Influence of Proportional Number Relationships on Item Accessibility and Students Strategies

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Abstract

Proportional reasoning is key to students' future success in mathematics and science endeavors. More specifically, students' fluent and flexible use of scalar and functional relationships to solve problems is key to their ability to proportionally reason. The purpose of this study is to investigate the influence of systematically manipulating the location of an integer multiplier – to press the scalar or functional relationship – on item difficulty and student solution strategies. We found that manipulating the location of the integer multiplier will encourage students to make use of different aspects of proportional relationships without decreasing item accessibility. Implications for proportional reasoning curricular materials, instruction and assessment are addressed.

Introduction

Extensive evidence points to the need for mathematics instruction to tap into students' informal understandings in order to conceptually develop formal mathematical ideas (Ahl, Moore, & Dixon, 1992; Freudenthal, 1973, 1991; Treffers, 1987). Contextual problems are a common means of helping students access their informal mathematical ideas (Lamon, 1993; Moore & Carlson, 2012). However, to successfully use contextual problems in this manner, we must ensure that they are accessible to students and have the potential to promote connections to deeper or more formal mathematics (Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013; Stein, Smith, Henningsen, & Silver, 2000). Thus, there is need for research to identify what characteristics of contextual tasks make them both accessible to students as a point of entry and useful for educators in analyzing and pressing students' thinking.

We have selected to investigate proportional reasoning due to its relationship to students' future success in mathematics and science classes (Heller, Ahlgren, Post, Behr, & Lesh, 1989; Johnson, 2015; Lesh, Post, & Behr, 1988; Ramful & Narod, 2014), careers (e.g., Hoyles, Noss, & Pozzi, 2001) and life in general (e.g., Capon & Kuhn, 1979). Proportional reasoning is an extremely complex topic with a multitude of relationships and understandings that students must acquire in order to meaningfully utilize ratios across various mathematics and science situations. Given its multifaceted nature it is not surprising there is evidence that students are not developing proportional reasoning or the ability to apply this reasoning to other topics during their school experiences (e.g., Brahmia, Boudreaux, & Kanim, 2016; Cohen, Anat Ben, & Chayoth, 1999; Gabel, 1984). In addition to complexity as potential cause of student difficulties, previous research has demonstrated that proportional reasoning instruction and curricular materials have tended to focus on procedural knowledge and lack depth in terms of developing students' understanding of important multiplicative relationships (Dole, Clarke, Wright, Hilton, & Roche, 2008; Heller, Ahlgren, Post, Behr, & Lesh, 1989). Fortunately, due to its importance in students' future success, proportional reasoning is also an area where extensive research has been conducted related to understanding students' thinking and development of key ideas (see Lamon, 2007 for a summary). In particular, there is research around the characteristics of contextual proportional reasoning tasks that influence their difficulty (Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2011; Karplus, Pulos, & Stage, 1983b; Lamon, 1993; Tourniaire & Pulos, 1985). The purpose of our work is to further investigate two particular characteristics of contextual proportional reasoning tasks that could influence students' initial proportional reasoning, the influence of number and unit relationships on task accessibility and student strategies.

Theoretical Framework

Our focus on using students' thinking as the basis for formal mathematics instruction is rooted in progressive formalization, an aspect of the Realistic Mathematics Education philosophy (Freudenthal, 1973, 1991; Treffers, 1987). In progressive formalization, students initially apply their existing mathematical knowledge and intuition to solve a problem, or to mathematize the situation (Freudenthal, 1991). Students continue to solve problems by refining and formalizing their understanding under the guidance of their teacher. Through this process they reinvent progressively more formal mathematical ideas and connect them to established conventions.

Related to progressive formalization, hypothetical learning trajectories (HLTs) (Simon, 1995; Simon & Tzur, 2004) articulate the goal(s) for instruction, ideas about how students develop understanding of the topic, and tasks designed to stimulate students' development of the articulated goal for instruction. HLTs provide a structure for reasoning about progressively formalizing students' understanding. Lastly, but perhaps most relevant to our present work, the construct of key developmental understandings (KDU) can be used to assist in identifying the important goals for instruction articulated in an HLT (Simon, 2006). Articulation of a KDU provides an overarching target to which we can relate our research findings and can then be used to inform the HLT. We see students' fluent and flexible use of the scalar and functional relationships within proportional reasoning situations as a KDU that should be a point of focus from the very beginning of formal proportional reasoning instruction (Lamon, 2007; Lobato, Ellis, & Charles, 2010; Simon & Placa, 2012). Below we further articulate the terms *scalar* and *functional relationships* and how students would demonstrate evidence of this KDU.

Scalar and Functional Proportional Relationships

Proportional situations are those involving an equivalent relationship between ratios, such that $\frac{a}{b} = \frac{c}{d}$. Because of this definition, two different multiplicative relationships can be seen within any proportion. Imagine the situation "Callie bought 6 cookies for \$3. How many cookies can Callie buy for \$12?" as represented by the proportion in Figure 1. One can solve this problem by scaling up both elements of the original ratio by a factor of 4 to find 24 cookies for \$12. We will refer to this as the *scalar relationship* because we are scaling up both quantities in the ratio by a scale factor to create a new equivalent ratio. Alternatively one might recognize that the number of cookies is always 2 times the number of dollars spent (or each cookie is ½ dollar) to determine that the number of cookies should be 2 x \$12 or \$24. We refer to this as the *functional relationship* because one quantity (cookies) is defined in terms of the other (dollars) multiplicatively.

Insert Figure 1 Here

Proportion-based problems involving ratios and rates[1] can be solved using both the scalar and functional relationships. However, the context and the numbers therein may make the use of either the scalar or the functional relationship more apparent. In figure 1 a whole number multiplier can be used with both the scalar and functional relationships. If the number relationships changed to "*Callie bought 3 cookies for $6. How many cookies can Callie buy for $15*?" $\left(i.e., \frac{3\ C}{\$6} = \frac{?}{\$15}\right)$ the scalar relationship may become more difficult to utilize due to the lack of a whole number multiplier. On the other hand, it may be the relationship between the units (i.e., $ to $ or cookies to cookies for scalar) is a more relevant factor. For example, the $6 to $15 dollar relationship may be more accessible due to having the same units on the quantities involved. A question of interest to the research community, curriculum designers, and classroom teachers would be, what influence does manipulating the location of an integer multiplicative relationship in favor of either a scalar or functional perspective, have on item accessibility and student strategies?

For this project we are not focused on how students' conceive of these relationships – from a composed unit or multiplicative comparison perspective (Lobato et al., 2010), rather we focus on how they use the relationships from an arithmetic perspective. While we see students' conceptions of these relationships as a very important area of study, we find it valuable to separate students' use of the relationship in their mathematical processes from students' conception of these relationships.

[1] We use Lobato et al. (2010) definition of rate as a "…set of infinitely many equivalent ratios (p.13)".

Task Characteristics

We know from research, such as Cognitively Guided Instruction (CGI), that the structure of contextual tasks can influence students' thinking and strategies (Carpenter, Fennema, Franke, Levi, & Empson, 2004). In the area of proportional reasoning, there have been multiple investigations of the influence of the task characteristics on students' strategies (e.g., Karplus, Pulos, & Stage, 1983a) and ability to solve problems (e.g., Fernández et al., 2011). The major areas of investigation related to task characteristics that influences students' proportional reasoning are: number relationships, familiarity with contextual situation, units of measure, and item type (e.g., missing value or comparison problems). Due to our focus on identifying task characteristics that influence the accessibility of initial, informal proportional reasoning tasks and the need to minimize the number of variables we manipulated, we chose to focus on number relationships as the primary variable of interest and held the other three areas constant through: (1) use a consistent familiar context (food items:dollars) (Ben-Chaim, Fey, Fitzgerald, Benedetto, & Miller, 1998; Heller et al., 1989; Saunders & Jesunathadas, 1988), (2) discrete, visually distinct units of measure (Behr, Harel, Post, & Lesh, 1992; Lawton, 1993; Tourniaire & Pulos, 1985), and (3) missing value format (Ahl et al., 1992; Modestou & Gagatsis, 2010; Tourniaire & Pulos, 1985). The choice to hold these particular characteristics constant was based on research (cited above) indicating these selections would decrease item complexity and therefore increase students' access to the items (i.e., they were intended to make the item as easy as possible so the focus could be on the outcome of the manipulation of the number relationships). We provide further description of the research related to number relationships below.

Number Relationships. Number relationships are known to heavily influence students' choice of strategies for and success in solving proportional reasoning situations. Manipulation of number relationships in missing value proportional reasoning situations commonly involves the presence of integer versus non-integer multiplicative relationships and the location of the multiplicative relationship (Karplus et al., 1983b). There is evidence indicating integer multipliers (e.g., *x*4) are more accessible than non-integer multipliers (e.g., *x*4.25) (Schwartz & Moore, 1998; Tourniaire & Pulos, 1985). Therefore, providing an integer multiplier for one relationship (e.g., scalar) and non-integer for the other relationship (e.g., functional) may influence students to make use of that relationship. Relatedly, location refers to whether the number relationships are designed to press for a focus on the scalar (quantities with the same units) or functional (quantities with different units) relationship by intentionally making one an integer relationship and the other a non-integer relationship.

The evidence is mixed regarding situations with a scalar integer multiplier in terms of whether or not they tend to be easier (i.e., more accessible) for students than those with a functional integer multiplier. Tjoe and de la Torre (2014) found significant differences in students' success in solving problems utilizing a scalar versus functional integer relationship for low-achieving 8th grade students. Similarly, Lamon (1993) found that strategies involving scalar relations were more readily accessible to students than those involving the functional relationships. Steinthorsdottir and Sriraman (2009) placed use of the scalar relationship earlier on a developmental trajectory than functional. However, Karplus et al. (1983a) found when an integer multiplier was provided for both the scalar and functional relationship, students did not have a particular preference for use of one relationship over another and in a follow-up study (Karplus et al., 1983b) they found when an integer multiplier was provided for both the scalar and functional relationship, students showed preference for the functional relationship. Similarly, Misailidou and Williams (2003) identified use of both the scalar and functional relationship in all three level of their diagnostic framework. Therefore, a question of interest is what influence does manipulating the location of an integer multiplicative relationship to press either a scalar or functional perspective, have on item accessibility and student strategies?

Based on our analysis of the previous research, we developed two models to investigate the influence of manipulating the location of the integer multiplier – to press either the scalar or functional relationship - on item accessibility and student strategies. These models are presented in Figure 2. The two models related to *item accessibility* are focused on determining whether items designed to press the scalar relationship are more accessible than items designed to press the functional relationship or if they have similar levels of accessibility. The two models related to *student strategies* are focused on determining whether particular solution strategies are associated with particular item types or if the type of solution strategies used are consistent across the two item types.

Insert Figure 2 Here

The next section describes the assessment framework we created to empirically examine these models, followed by the rationale for using Rasch analysis to examine item accessibility through item difficulty measure scores.

Assessment Framework

Based on our focus on students' initial proportional reasoning instruction and strategies and our analysis of the literature around task characteristics influencing item difficulty and student strategies, we held constant the use of a familiar context (food items: dollars) involving visually distinct units of measure with a missing value format. This avoids the conflation of multiple task characteristics influencing item accessibility experienced in other research (as described in Karplus et al., 1983b). We focused on manipulating the location of the whole number multiplier - to press for use of the scalar or functional relationship - so we could examine influence of these attributes on item accessibility and to explore their impact on student strategies. We focused on investigating a specific aspect of the domain of proportional reasoning with the intent of better understanding how initial proportional reasoning may develop in order to inform creation of tasks and activities for an HLT.

The operationalization of our assessment framework is presented in Table 1. The manipulation of the whole number multiplier to press either scalar or functional understanding is presented along the left hand side of the table. We differentiated between items that involved application of scalar or functional understanding in situations where the missing value involved generating an equivalent ratio larger than the original ratio or smaller than the original ratio. Along the top of the table the manipulation of the magnitude of the multiplier is represented (i.e., 2 with picture, 4, 3, 7, 8). Multiple assessment forms were created from the assessment framework. Form development is further described in the methods section.

Insert Table 1 Here

Rasch Analysis

Researchers (e.g., Andrich, De Jong, & Sheridan, 1997; Callingham & Bond, 2006; Long, Wendt, & Dunne, 2011) have argued for the use of Rasch methodology in mathematics education due to its usefulness in examining test performance in relationship to a cognitive model (Bond & Fox, 2013). Most often, assessments created to fit the Rasch model consist of items designed to assess a single (unidimensional) theoretical construct (Wilson, 2004) although multidimensional Rasch models are available. The estimates of student ability and item difficulty obtained from a Rasch analysis situate test takers' understanding and item difficulty along a common equal interval scale when the data adhere to Rasch model requirements (Bond & Fox, 2013). As a result, student ability and item difficulty can be interpreted in relation to one another through probabilistic language.

The simplified version of the dichotomous Rasch model is

$$L = ln\left(\frac{P}{1-P}\right) = B_n - D_i$$

where L is the natural logarithm of the probability of success over the probability of failure. B_n is a student's ability and D_i is an item's difficulty. The equation states that the log-likelihood for a student to answer an item correctly is a function of the difference between the item difficulty and the student ability. The greater the positive difference (B - D) the more likely a student is to respond correctly to an item. The greater the negative difference the more likely a student is to respond incorrectly to an item. In situations involving dichotomous scoring (0=incorrect, 1=correct), when student ability and item difficulty are the same this indicates a 50% probability that the individual would respond correctly (or incorrectly).

The results from applying Rasch models lends themselves towards use as an investigatory tool for student cognition (Callingham & Bond, 2006; Long et al., 2011). For example, examination of the hierarchical relationship among item types on a common interval scale lends itself to validation efforts (Wolfe & Smith Jr., 2006a; Wolfe & Smith Jr., 2006b) with respect to a priori cognitive models and the empirical item hierarchy. For example, we wanted to determine if items pressing for use of the scalar relationship will be easier than items with the same multiplier magnitude pressing for the functional relationship. Comparison across item types and examination of patterns in the Rasch item difficulty scores will allow us to make that comparison. In addition, as mentioned previously, when data meet Rasch model requirements, the model transforms ordinal observations into an equal interval scale, meaning differences in items are represented as an interval relationship versus the traditional ordinal ranking received from totaling scores or calculating a percent correct (Merbitz, Morris, & Grip, 1989; Wright & Linacre, 1989).

Previous research involving proportional reasoning assessments has often used a total score or percent correct to examine the relationship between task characteristics and accessibility (e.g., Boyer et al., 2008; Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2011; Van Dooren, De Bock, & Verschaffel, 2010). But total scores and percents present a potential shortcoming in that equal differences between different sets of data points do not represent equal amounts of the construct under investigation due to the ordinal nature of the data (Wright & Linacre, 1989). As such, we opted to use Rasch methodology over the potentially more easily understood total score or percent correct based on the ability to transform the data into an equal internal scale if the data meet model requirements. This transformation then allows the valid application of parametric statistics that assume at least an interval scale. However, it may be important to note, for those less familiar with Rasch methodologies, that increases or decreases in item difficulty result in respective decreases and increase in percent correct (i.e., as item difficulty increases the number of students who answer that item correctly decreases).

Research Questions

To investigate the development of students' fluent and flexible use of the scalar and functional relationships within proportional reasoning situations, we examined the influence of the location (i.e., pressing the scalar or functional relationship) of the integer multiplier on item accessibility and student strategies. Our overall research question is, what influence does manipulating the location of an integer multiplicative relationship to press either a scalar or functional perspective, have on item accessibility and student strategies? More specifically, we sought to address this question through the following two questions:

1. Is there a difference in item accessibility between scalar and functional item types?
2. Is there a difference in student strategy use between the scalar and functional item types?

Methods

Our intent was to design an instrument that assessed students' informal proportional reasoning understanding. Therefore, we wanted to assess students at the beginning of the school year prior to formal instruction in proportional reasoning to focus on how students initially demonstrate cognitive understanding (versus procedural knowledge) in these situations. While the assessment items were not designed through the lens of the Common Core State Standards[2] - examination of the standards indicated the assessment framework primarily addressed aspects of the content from the grade 6 standards.

Instrument

Four different forms of the assessment were created from the items presented in Table 1. There were a total of 12 items per form with the first six items the same across all four forms and

[2] The Common Core State Standards have been widely adopted in the United States and provide guidance to teachers and school districts related to the mathematics content taught at each grade level.

the remaining 24 items were distributed with six items per form. The six items that were consistent across the four forms were selected to represent an anticipated range of item difficulties and different types of items with three problems each for the scalar and functional perspectives. The remaining 24 items were distributed across the forms with the intent of providing a relatively equal spread in anticipated item difficulties and types.

The items all maintained a consistent format and spacing. There were six items per page. The problems all had a blank line for students to indicate their answer and a space to show their work (see Appendix A for example of format from the first page of the assessment).

Participants

We opted to use students in grades 6-8 (approximately ages 11-14) to ensure we had a broad range of abilities within the sample. Older students in our sample should have received instruction around proportional reasoning. However, review of previous state standards, and contact with teachers in our study indicated that instruction was based primarily on algorithmic implementation of cross-multiplication, with little or no instruction emphasizing a scalar or functional perspectives.

The teachers of the students in our sample were participants in a one-day proportional reasoning professional development workshop in the summer of 2014. They came from two different regions within our state, representing a mix of urban, suburban and rural school districts.

Instrument Administration

Teachers were asked to volunteer to administer the assessment as close to the start of the school year as possible (within the first 1-3 weeks) prior to any formal proportional reasoning instruction. There was no time limit for the assessment but we informed teachers we anticipated it would take students about 30 minutes. We requested that students not be allowed to use calculators. In the directions we asked teachers to remind student to show or explain their thinking for each problem. Teachers then used the pre-paid postage mailing envelopes to return the assessments. A total of 473 assessments were returned. Students responded to one of the four assessment forms with the following number of students for each grade; grade six – 313, grade seven – 45, grade eight – 103, no grade indicated – 12.

Quantitative Data Analysis

Initial data analysis involved application of *dichotomous scoring* (0=incorrect, 1=correct) using the Rasch model in the WinSteps version 3.70.0.5 (Linacre, 2010). Each form of the test was first analyzed independently with a focus on examination of item fit for that form. Fit indices ranging from .7 to 1.3 for Infit and Outfit MNSQ were considered acceptable (Bond & Fox, 2013). Items that are not consistent with the Rasch model requirements fall outside these indices and were flagged for further investigation. This involved a person with subject matter expertise conducting qualitative investigation of the data. For example, further investigation of responses to misfitting items may indicate mis-scoring of the item or the presence of the correct answer but the coder missed it because it was not placed on the answer line provided. Once these abnormalities in the data were corrected, the data from the four forms were combined and analyzed through concurrent calibration.

The Rasch model sets the mean of the item difficulties to zero (SD = 1.14) (for identification purposes related to estimation of the model parameters) and the student mean, estimated in relation to the item mean, was .48 (SD = 2.00), indicating the sample was slightly more able then the items were difficult. While the student separation reliability of .72 (analogus to KR20 in classical test theory - see Smith Jr, 2001) was not as good as the item separation reliability of .95 (on a scale of 0-1), the intent of this aspect of our research is to better understand item characteristics. Our high item separation reliability statistic indicates a spread in item difficulties on the logit scale and supports comparisons between item scores (Wolfe & Smith Jr., 2006a; Wolfe & Smith Jr., 2006b).

Student Solution Strategies Analysis

Student strategies for items 2-6 (see table 2) were coded by solution strategy. These problems were selected because they were administered to all students in the sample and represented a range of item difficulties and number relationship structures to allow for investigation of students' solution strategies on scalar and functional item types.

Insert Table 2 Here

We analyzed students' correct solution strategies for these five problems. Our coding involved identifying whether students' <u>first step</u> in their solution strategy made use of the scalar or functional relationship. Demonstrating evidence of the use of the scalar relationship involved (1) iterating or partitioning the initial ratio – typically through doubling or halving – to determine the quantity of the missing value, or (2) determining the scale factor that scales the initial ratio to the quantity of the missing component. Either method involved calculations among quantities with the same units. Demonstrating evidence of use of the functional relationship involved identification of the multiplier between quantities with different units, typically this involved dividing (or multiplying) one component of the initial ratio by the other. This was followed by either iterating the resulting unit ratio to generate the unknown value or applying the functional relationship in a single (scalar) step to generate the unknown value. Table 3 provides the coding rubric with multiple exemplar strategies for items 2 and 6. As evidenced by the multiple examples provided in table 3, there were different paths that followed students' initial first step in their solution. These paths were primarily additive or multiplicative in nature. For the purpose of answering our research question related to students' strategies, further breakdown of the students' solution strategies was not necessary. However, our future work will further examine the hierarchy amongst these strategies.

Insert Table 3 Here

Results

In this section we describe and interpret the results of our investigation into item accessibility and student strategy use as related to the two item types; scalar and functional. We first examine item accessibility through the Rasch item difficulty scores. We then examine student strategy use through the distributions of the frequency of their use by item type.

Scalar vs. Functional Item Difficulty

To examine potential differences in item accessibility between scalar and functional item types, we first present the item difficulties measures across all the forms within the perspective of the assessment framework (see table 4). Beyond the increasing difficulty measures for the first row of the scalar items, we could discern no specific pattern at the item level related to; size of multiplier, whether the missing quantity involved an increasing or decreasing ratio, or item type.

Insert Table 4 Here

Our research question focused on examining potential differences in item accessibility by item type. Figure 3 presents box-plots of the item difficulty measures by item type. The box-plots demonstrate the variance in the functional items was less than the variance in the scalar items but do not seem to indicate a difference in item difficulties. To confirm the visual examination of the data, an independent-samples t-test was conducted to determine if the scalar and functional item type item difficulty measures were significantly different. There was not a significant difference in the scores for scalar ($M = 0.49$, $SD = 1.69$) and functional ($M = -0.22$, $SD = 0.77$) item types; $t(10) = 1.21$, $p = .26$. Levene's test indicated unequal variances ($F = 7.99$, $p = .01$), so degrees of freedom were adjusted from 27 to 10. These results suggest there is no difference in difficulty between missing value items with single digit multipliers that press for the scalar versus the functional relationship. There is also some indication that the scalar items had more variance in their item difficulties when compared to the functional items.

Insert Figure 3 Here

Analysis of Student Strategies

To examine potential differences in strategy between scalar and functional item types, we examined students' initial solution strategy. To investigate, we selected the two scalar and three functional item types that all students in the sample solved (n=475). The selected items and coding rubric were previously provided in tables 2 and 3, respectively. Table 5 and figure 4 provides the frequency and percent of each solution strategy by item type for items 2-6, respectively.

Insert Table 5 Here

Insert Figure 4 Here

The indicate students' first step in their solution strategy was strongly influenced by item type. On scalar items students preferred to use the scalar relationship as the first step in their solution process and on functional items students preferred to use the functional relationship as the first step in their solution process. These results provide strong evidence the ease of the number relationship drives students' solution strategies for our particular context.

Discussion

The focus of this research was to investigate the influence of manipulating the location of an integer multiplicative relationship to press either a scalar or functional perspective on item accessibility and student strategies with the primary purpose of informing initial proportional reasoning instruction. Our process involved developing and testing models for item accessibility and student strategies.

The two models related to item accessibility (see Figure 2) centered on determining whether items designed to press the scalar relationship are more accessible than items designed to press the functional relationship (IA Model 1) or if they have similar levels of accessibility (IA Model 2). Our results indicate they were equally accessible in terms of item difficulty, providing support for IA model 2 for our particular proportional reasoning context. When considered in relation to other research that indicated a preference for the scalar (Lamon, 1993) or functional (Karplus et al., 1983b) relationship, these results provide support for the notion that while students may demonstrate preference for one relationship over another when both relationships have an integer multiplier, the level of accessibility for students is the same across both relationships.

The two models related to student strategies (see Figure 2) centered on determining whether particular solution strategies are associated with particular item types (SS Model 1) or if the type of solution strategies used are consistent across the two item types (SS Model 2). In particular, we wanted to know if manipulating the location of the integer multiplier could be used to encourage students to focus on either the scalar or functional relationship. Our results indicate students' first step in their solution strategy was strongly influenced by item type, thus supporting SS Model 1 for our particular proportional reasoning context. When considered in relation to other research and the findings on item accessibility, this indicates that while the items are roughly equivalent in difficulty, pressing students to make use of a particular relationship by manipulating the location of the integer multiplier does encourage them to focus on that particular relationship.

Our findings are relevant to initial proportional reasoning instruction primarily through application to curricular materials and instruction. Our findings indicate that development of curricular materials that focus on the intentional manipulation of an integer multiplier will encourage students to focus on different aspects of the mathematical relationships that exist in proportional situations while maintaining a similar level of accessibility. These types of materials in conjunction with classroom discussion articulating the similarities and differences in solution strategies focused on the two relationships could assist students in developing strong arithmetic and conceptual understanding of the scalar and functional relationships. While our present research does not focus on students’ conceptual understanding, we see this as the next step in our research around students’ initial proportional reasoning. In particular, how students’ conceive of the scalar and functional relationships – from a composed unit or multiplicative comparison perspective (Lobato et al., 2010) - is an important extension to the present work. In the meantime, the current results support the notion of developing materials that intentionally press both relationships from the start of proportional reasoning instruction, as called for by others (e.g., Simon & Placa, 2012)

There are factors that may have impacted our findings, such as the use of a discrete, easy to visualize context and missing value problem types. It is possible these factors influence the level of accessibility and/or students’ strategies. For example, Karplus et al. (1983b) mentioned it may have been their context that “…emphasized the Within [functional] relationship because each recipe was described as an entity (p. 58)” and this same preference may not be displayed in other proportional reasoning tasks. Future research could focus on intentionally manipulating the contextual situation to determine if particular contexts are useful for encouraging students to focus on either the scalar or functional relationship.

Lastly, returning to the idea that contextual problems can tap into students’ informal mathematical understanding and serve as a basis for progressive formalization, research such as this addresses the details of the day-to-day decisions teachers must make to successfully implement this type of instruction. By systematically investigating factors that influence task accessibility, we provide teachers and curriculum designers with information on students’ thinking around a particular topic and ways of modifying or creating tasks to scaffold students throughout instruction.

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Figure 1: Scalar and functional solution paths for “Callie bought 6 cookies for $3. How many cookies can Callie buy for $12?”

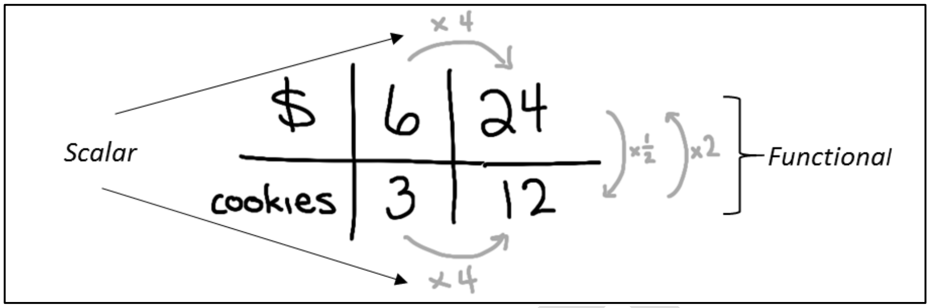


Figure 2: Models for investigating item accessibility and student solution strategies.

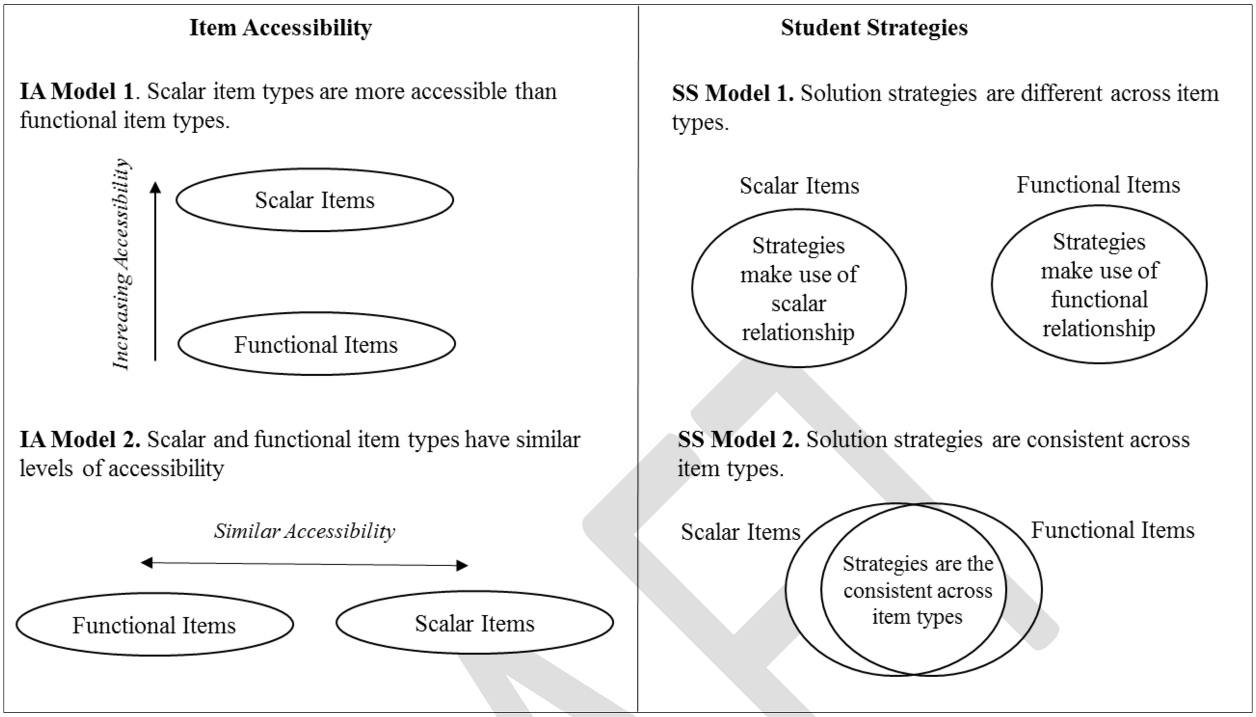


Figure 3: Box-plot of item difficulty measures by problem type.

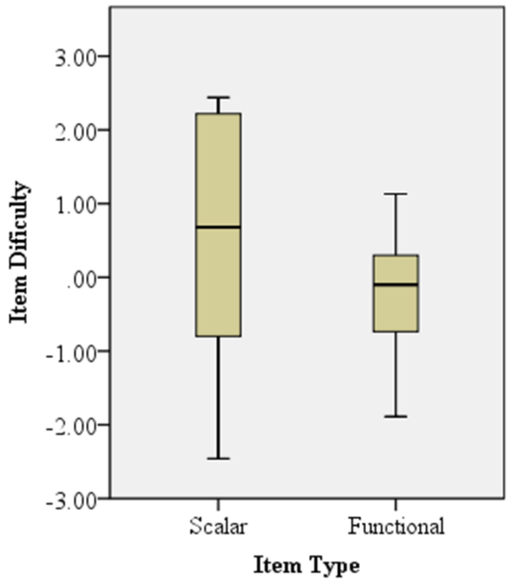


Figure 4: Histogram of percentage of students who made use of a scalar, functional, or ‘other’ approach as their first step in their solution strategy.

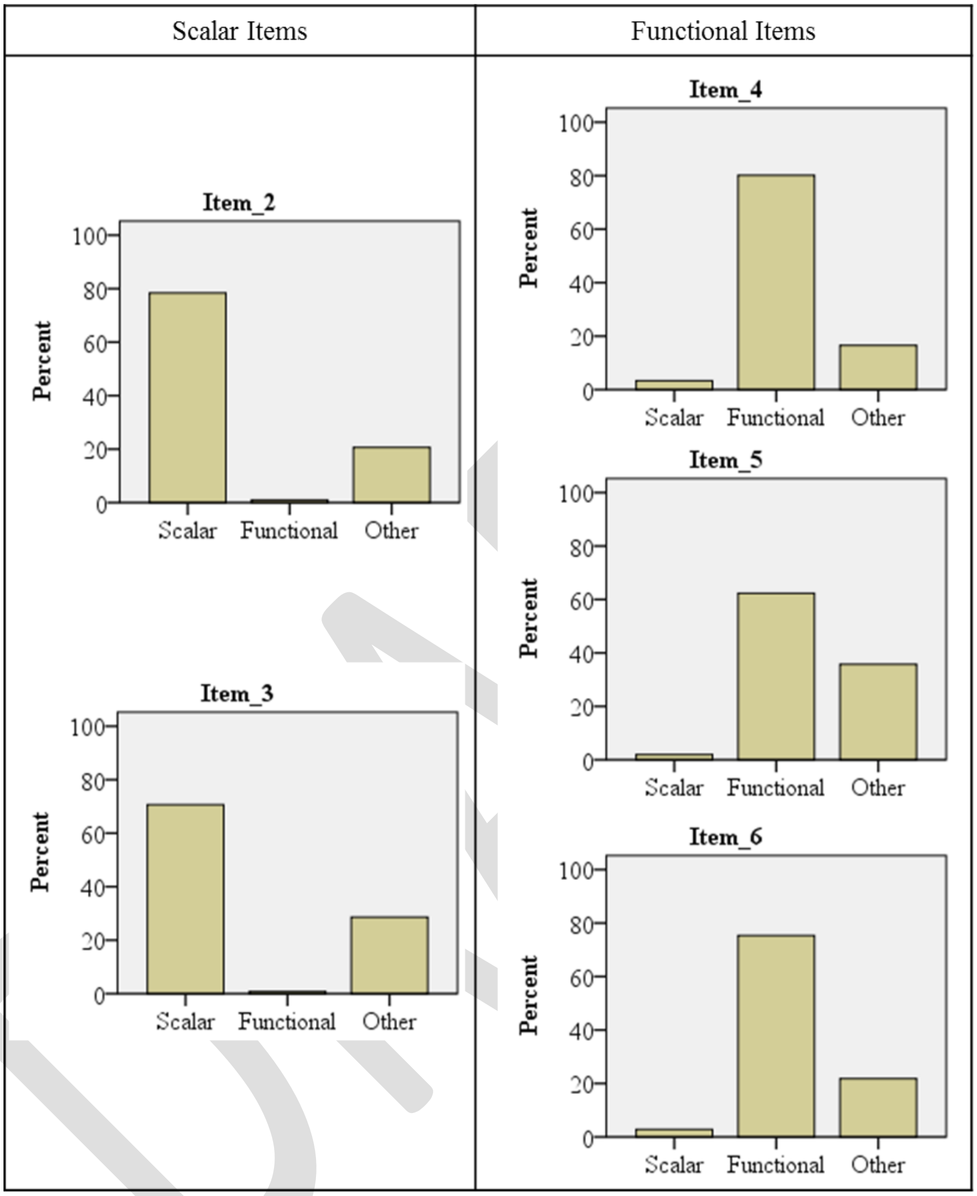


Table 1: Assessment framework for systematic manipulation of the magnitude and location of the integer multiplier.

<table>
<tr><th colspan="3" rowspan="2">Number Relationships</th><th colspan="15">Integral Multiplier</th></tr>
<tr><th colspan="3">2 with pic</th><th colspan="3">4</th><th colspan="3">3</th><th colspan="3">7</th><th colspan="3">8</th></tr>
<tr><td colspan="2" rowspan="4">Scalar</td><td rowspan="2">x</td><td>Cookies</td><td>5</td><td></td><td>Cookies</td><td>5</td><td>20</td><td>Brownies</td><td>3</td><td>?</td><td>Cookies</td><td>5</td><td>35</td><td>Brownies</td><td>2</td><td>16</td></tr>
<tr><td>$</td><td>4</td><td>8</td><td>$</td><td>2</td><td>?</td><td>$</td><td>7</td><td>21</td><td>$</td><td>6</td><td></td><td>$</td><td>5</td><td>?</td></tr>
<tr><td rowspan="2">÷</td><td>Cookies</td><td></td><td>10</td><td>Cookies</td><td></td><td>12</td><td>Brownies</td><td>?</td><td>12</td><td></td><td></td><td></td><td>Brownies</td><td>?</td><td>24</td></tr>
<tr><td>$</td><td>6</td><td>12</td><td>$</td><td>2</td><td>8</td><td>$</td><td>5</td><td>15</td><td></td><td></td><td></td><td>$</td><td>5</td><td>40</td></tr>
<tr><td rowspan="8">Functional</td><td rowspan="4">Increasing</td><td rowspan="2">x</td><td>Cookies</td><td>4</td><td>6</td><td>Burger</td><td>4</td><td>6</td><td>Brownies</td><td>15</td><td>?</td><td>Burgers</td><td>4</td><td>5</td><td>Burgers</td><td>2</td><td>5</td></tr>
<tr><td>$</td><td>8</td><td></td><td>$</td><td>16</td><td></td><td>$</td><td>5</td><td>7</td><td>$</td><td>28</td><td>?</td><td>$</td><td>16</td><td>?</td></tr>
<tr><td rowspan="2">÷</td><td>Cookies</td><td>4</td><td></td><td>Burgers</td><td>4</td><td></td><td>Brownies</td><td>3</td><td>?</td><td>Burgers</td><td>6</td><td>?</td><td>Burgers</td><td>4</td><td>?</td></tr>
<tr><td>$</td><td>8</td><td>10</td><td>$</td><td>16</td><td>20</td><td>$</td><td>9</td><td>12</td><td>$</td><td>42</td><td>63</td><td>$</td><td>32</td><td>40</td></tr>
<tr><td rowspan="4">Decreasing</td><td rowspan="2">x</td><td>Cookies</td><td>6</td><td>8</td><td>Burgers</td><td>5</td><td>8</td><td>Brownies</td><td>5</td><td>6</td><td>Burgers</td><td>3</td><td>4</td><td>Burgers</td><td>3</td><td>4</td></tr>
<tr><td>$</td><td></td><td>16</td><td>$</td><td>?</td><td>32</td><td>$</td><td></td><td>18</td><td>$</td><td>?</td><td>28</td><td>$</td><td>?</td><td>32</td></tr>
<tr><td rowspan="2">÷</td><td>Cookies</td><td>6</td><td>16</td><td>Burgers</td><td></td><td>4</td><td>Brownies</td><td>?</td><td>7</td><td>Burgers</td><td>?</td><td>8</td><td>Cookies</td><td></td><td>8</td></tr>
<tr><td>$</td><td></td><td>8</td><td>$</td><td>12</td><td>16</td><td>$</td><td>12</td><td>21</td><td>$</td><td>35</td><td>56</td><td>$</td><td>24</td><td>64</td></tr>
</table>

There was a mistake in the number relationship structure for the ÷7 scalar item. Therefore, it was eliminated from analyses.

Table 2: Item description, difficulty, context and number relationships for the 5 items selected for strategy analysis.

Relationship	Item #	Difficulty(SE)	Problem Context	Number Relationship Structure
Scalar	2	-.80(.13)	Hillary found a cookie deal with 5 chocolate chip cookies for $2. How much will it cost to buy 20 chocolate chip cookies?	Cookies 5 20 $ 2 ?
	3	2.22(.14)	Angie found a brownie deal with 24 brownies for $40. How many brownies can she get with $5?	Brownies ? 24 $ 5 40
Functional	4	-.85(.13)	Marta found brownie deal with 3 brownies for $9. How many brownies can she buy with $12?	Brownies 3 ? $ 9 12
	5	.09(.12)	Tomas found a hamburger deal with 4 hamburgers for $28. How much will 5 hamburgers cost?	Burgers 4 5 $ 28 ?
	6	.65(.12)	Mark found a hamburger deal with 8 hamburgers for $32. How much will it cost to buy 5 hamburgers?	Burgers 5 8 $? 32

Table 3: Coding rubric for students’ first step in their solution strategy.

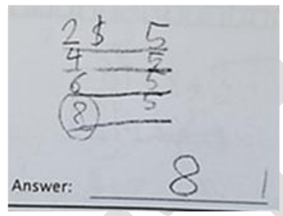
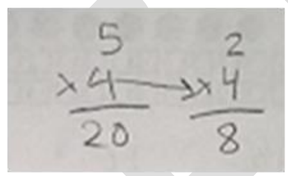
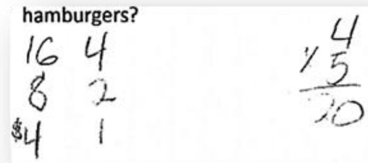
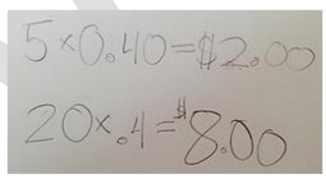
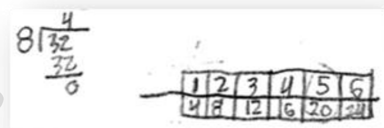
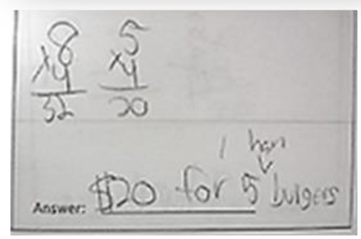
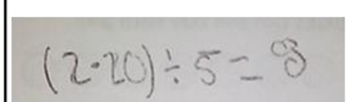
Initial Focus	**Strategy Description**	**Exemplar Solution Strategy by Problem Type**	
		Scalar Cookies: 5, 20; $: 2, ?	**Functional** Burgers: 5, 8; $: ?, 32
Scalar	- Iterates or partitions the initial ratio – typically through doubling or halving - to determine the quantity of the missing value. - Determine scale factor that scales initial ratio to the quantity of the missing component.	 	 8 ÷ 5 = 1.6 32 ÷ 1.6 = 20
Functional	Identifies the multiplier between quantities with different units, typically this involved dividing (or multiplying) one component of the initial ratio by the other - Then iterates resulting unit ratio to generate the unknown value, or - Applies the result in a single step to generate the unknown value.	5 ÷ 2 = 2.5 $\frac{2.5}{\$1} = \frac{5}{\$2} = \frac{10}{\$4} = \frac{20}{\$8}$ 	 
Other	Makes use of the cross-multiplication procedure, provided no work, or the initial solution approach was indeterminate.		32*5 = 8 * x

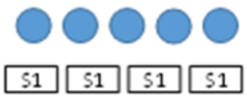
Table 4: Item difficulty measures (and standard errors) presented within the original assessment framework.

Number Relationships			Difficulty(SE) by Integral Multiplier				
			2 with pic	4	3	7	8
Scalar		x	-2.46(.17)	-.80(.13)	.27(.25)	.68(.27)	.84(.24)
		÷	-1.04(.29)	2.44(.27)	2.29(.24)		2.22(.14)
Functional	Increasing	x	-1.36(.30)	-.27(.24)	1.13(.26)	.09(.12)	-.41(.25)
		÷	-.61(.24)	-.82(.29)	-.85(.13)	.07(.25)	.56(.26)
	Decreasing	x	-1.29(.26)	.65(.12)	-.65(.29)	-.44(.27)	.26(.27)
		÷	.58(24)	-1.89(.34)	.27(.28)	.20(.25)	.33(.25)

Table 5: Frequency of students who made use of a scalar, functional, or 'other' approach as the first step in their solution strategy.

Item	Correct			Incorrect or Missing
	Scalar	Functional	Other	
2	243	3	64	164
3	89	1	36	348
4	8	194	40	232
5	6	192	110	166
6	6	159	46	263

Appendix.

1.) Callie bought 5 sugar cookies for $4. How many cookies can she buy with $8?  Answer: ________________	2.) Hillary found a cookie deal with 5 chocolate chip cookies for $2. How much will it cost to buy 20 chocolate chip cookies? Answer: ________________
3.) Angie found a brownie deal with 24 brownies for $40. How many brownies can she get with $5? Answer: ________________	4.) Tomas found a hamburger deal with 4 hamburgers for $28. How much will 5 hamburgers cost? Answer: ________________
5.) Marta found brownie deal with 3 brownies for $9. How many brownies can she buy with $12? Answer: ________________	6.) Mark found a hamburger deal with 8 hamburgers for $32. How much will it cost to buy 5 hamburgers? Answer: ________________