If you are a team, work together and make sure that everyone understands before you move forward.

1 We want to pair students so that they can support each other, but we also need to make sure both students have the opportunity/time/space to grapple with the problem in their own way.

An open approach to looking at visual patterns can allow advanced students to tackle more difficult problems, while other students work on more accessible questions. If the groups are more homogenous, students can choose questions that seem doable to them.

2 Post easel paper with visual pattern #141 up on the wall. You might hang it before class and fold it up with tape in order to keep the

activity a surprise. Ask students to look at the pattern and think about what is happening with these figures. There isn't anything in particular for students to notice at this point in the lesson, but they may come up with some ideas that help other students get started with the worksheet that follows. This also might be a good time to define vocabulary they will be using in the rest of the lesson: *pattern, figure, stage, series.*

This line of questioning can be helpful to introduce students to the new visual pattern before working independently:

- *What do you notice?*
- *What patterns do you see?*
- *What could you say about the figures as they grow?*
- *How would you describe the series to someone who couldn't see what you're looking at (over the telephone, for example)?*

3 Give out the VISUAL PATTERN #141 handout. Ask students to work individually. They should start by describing patterns that they see: *Do you see a pattern? Describe in words what you see.* Encourage students to write sentences, or at least notes, with details about the figures. Some preliminary examples might include:

- *The drawing is getting bigger in every stage.*
- *There are more squares in each figure.*
- *There are more missing squares each time.*

Encourage students to include details: *In what way is it growing bigger? How many more squares do you see? Which squares are missing? Can you show me how the figure is growing? How can you include these details in your written description?* More precise, rewritten sentences might include:

- *Each drawing is one row taller than the last one.*
- *Stage 2 has 10 more squares than Stage 1. Stage 3 has 14 more squares than Stage 2.*
- *There is one more missing square each time. For example, Stage 1 has one missing square and Stage 2 has two missing squares.*

4 After most students have written some descriptions, encourage them to start on the second prompt: Think of some questions you could ask about this series of figures. Write 2-3 questions below.

5 After students have had a few minutes to write questions, ask them to get into their small groups. Say, *Share some of your questions with a partner and then write more questions.* The goal is to come up with as many questions as possible.

2

Remind students that we will be spending time in group discussion today and should remember to be respectful. Ask student to try these sentence starters:
"I agree with ____ because…" or "I respectfully disagree with ____ because…"

5b

We want students to struggle, but we don't want students to shut down with the openness of this task. If the struggle becomes unproductive, we should talk to students individually and explain, "Here's what I mean..." and offer some model questions.

What would the next figure look like if I drew it?

How high would the next figure be?

How many squares would there be in the 8th figure?

What questions can you think of?

5d

There are different ways to structure a class around student questions. You might ask the group to work on one particular question first, and, if there is time or for homework, students can move on to other questions that were generated. Most importantly, students have more ownership on the work they do on the problem. Asking questions about patterns and working to answer these questions is way for students to act as mathematicians.

After a few more minutes of walking around, and listening to students' conversations, offer a few questions that occurred to you, such as, *What would Stage 5 look like?* and *Would Stage 10 have an even number or odd number of squares?*

If this is the first time your students have approached a problem like this in an open way, they will most likely need some guidance in forming questions, but let them struggle for a bit before you help them.

a. If students have trouble coming up with questions, you might write the following phrase under the visual pattern on newsprint:

> In the figures below, as the stage changes, _______ also changes.*

Ask students to fill in the blank with as many different words as they can. Possibilities include:

- the number of squares
- the number of columns
- the number of rows
- the width
- the height
- the perimeter
- the area
- the number of line segments
- the number of intersections

b. Now, ask students to use these observations in order to form questions about the figures. For example, they might ask: "How many columns will the Stage 10 figure have?" Give students a few more minutes to brainstorm questions in their groups.

c. When each group has at least a few sentences completed, say, *Together with your group, decide which questions you'd like to share with the class and write them on these large strips of paper.* Hand out a couple strips of paper and a marker to each group.

d. Collect the strips as they are written and tape them up on the wall near the newsprint with the visual pattern. You might tape them in rows at the front of the room so that everyone can see them. You can hang the questions as they come in, or you can try to group them in categories. Make sure that each group has contributed at least one question.

*FOOTNOTE: Thank you to Blake Peterson (Brigham Young University) for suggesting this prompt and brainstorming activity in his workshop at the National Council of Teachers of Mathematics Conference, 2015. You can see the presentation for his workshop, "Intellectually Engaging Problems: The Heart of a Good Lesson," at http://nctm.confex.com/nctm/2015AM/webprogram/Session31618.html.

There are many possible questions that students could ask. Here are is a list of questions brainstormed by teachers in a professional development workshop:

What would the 5th stage look like?

What will the 50th stage look like?

How many squares would there be in stage 5?

How many squares are there at stage 10?

How many squares do you have to add to get the next figure each time?

Will the 10th stage have an odd or even number of squares?

Will the bottom row always increase by 2?

How many columns will stage 25 have?

Will it always get one taller each time?

How many rows will stage 25 have?

If these were tiles on a kitchen floor, how many tiles would you need for kitchen 5?

What is the area and perimeter at each stage?

If you had to find out the area of the floor after taking out a piece, what would the area be?

How much does the perimeter increase at each stage?

How can we predict the area at stage 10?

Imagine that these figures were made out of toothpicks. How can you find the number of toothpicks in stage 5?

Is this a linear, exponential or quadratic relationship? Or all of the above?

What would stage 0 look like?

6 When all the groups have contributed questions for the wall, read the questions aloud to the group one by one. If students can't see the questions from their seats, you might ask them to come up to the board to look at all the questions while you read them aloud. Say, *As I read the questions, look at the figures on the board and see if you understand the question.* Read questions aloud one at a time. After reading all the questions, engage students in the following line of questioning:

6

One goal of this discussion is to help students think about audience while writing. In hearing other students express their understanding of a question, the writer will learn whether they have communicated clearly enough to be understood. After this discussion, students should be given an opportunity to rewrite their questions if they need to clarify their intention. This section of reading and rewriting questions is not essential to the larger activity, and can be skipped if time is a constraint.

a. **To the class:** *We're going to spend some time looking at a couple of the questions you wrote. We won't have time to look at all of them in detail, but I want us to practice writing with precision so that others can understand our thinking. When we write something, we know what we mean, but do others have the same idea? This is an important skill for the HSE exam, in which you will have to write short explanations on all subjects of the test, including math.*

We can help each other improve our abilities to communicate our ideas clearly. We want to be understood when we talk about our mathematical ideas. Other people can help us do that by sharing what ideas they have after reading what we write, and giving us a chance to change our words to get our ideas across even more clearly. So, this is an opportunity for us to practice writing as a way to communicate mathematically. Which of these questions are interesting to you? Are there any questions here that you are not sure what they mean? Give an example of a question you like that isn't your own.

b. **To a volunteer:** *Could you read the question aloud to the group?*

c. **To the same student:** *Looking at the visual pattern (the large drawing on newsprint), please explain your understanding of the question.* (Teacher note: Ask the writer of the question to listen carefully.)

Let's say, for example, that the question chosen was: Will the bottom row always increase by 2? We would expect the volunteer who read this question to be able to show the class how the number of squares across the bottom is 2 more in each stage. Another way to say this is that there are 2 more columns in each stage of the figure. Ask the student to show their understanding of what the question is asking by demonstrating visually with the figures on newsprint.

d. **To the class:** Before we ask the writer to respond, *does anyone have thoughts about this question? Does anyone have a different idea about what this question is asking? What gave you that idea? If you could ask the writer one question, what would it be? What would you like to be explained?*

e. **To the writer/s:** *Is this what you meant by your question? Is there anything that you want to explain about the change you see in the figures? Would you change anything about how you wrote the question?*

f. **To the class:** *What's another question someone finds interesting? What's another question that you're not sure what it means?*

ACTIVITY 3 Work on Visual Pattern #141

1. **Individual Work:** Students should now start to work on one of the questions. The groups should each choose questions that are interesting to them and they feel like they might be able to answer. Each group should choose a question to work on, but to start, each student should work on the question independently. Make it clear to students that they should work alone at first, without help or conversation with each other. Say, *Choose a problem/pick a question, with your partner, and then work on the question alone.*

2. **Pair Work:** Give the groups time to talk about what individual students each did. While students are sharing with their partners, visit each group to get a sense of the work each person had done and the conversation of the pair. This will allow you to guide the whole-class discussion that follows.

3. **Whole Class Discussion:** One way to organize the whole group activity is through student posters. Having posters can help other students focus on other students' thinking and allow them to ask targeted questions. Here is a great way to organize a gallery walk of student work:

   a. Groups create posters on newsprint or a section of the whiteboard.

   b. One partner stands by the poster and fields questions while the other walks around to look at the other posters, then students switch.

   c. Students annotate each others' posters with questions and comments on stickies.

   You can also ask each group to report out and give a general sense of what they discussed. Since students will have worked on different questions, it is important that other students be given time to understand their questions, the patterns they saw and the conclusions they came to.

1

This set of instructions gives problem-solvers time to try some things on their own, but with the expectation that they are going to have to explain their ideas and talk about what they did with a partner. Individual students should come up with some of their own ideas first before talking with a partner. We want to insure different student voices are heard. Students need to be in a place where they are ready to contribute ideas and hear the ideas of a partner.

ACTIVITY 4 Extension Questions

MATERIALS:

- **Brainstormed questions about Visual Pattern #141**
- **graph paper**
- **scaffolded worksheet on Visual Pattern #141**

STEPS:

1. Some groups will finish answering their questions sooner than others. One of the nice things about this approach to using visual patterns is that you should now have a long list of questions that can be used as extensions. You can simply ask the group to choose another question from the wall to work on.

2. Another way to think about extensions is to ask groups to represent their work in different ways and generalize the answers they have discovered. You might give them newsprint to present their work, and respond to some of the following extensions.

   a. If the group has worked on the question, "How many columns will stage 25 have?", you might ask them to organize their results so that someone else can see the pattern they see. How many columns are there at each stage? This might lead to a table or a chart, but we recommend letting students decide how to present the pattern.

   b. If a group has a chart or table: Figure out a way to determine the number of columns in any stage. What would a rule in words be for this pattern? The answer for this question might be, *multiply the stage number by two and add one.*

   c. If the group has a rule in words: Find an algebraic equation that works for the nth stage. The answer in this case might be, $2n + 1$.

   d. If the group has an equation:

   - Graph the pattern you are investigating on graph paper.
   - Find the slope (or rate of change) of the column growth. Two, in this case.
   - Find the y-intercept (or starting amount) of the column growth. One, in this case.

2

\One challenge to using an open approach to visual patterns can be that it is difficult to predict the questions that students will choose and have the answers prepared. We recommend spending some time working on the visual pattern before class, ideally with other teachers, so that multiple ways of seeing patterns and related algebra are revealed. It is certainly possible to do this work on your own as well. Put yourself in the position of a student, write a few questions about the visual pattern, then work to answer them. In the process of looking at patterns, asking questions and finding rules, you will discover connections that your students will make as well.

3 Finally, you might prepare a scaffolded worksheet with a set of questions related to a particular kind of change in the visual pattern. For example, you could create a worksheet related to the number of squares in each figure or, for more of a challenge, the number of toothpicks necessary to build each figure. Whatever kind of change you identify, you can ask the following questions in a worksheet:

a. *Draw the next two figures.*

b. *What patterns do you see?*

c. *Create a chart with the data you have gathered. (Insert a blank table with headers of Stage # and Squares (or toothpicks, columns, or other change.)*

d. *In a few sentences, describe what the 10th figure would look like.*

e. *Explain how you would figure out the number of _____ in the 25th figure.*

f. *In a few sentences, describe how you would figure out how many _____ there would be in any figure in this pattern.*

g. *Which figure will have (some number) _____?*

Check-Out/Exit Ticket

This is a process that can be used at the end of class to get quick feedback from students and encourage discussion about ways of learning.

- *Think about today's class. In your head, think of a number 1–10 that represents how you feel that class went and what you learned. Are you ready? Don't say it yet. One, two, three. Say the number.* (Listen to see if you can hear more than one number at once.)
- *I didn't quite hear everyone. Can we try again? One, two, three. Everyone at the same time. Say the number.*

Ask a couple volunteers to explain their number. You might start with a couple lower numbers, like a 5, to hear from someone who wasn't so positive, then hear from a couple people who had higher numbers. This should be a short discussion, though, and just enough so students feel comfortable giving feedback. End session.

Visual Pattern #141

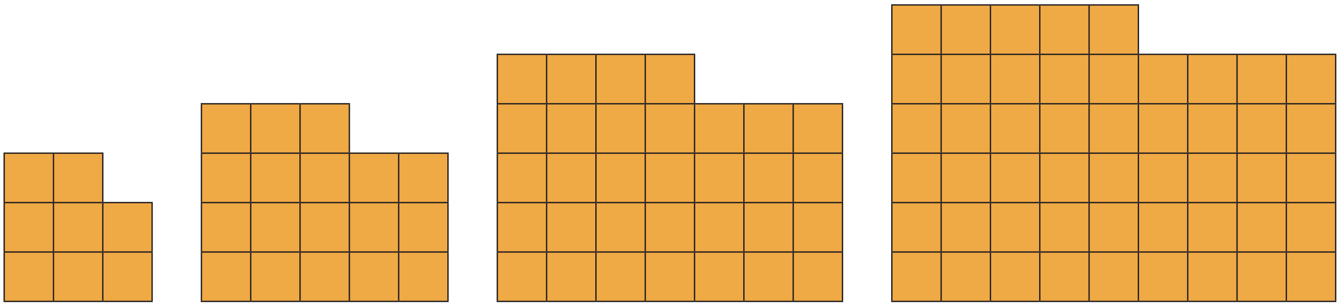


Figure 1 Figure 2 Figure 3 Figure 4

1. Do you see a pattern? Describe in words what you see.

2. Think of some questions you could ask about this series of figures.
Write 2-3 questions below.

A Sample Progression of Visual Patterns

Visual Pattern 1: The Upside Down T Problem

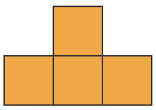
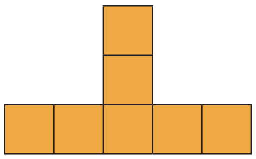


Figure 1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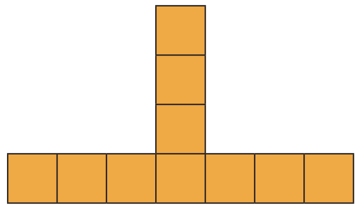
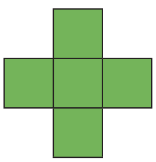


Figure 3

Visual Pattern 2: The Cross Problem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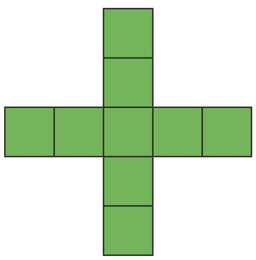
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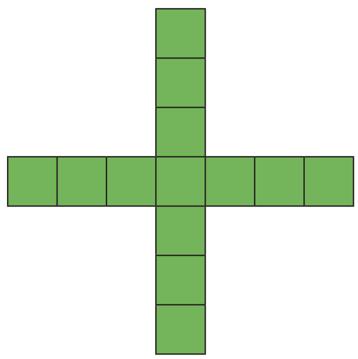
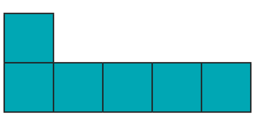


Figure 3

Visual Pattern 3: The Factory Problem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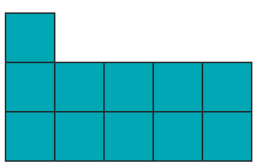
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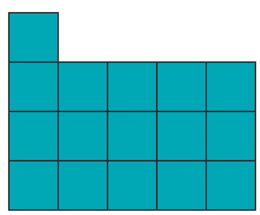
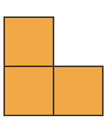


Figure 3

Visual Pattern 4: The Zig Zag Problem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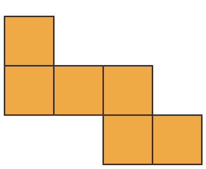
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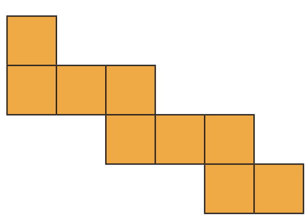
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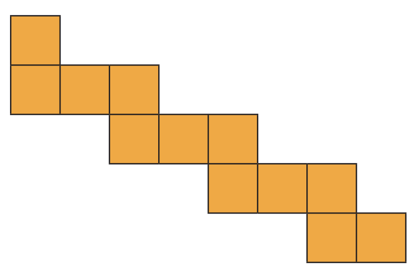
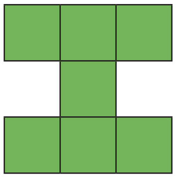


Figure 4

Visual Pattern 5: The I Problem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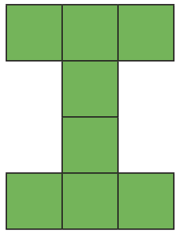
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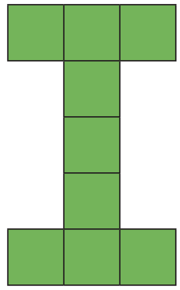
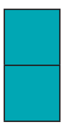


Figure 3

Visual Pattern 6: The Columns Problem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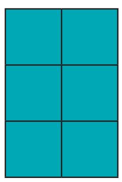
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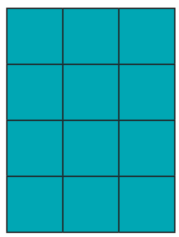
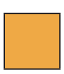


Figure 3

Visual Pattern 7: The Steps Problem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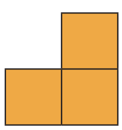
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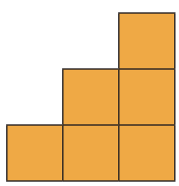
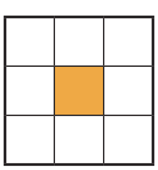


Figure 3

Visual Pattern 8: The Pool Problem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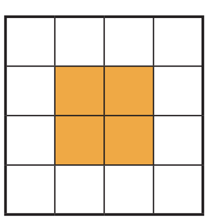
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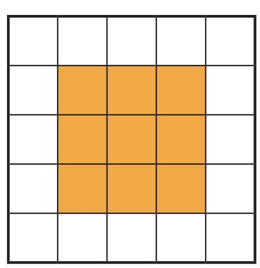
Figure 3

Visual Pattern 9: Square and Triangular Numbers

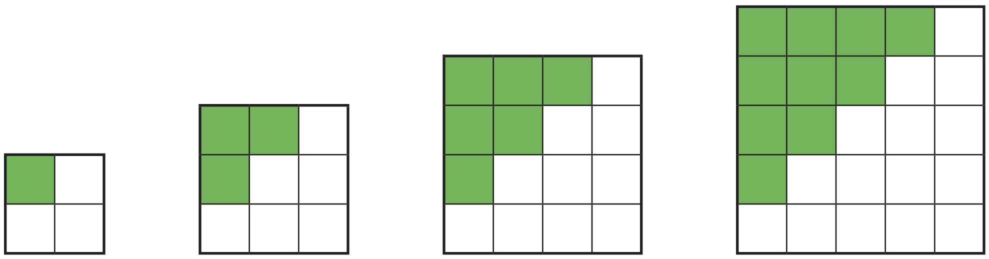

Figure 1 Figure 2 Figure 3 Figure 4

Visual Pattern 10: The Lunar Lander Problem

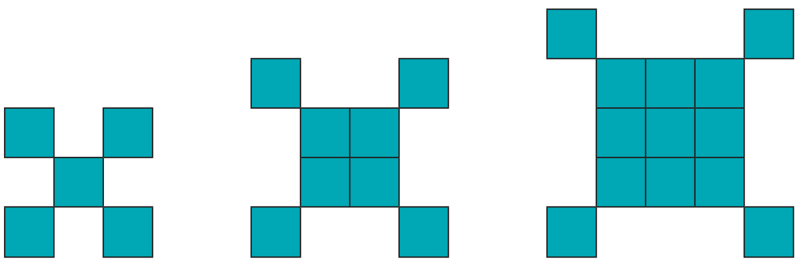

Figure 1 Figure 2 Figure 3

Visual Pattern 11: The Lobster Claw Problem

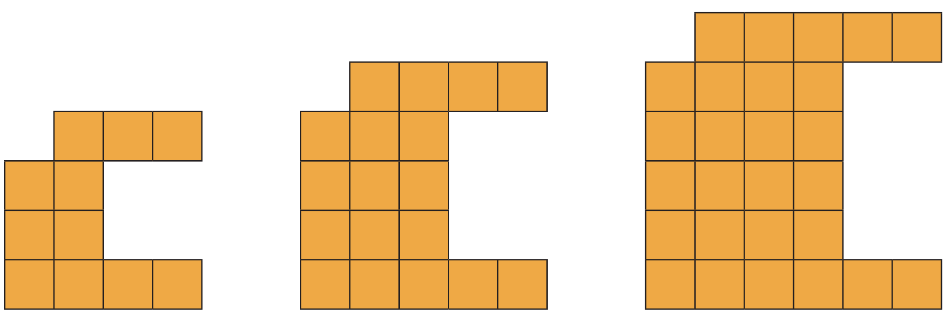

Figure 1 Figure 2 Figure 3

Visual Pattern 12: The Super Columns Problem

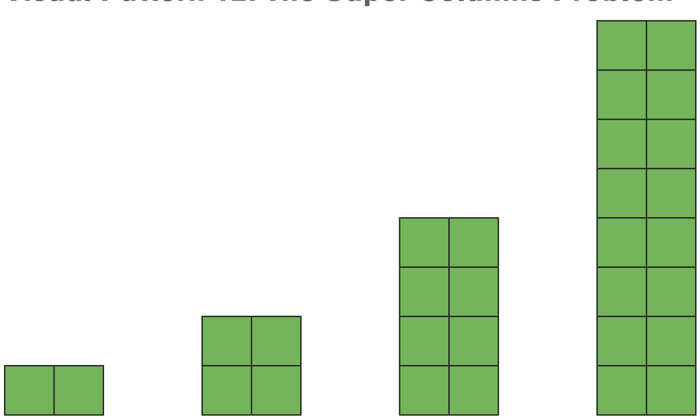

Figure 1 Figure 2 Figure 3 Figure 4

Additional Recommended Resources for Visual Patterns

Visualpatterns.org

This is a great resource for exploring visual patterns with your students. It is a very simple and wonderful website, created by Fawn Nguyen, a public middle school teacher in Southern California. The site is a collection of 180 different visual patterns (with new ones posted all the time). For each pattern, you are given the first three figures/stages of the pattern. You are also told the number of squares in the 43rd figure in the pattern (as a way to check whether you have the correct equation). You can email Fawn and she will send you the "answers"—the function equations that go with each visual pattern. This includes many of the patterns in the suggested progression, many of which were taken or adapted from this website.

Grade 6 Rocks Visual Patterns
http://fawnnguyen.com/grade-6-rocks-visual-patterns/

Fawn Nguyen (creator of visualpatterns.org) also has a great blog (http://fawnnguyen.com/) in which she tells stories about teaching and learning mathematics in the classroom. For more ideas about how to use visual patterns to develop your students' algebraic thinking and their understanding of functions, check out her post called "Grade 6 Rocks Visual Patterns." The post has some great ideas from other teachers about how they have been using visualpatterns.org, as well as some examples from Fawn's own class.

Animating Patterns
http://musingmathematically.blogspot.com/2013/08/animating-patterns.html

This is a blog post by Nat Banting about using visual patterns with his students to support their work with linear functions. In the post, Banting reports on how he created vines (7-second looping videos) to bring visual patterns to life and help students see that visual patterns are about change.

The Border Problem

This problem is a classic and can be used in adult education classrooms at any level. Similar to the lesson on the Arch Problem, the border problem can be used in a variety of ways, including asking students to create their own equations and explore the equivalency of those equations. To learn more about the problem, check out these resources:

- **Youtube video of a teacher doing the border problem with students**
  https://www.youtube.com/watch?v=I6BJXKp2Sag
- **Math Snacks—a brief video of two teachers talking about the problem**
  https://www.youtube.com/watch?v=Tgaah_0Urvs

- **A brief blog post describing how a teacher used the problem in class** https://themathletes.wordpress.com/2013/10/07/the-border-problem/
- **A Collection of Math Lessons, from Grades 6 through 8** by Marilyn Burns and Cathy Humphreys, Chapter 2: Introducing Algebra

You can also look at the border problem as a quadratic function, if you focus either on the total number of squares, or on the number of squares contained within the border.

"Developing Algebraic Thinking Through Pattern Exploration"

An article by Leslee Lee & Viktor Freiman, from *Mathematics Teaching in the Middle School*, This article models an exploration of a particular pattern and also describes different questions you could ask with a wide variety of visual patterns. It makes connections between the specific questions and the algebra content each develops.

Toothpicks by Dan Meyer—http://threeacts.mrmeyer.com/toothpicks/

This is a Three-Act Math Task by Dan Meyer that looks at toothpicks arranged in an interesting triangular pattern.