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Paper Title: What does perseverance in problem solving mean for collaborative learning
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Abstract

Students are expected to learn mathematics such that when they encounter challenging problems they will persist. In the U.S., creating opportunities for students to persist in problem solving individually *and collectively* is a tenet of effective teaching, an aspect of developing positive dispositions, and central to learning. This analysis progresses the idea of perseverance as collective enterprise (and not individual capacity), through empirical illustrations of it as a joint accomplishment of peers in context. Based on video of primary school children in two U.S. classrooms (n=52), this paper offers: 1) empirical examples to define perseverance as collective enterprise; 2) indicators of perseverance that identify opportunities for teachers (and researchers) to support (or study) it; and 3) analyses that evidence how the task, peer dynamics, and student-teacher interactions afford or constrain its occurrence. The significance of perseverance as collective enterprise for promoting relational equity, proficiency, and mathematical learning is discussed.

Keywords: problem solving, perseverance, productive struggle, collaborative learning, discourse communities

Introduction

Mathematics learning involves more than mastery of content, which is perhaps most clear in collaborative learning environments where children learn concepts and procedures inasmuch as they learn how to listen and negotiate ideas with others, justify their reasoning, support or critique others' ideas, forge supportive communities of respect, and navigate issues of power and status among peers (Sengupta-Irving, 2014; Barron, 2003; Boaler, 2008; Cohen & Lotan, 2014; Engle, Langer-Osuna, & McKinney de Royston, 2014; Sfard & Kieran, 2001). In the U.S., this idea of mathematical learning as involving more than disciplinary knowledge and practices appears in guiding texts like *Adding It Up* (2001), where the National Research Council (NRC) argues that mathematically proficient children exhibit *productive dispositions* (Kilpatrick, Swafford & Findell, 2001). The NRC defines productive dispositions as the "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (p. 5). Using the imagery of a five-strand rope, where one strand is productive dispositions, the council ratifies the idea that mathematics proficiency extends past what children know and do, to include what perspective or stance they take toward the discipline. This exploratory analysis investigates one aspect of productive dispositions, which the NRC refers to as "diligence" or "steady effort" (p. 131). This idea, often articulated as *perseverance* or *persistence* or *productive struggle,* is increasingly prevalent in mathematics frameworks in and beyond the U.S.

In mathematics education, children are expected to learn content and practices such that when they encounter challenging problems they will persist (CCSI, 2010; DfEE, 1999; HRDWG, 2009; Kilpatrick, Swafford & Findell, 2001; NCTM, 2014). In Singapore, for example, the primary and secondary mathematics frameworks center on problem solving and students developing an "attitude of perseverance" (HRDWG, 2009). A similar sentiment is found in the English National Curriculum where students are expected to "overcome difficulties in problem solving" (DfEE, 1999, p. 5). In the U.S., where this analysis takes place, creating opportunities for students to persist in problem solving is considered a tenet of effective teaching (NCTM, 2014). Further, productive struggle – students grappling with important mathematical ideas or relationships that are within reach but not immediately apparent –is necessary for advancing conceptual understanding (Hiebert & Grouws, 2007). Most recently, mathematics education reforms in the U.S. include practice standards that articulate children's perseverance as making sense of problems, modeling and reasoning about them, as well as monitoring or changing directions in their problem solving approach (CCSI, 2010). Given the national and global insistence on perseverance in problem solving as a critical element of mathematical learning, we argue it is an idea worthy of greater empirical attention[1].

Studying perseverance in problem solving as a phenomenon unto itself is important for at least two reasons related teaching and learning. First, the NRC argues persevering is what makes children believe they are doers and learners of the discipline (p. 131). This is an important departure from what children (and teachers) often mistake in school: students who master ideas quickly and accurately with little or no effort are the doers and learners of the discipline (Horn, 2007; Schoenfeld, 1992; Stevenson & Stigler, 1992). Thus perseverance raises important issues relevant to practice, including how teachers create opportunities for perseverance, what they look and listen for to support children when they are persevering, and understanding what constrains children from persisting. Second, as will be elaborated in the prior literature section, perseverance in problem solving offers a new vantage point from which to study what affords or constrains children from building community with one another. Boaler (2008), for example, argues that structuring and supporting groupwork in mathematics helps cultivate relational equity as children learn to respect one another and one another's ideas. Elsewhere, the first author shows how peer collaboration in racially and ethnically diverse classrooms provides opportunities for children to forge generative and unique relationships, which contribute to a sense of democratic community. One can imagine therefore, that building respectful, egalitarian learning communities includes cultivating students' collective capacity to persevere in problem solving. Thus while scholars are making great strides in articulating the value of perseverance for individual outcomes (e.g., Bass & Ball, 2015; Duckworth, Peterson, Matthews & Kelly, 2007) we argue the significance of perseverance *for the*

[1] Notably, in 2015 the Spencer Foundation sponsored a collection of papers on perseverance in mathematics that include (Bass & Ball, 2015; Berry & Thunder, 2015; Middleton, Tallman, Hatfield & Davis, 2015; Star, 2015; Taylor, 2015) .

collective. We see generative potential and significance in understanding perseverance as an interdependently individual and collective enterprise of children in collaborative learning environments.

The purpose of this exploratory analysis is to progress the idea of perseverance as collective enterprise of mathematics learning. As will be elaborated in our conceptual framework, our efforts to advance this concept begin with resisting the analytic stance taken in previous studies of perseverance – that is, perseverance as an individual trait or capacity (e.g., Dicerbo, 2014; Feather, 1961, 1966; Ryans, 1938ab). Here, perseverance is treated as a joint accomplishment of peers responding to opportunities for productive struggle with effort over time. As previously defined, productive struggle refers to students grappling with important mathematical ideas. Also discussed, effort refers to children modeling or representing their ideas to others, proffering or considering one another's explanations, and monitoring their progress or changing their problem solving approach (CCSI, 2010). Drawing on video of primary school children solving non-routine algebraic tasks in groups, this analysis makes three contributions. First, it offers empirical examples of what constitutes perseverance and what is meant by perseverance as a collective. We see this as meaningful for defining these phenomena more specifically, and for troubling the idea that whenever children solve problems they have *de facto* persevered. Second, the methodological approach used in this analysis offers indicators of perseverance, which can be leveraged as a basis for anticipating *in situ* opportunities to support children's efforts (i.e., How do I know they are persisting?) rather than rely on retrospective accounts to assess its occurrence (i.e., Did they persist?). Third and finally, we offer an extended case analysis of perseverance as collective enterprise to show how the task, peer dynamics, and student-teacher interactions afford (or constrain) what transpires.

Literature

We argue that perseverance as collective enterprise be seen as an important dimension of creating equitable and effective opportunities for learning in mathematics. Here we offer two related areas of research that support this possibility through their emphasis on community, collectivity, and learning. First, there is growing consensus that effective mathematics learning environments are those that are organized as "math-talk learning communities" (Hufferd-Ackles, Fuson & Sherin 2004; Sfard, 2002;

Sherin, 2002; Silver & Smith, 1996). Yet, traditional classrooms continue to persist in which teachers ask known information questions, offer explanations that state rules to follow, students practice tasks individually, and students expect to find the one and only solution quickly (Ball, 1988; Schoenfeld, 1985; Stodolsky, 1985). Cultivating perseverance as collective enterprise contributes therefore to the decades-long goal of mathematics education in the U.S.: to create learning communities in which students readily wrestle with concepts *together* by formulating and representing their ideas, explaining their thinking, evaluating assertions, and questioning themselves and others (CCSI, 2010; NCTM, 1989; NCTM, 2014; Kilpatrick, Swafford & Findell, 2001). Creating such learning environments can be challenging because they require teachers to establish and maintain supportive norms of communication, to design tasks and assessments that create opportunities for rigorous and meaningful debate, and to maintain focus on mathematical content in students' talk (Hufferd-Ackles, Fuson & Sherin 2004; Silver & Smith, 1996; Sherin, 2002; Stein, 2000; Stein & Smith, 2011). This is to say, the challenge in creating such discourse communities relates not only to creating opportunities for mathematical discourse but to also create opportunities that are constructive and empowering to all students. Borrowing from what feminist philosopher and researcher Jane Roland Martin (1985, p. 10) described as a good conversation, an equitable and effective discourse community is:

> Circular in form, co-operative in manner, and constructive in intent: it is an interchange of ideas by those who see themselves not as adversaries but as human beings coming together to talk and listen and learn from one another.

Creating opportunities for the kind of "good conversation" that Martin describes entails more than simply cultivating perseverance as collective enterprise. However, perseverance as collective enterprise is part of this vision of creating supportive and trusting discourse communities (Silver, Smith & Nelson, 1995) because it places a premium on children *collectively* engaging in productive struggle (NCTM, 2014); children engaging in opportunities to talk and listen and learn from one another as they grapple with mathematical ideas and relationships that are within reach but as yet unformulated (Hiebert & Grouws, 2007). Notably, bringing students together *not as adversaries* has proven a significant and enduring challenge of mathematics learning environments organized around collaborative learning.

The second area of research relevant to our proposition of perseverance as collective enterprise involves the study of children's collaborative learning. Barron (2000, 2003) proposed that collaborative in schools bears the promise for deep engagement in the discipline while also facilitating children's sense of agency through solving interesting problems together. Often conceived of as the simultaneous negotiation of persons and mathematical content (Sengupta-Irving, 2009; Barron 2003; Sfard & Kieran, 2001; Wood & Kalinec, 2012), collaborative learning reveals the discipline as an inherently social activity where justifying, negotiating meaning, and reasoning are jointly accomplished (Sengupta, 2014; Boaler, 2008; Cobb, Wood & Yackel, 1993; O'Connor, 1998). Students also express increased social and academic support in collaborative learning environments (Angier & Povey, 1999; Cohen & Lotan, 2014; Leikin & Zaslavsky, 1997; Sharan, 1990; Slavin, 1983), greater intellectual and social autonomy, and greater responsibility for their learning (Yackel, Cobb & Wood, 1991). Indeed, decades of research evidence the promise of structuring children's participation through collaboration (Johnson & Johnson, 1989; Johnson, Maruyama, Johnson, Nelson & Skon, 1981; Sharan, 1990; Slavin, 1995; Webb & Palinscar, 1996).

In collaborative contexts, children's moment-to-moment interactions demonstrate how they learn mathematical content, and also learn how to listen and negotiate ideas together, justify their reasoning, support or critique others, and navigate issues of status and power within the group (Sengupta, 2014; Barron, 2003; Cohen & Lotan, 2014; Engle, Langer-Osuna, McKinney de Royston, 2014; Gresalfi, 2009; Sfard & Kieran, 2001). These and similar studies that investigate what children learn mathematically and interpersonally during peer collaboration speak to outcomes of individual and collective significance. Moreover, this research reflects the growing consensus that social and affective aspects of learning are important to study (Boaler, 1997; Greeno, 1991; Gresalfi, 2009; Schoenfeld, 1988, 1992) because, in effect, children are never *only* learning mathematics when collaborating together. Indeed, the study of small group collaborations has drawn acute attention to how children relate to one another and the discipline. To wit, structuring opportunities for students to participate in ways that are equitable and respectful of one another, presents a meaningful challenge of practice (e.g., Sengupta-Irving, Redman & Enyedy, 2013; Langer-Osuna, 2011; Wood, 2013). Often, students who participate more also have relatively high academic status and those with lower status participate less (Sengupta-Irving, Redman & Enyedy, 2013; Chiu & Khoo, 2003; Mulryan 1992, 1994, 1995), which means important ideas can be perniciously and prematurely relegated the margins (e.g., Engle, Langer-Osuna, & McKinney de Royston, 2014; Turner, Dominguez & Maldonado, 2012). Research into ways of supporting equitable peer collaboration, which describes much of the research on complex instruction (Cohen & Lotan, 2004), suggests achieving equal status interactions (or a "good conversation") is a formidable challenge that *can be mitigated* by teachers engendering supportive relationships among students (Sengupta-Irving, 2014; Boaler, 2008; Boaler & Staples, 2008; Cohen & Lotan, 2004, 2014; Nasir, Cabana, Shreve, Woodbury & Louie, 2014). A teacher may, for example, elevate and highlight the ideas of a student who is being overlooked by his peers (Cohen & Lotan, 1995), or the teacher may gesture to the group when a student seeks her expert help and not help from her peers. As Boaler (2008) explains, teachers who structure and support opportunities for small group collaboration can help cultivate relational equity as children learn to respect a diversity of ideas from a diversity of people. Elsewhere, the first author (2014) illustrates how peer collaboration in a racially and ethnically diverse classroom provides opportunities for children to forge unique and generative relationships, which lends to a sense of democratic community. In this way, equitable and effective peer collaboration carries both individual and collective significance. Similarly, engaging students in productive struggle individually *and collectively* (NCTM, 2014) can be seen as a parallel and related effort of cultivating equitable learning opportunities in collaborative contexts.

To be clear, perseverance as collective enterprise is not the same as "good groupwork". Rather, perseverance as collective enterprise is one of many dimensions to teaching and learning that helps cultivate equitable learning opportunities and outcomes for communities of learners. By creating opportunities for students to engage in productive struggle, they may learn they are not one another's adversaries but rather, allies in taming a common enemy: the as-yet unformulated mathematical idea that with collective effort, could be within their reach.

Conceptual Framework

Perseverance as Individual and Collective Enterprise[2]

This analysis builds on prior research regarding children's mathematics dispositions in relation to classroom practices. Gresalfi (2009) operationalized dispositions in relation to two components – students working with content and with each other. As she explains, the analysis "did not consider more nuanced differences between students' engagement with content, such as giving up, trying multiple strategies or critiquing answers" (p. 336) and later suggests "many other aspects of mathematical dispositions could quite usefully be pursued" (p. 363). The current analysis considers a more specific aspect of dispositions: children *not* giving up on each other in grappling with difficult mathematical ideas or relationships together.

In framing her analysis of dispositions, Gresalfi (2009) delineates several perspectives on learning, two of which are relevant here. First, Gresalfi explains how an *individual* perspective focuses primarily on traits or characteristics. We find this a useful way to describe how perseverance has been typically studied (e.g., Dicerbo, 2014; Feather, 1961, 1966; Ryans, 1938ab). Early on, for example, the notion of *perseverance* or *persistence* was described through popular aphorisms like "If at first you don't succeed, try, try, try again," (Ryans, 1938a, pp. 80-81). This literary trope conveys a view of persistence as an individual capacity ("*you* don't succeed") that is then measured as a function of time spent on task (Ryans, 1938b). This look to an individual's time on task or trials permeates much of the empirical history of this construct (e.g., Altshuler & Kassinove, 1975; Briggs & Johnson, 1942; Schofield, 1943;

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[2] Productive struggle refers to students grappling with key mathematical ideas that are yet unformulated (Hiebert et al., 1996); it is the effort expended to make sense of mathematical ideas that are within reach but not immediately apparent (Hiebert & Grouws, 2007). Similarly, perseverance and persistence are said to occur as the result of opposition and struggle (Ryans, 1938a, p. 83). Thus, we conceptually equate productive struggle with perseverance and persistence, and treat them as descriptive equals in analyzing children's collaborative mathematical problem solving in this analysis.

Sigman, Cohen, Beckwith & Topinka, 1987). As recently as 2014 in a study of video gaming, for example, persistence is measured in relation to time, number of restarts, and task difficulty (e.g., DiCerbo, 2014).

The problem with an individual perspective on perseverance is that it presumes the child is the cause of success/failure (i.e., she *lacks* perseverance) without considering what role the context plays in supporting or constraining her efforts. Gresalfi (2009) refers to this more situated and contextualized accounting of dispositions as taking an *individual-with-context* perspective. Such a perspective identifies how the task, physical materials, peer interactions or teacher intervention constrains or affords students from grappling with challenging ideas. Indeed, effective mathematics classrooms readily engage students in thinking and working together in formulating ideas, explaining and justifying assertions, questioning or critiquing each other's reasoning (CCSI, 2010; Hufferd-Ackles, Fuson & Sherin, 2004; NCTM, 2014; Sfard, 2002; Sherin, 2002; Silver & Smith, 1996). Despite these mathematical practices being joint, interactive, or shared among some collection of students (e.g., pairs, groups, or whole-class), they are seldom explicitly understood in relation to perseverance and more often evidence of achievement or learning. Further, their significance is traced back to the individual – how the task, peers, or the teacher advanced an individual child's mathematical understanding or disposition over time. This latter tracing of outcomes back to the individual, while important, is not the goal of this analysis. The goal of this analysis is to articulate what transpires within the collective in relation to a particular task, how what transpires is shaped by the context (peers, teacher, task), and what it evidences of perseverance *for the collective* in moving forward. We find warrant for this line of inquiry in the language of current mathematics education reforms. In *Principles to Action* (2014), a guide to current mathematics reforms, for example, the National Council for Teachers of Mathematics asserts that effective teaching should consider perseverance an individual and collective enterprise:

> Effective teaching of mathematics consistently provides students, individually *and collectively*, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships (NCTM, 2014, p. 48, our emphasis).

This passage invites the study of children grappling with important mathematical ideas and relationships *collectively* (i.e., perseverance as a collective enterprise), where the unit of analysis is *the group's problem solving efforts in context*, apart from any analysis of one child's efforts on her own or within a group. Thus we coordinate attention to the *in situ* opportunities for productive struggle, how such opportunities are taken up (or not) in the group, and how efforts at joint problem solving are sustained, advanced, or foreclosed on for the collective.

Relating Learning and Productive Struggle

When Vygotsky (1978) formulated the idea of a Zone of Proximal Development (ZPD) he was grappling with the paradoxical outcomes of seemingly more mature children being slowed by their experiences of schooling, while the opposite was occurring for seemingly less mature children (Del Rio & Alvarez, 2007). Casting development as the outcome of learning, Vygotsky formulated the concept as a way to anticipate development but also recognize its open, divergent, and idiosyncratic nature. Much of this openness, in Vygotsky's conception, hinged on what cultural and societal tools were at a child's disposal loaned through the shared discourse and activity of the child with his peers or adults. The resulting articulation of the ZPD therefore describes the difference in what a child can achieve alone and what is possible when cooperating with an expert adult or more capable peer. The initial and terminal thresholds of development – i.e., the zone – together make for a "leading window" of learning. Thus the ZPD is a prospective and anticipatory look to what a child can accomplish with others. As Vygotsky explains (1978, pp. 89-90, italics added, as cited in Del Rio & Alvarez, 2007, p. 279):

> We propose that an essential feature of learning is that it *creates* the zone of proximal development; that is, learning awakens a variety of internal developmental processes that are able to operate *only* when the child is interacting with people in his environment and in cooperation with his peers.

The idea that students can be awakened to new possibilities together that are unlike what happens when working alone, reverberates through numerous studies of collaborative learning (e.g., Doolittle, 1997; De Guerrero & Villamil, 2000; Nyikos & Hashimoto, 1997; Slavin, 2011). Studies of collaborative problem solving have even extended the idea of ZPD in describing how peers negotiate the definition and goals of a task, or develop shared expectations for communication and interaction (e.g., Forman, 1989, 1992; Miell & MacDonald, 2000; Newman, Griffin & Cole, 1984; Saxe, 1991). Forman (Forman, 1989; Forman and McPhail, 1993), for example, describes how peers create a bi-directional ZPD where mathematics learners construct and coordinate different views in order to problem solve together. Goos, Galbraith & Renshaw (2002) leverage a similar understanding in what they refer to as "collaborative zones of proximal development" in their analyses of successful and unsuccessful cases of small group problem solving. The reach for new language such as 'bidirectional' or 'collaborative' ZPD is meant to distinguish expert-novice and same status interactions. Indeed, which child is "more capable" in a group can vary from moment-to-moment or task-to-task and cannot be presumed a fixed ascription of particular children (Sengupta-Irving, Redman & Enyedy, 2013; Engle, Langer-Osuna, & McKinney de Royston, 2014; Langer-Osuna, 2011). In studying metacognition and the collaborative ZPD of more and less successful groups, Goos and colleagues explain how students encountering a challenge that stimulate their collective mathematical activity (p. 218). This is important to the current analysis insofar as they suggest that productive struggle made evident their meta-cognitive work and in turn, the development of collaborative ZPD.

Coordinating the notion of productive struggle and ZPD is important to establishing why the study of perseverance contributes to the study of children's mathematical learning. Productive struggles are those that signal students grappling with key mathematical ideas that are yet unformulated (Hiebert et al., 1996); it is the effort expended to make sense of mathematical ideas (Hiebert & Grouws, 2007). Productive struggle allows students to advance their conceptual understanding and apply what they learn to new and novel problem situations (NCTM, 2014; Kapur, 2010, 2014). Figure 1 depicts our conceptual coordination of productive struggle and the ZPD.

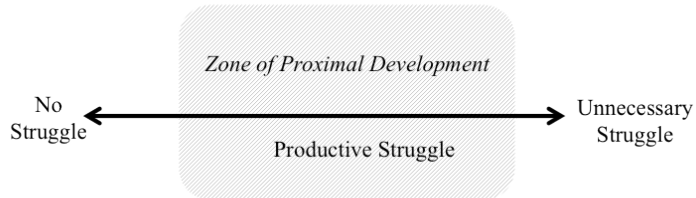

Figure 1: Conceptually coordinating productive struggle and the Zone of Proximal Development

In essence, since productive struggle marks the boundaries of what is yet mathematically unformulated but within reach, it too describes a zone of learning. In our view, this zone is bordered either by *no struggle* or *unnecessary struggle*. No struggle occurs when the solution or solution pathway is immediately obvious. It may be that the task is procedural and thus, if students merely follow the steps they achieve the outcome without grappling over deeper mathematical ideas. It may also be that the solution is obvious because it is easily within reach of what the group currently knows and can do. In some cases, the students quickly reach consensus on a correct solution, leaving no mathematical idea or relationship to grapple with. In other cases, students quickly reach consensus on a solution that misunderstands the mathematical idea and their supposition of its accuracy constrains them from grappling further. In any case, the difference between what they can do alone and together is imperceptible and in effect, no zone of proximal development is created.

On the right side of Figure 1 is the boundary between productive and unnecessary struggle, which also marks the terminal boundary of the zone of proximal development. Unnecessary struggles are those that involve extreme levels of challenge created by overly difficult problems (Hiebert & Grouws, 2007) or that do little to advance sense-making, explanations, or problem solving (Warshauer, 2014). We interpret this idea of extreme challenge as meaning that despite an expert adult or more capable peer, the mathematical idea or relationship is beyond what is possible for the students to grasp; beyond their ZPD.

At the center of Figure 1 is productive struggle, which reflects students grappling with mathematical ideas that are within reach but not immediately obvious. This notion of being within reach corresponds to the idea of working within the (collaborative) zone of proximal development and in such cases, we anticipate seeing students offering and considering one another's ideas for problem solving, attempting to model or represent one's thinking to each other, explaining ideas and critiquing each other's reasoning, and so on. In short, students working within their collective ZPD are also engaged in productive struggle.

Overall, neither the ZPD nor productive struggle exists in advance of the group or classroom context. What seems within the reach of one group may be perfectly evident to another without effort; then again, what seems an unnecessary struggle for some is squarely within reach of others. Even within a given task, students may grapple extensively with one part and by virtue of having done so, breeze through the remaining sections of the task. Similarly in terms of context, the norms and expectations for collaboration may differ between classrooms and thus children appear to persevere in some settings but not others. Ultimately, whether students are working within their ZPD and engaged in productive struggle is a function of what opportunities for productive struggle emerge, their past experiences and knowledge, what interactions they have, what tools are available to them, and what support they are offered by the teacher. To borrow and amend the language that Gresalfi (2009) offers: this is an analysis of perseverance that adopts an individual***s***-with-context perspective.

Methods

Setting and Participants

The data reported were collected at a progressive public charter elementary school in California where the teachers readily engage in research. The school was ethnically and racially diverse, reporting a student population that was 36% Caucasian, 32% Latino/a, 14% Asian, 10% African American, and 8% "Other" at the time of the study.

The study occurred over six days in two 5th grade classes. Each class period was 55 minutes in length. Fifty two of 54 students consented to all aspects of the study. Two groups of four students were selected at random in each class for videorecording (n=16). The teacher, Ms. Munroe, taught both classes. Ms. Munroe had 15 years of experience teaching math at the elementary and middle school levels. A typical lesson for Ms. Munroe had four parts: 1) a mini-lecture that directly instructs on the concept or topic; 2) small group discussion and problem solving time as she circulated; 3) whole-class discussion, including student presentations and questions; and 3) a teacher-led summary of the day's big ideas. As will be discussed in the next section, the study adhered to a similar four-part structure with some deliberate modifications.

Context of Study

The goal of the study was to engage students in problem solving together using tasks that would allow them to pursue their own (or multiple) lines of reasoning. The first author and Ms. Munroe had collaborated previously in a teaching experiment exploring students' affect and productive disciplinary engagement in a five week data and statistics unit (see Sengupta-Irving & Enyedy, 2014; Sengupta-Irving, Redman & Enyedy, 2013). In her experience of the teaching experiment, Ms. Munro was especially convinced of inquiry as a robust and rich learning opportunity. As a follow up, Ms. Munroe agreed to a closer focus on students' problem solving practices and thus the study became an opportunity to closely analyze children's moment-to-moment problem solving efforts together. The first author then designed six lessons in consultation with Ms. Munroe to advance conceptual learning of algebra through the use of groupworthy tasks (Lotan, 2003). Notably, conceptual learning is a better goal for peer collaboration than procedural learning (Pai, Sears & Maeda, 2014; Phelps & Damon, 1989) and groupworthy tasks afforded students multiple entry points and pathways to solving a problem together.

Lessons during the study were similar in structure to Ms. Munroe's typical lessons, as described previously. However, the opening minutes of class involved a Number Talk (Boaler, 2009; Parrish, 2010) or a brief introduction to the task, rather than a mini-lesson that directly instructed on the relevant mathematical ideas or relationships. In Number Talks, students are presented with a relatively simple problem (e.g., 223+119) to solve in their minds. Students then describe their varied approaches to visualizing and solving the problem, as the teacher records their approach on the board and facilitates discussions between the methods, which sometimes number as many as six or eight different ways. This visual and verbal affirmation of multiple solution pathways was important for the study because it communicated the plurality of how we see and think mathematically, and how there may be a range of valid and viable approaches emerging in small group discussions. Additionally, as Ms. Munroe had grown accustomed to from the first teaching experiment, her goal in supporting groups was to encourage them to listen to one another, offer their ideas, clarify their expectations, and generally look to the collective as a resource for problem solving. The other components of the lesson were largely the same (i.e., 20-25' of small group deliberations, 10-15' whole-class discussion), though whole-class discussions were more explicitly focused on students describing multiple ways of solving problems.

The six algebraic tasks used in this study had multiple entry points, solution pathways, and opportunities for interdependence and individual accountability (see Table 2). The tasks also represented what Stein and Smith (2011) describe as having high cognitive demand, which means they require complex and non-algorithmic thinking; an exploration and understanding of concepts, processes or relationships; self-monitoring of one's own cognitive engagement; and actively analyzing and examining constraints that limit strategies and solutions (p. 16). These tasks were selected to increase students' opportunities to develop and share various strategies, critique and evaluate the utility of strategies, and engage in problem-solving activities like patterning, working with a smaller case, and so on. The tasks came from published curricular resources – e.g., Interactive Mathematics Project (IMP), College Preparatory Mathematics (CPM), and Mathematics Education Collaborative (MEC) patterning tasks (see Parker, 2009). Ms. Munroe requested that some tasks be done in pairs and some in table groups of four so she could later assess how children's participation may vary by group size. This analysis focuses on the two tasks (*Cowpens* and *Handshake*) in Table 1 that were designed and implemented for use with *all four* students at the table.

Table 2

Overview of algebraic problem solving tasks in order of day presented

Day	Task	Description
1	Border Problem	Develop multiple strategies to find total number of 1x1 squares in 10x10 square border. Apply to larger cases until *B x B* border.
2	Cowpens &	Pictorial growth pattern. See below for description of *Cowpens. Bullpens* is a more

	Bullpens	challenging variation on *Cowpens* done in pairs.
3	Pile Patterns I	Pictorial growth patterns. Develop a strategy to predict the number of shapes in growing pattern given smaller cases. Students generalize for case A. Difficulty increases from additive to multiplicative reasoning.
4	Pile Patterns II	Same as previous but with more complex growth patterns. Task also includes designing (and solving) a growth pattern for a others.
5	Handshake	Permutations. Students evaluate whether the number of handshakes is always half the number of guests before developing their own strategy to determine the number of handshakes for 86 people.
6	Trains & Ways	Combinatorics. Students develop a strategy and solve for different ways a train of length 10 can be created from boxcars of lengths 1-4.

Handshake problem. This task required the group to represent their mathematical reasoning in words or images. The underlying concepts for this problem were permutations, graph theory, and discrete mathematics. The students must consider if the number of handshakes at a party is equal to half the number of people (i.e., 100 people, 50 handshakes). If true, how many handshakes for 86 people? If false, they must find a strategy to determine the number of handshakes for 86 people using pictures or words ***without*** solving. We anticipated students would agree that the proposition is false and develop a strategy for 86 people based on smaller cases (e.g., 2 people, 3 people). The students were likely to draw or write a strategy that could be represented functionally as $H(N) = (N-1) + (N-2) + \cdots + 1$, which reflects the sum of handshakes (H) as a function of people (N). Figure 2 visually represents this solution beginning with N = 2 people.

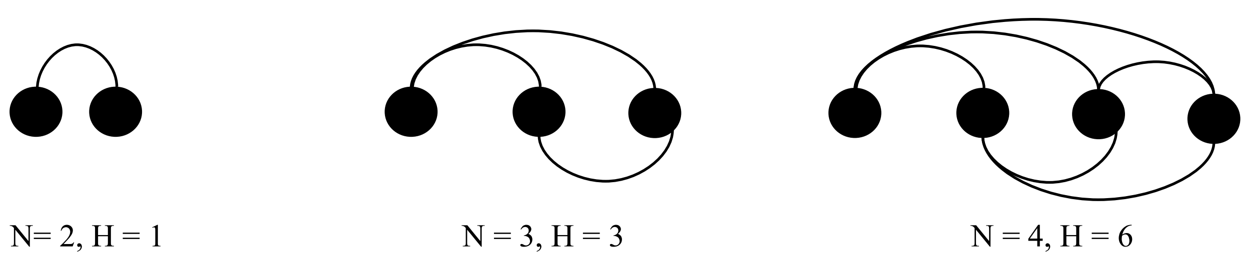

Figure 2: A deconstruction of cowpens task and corresponding algebraic function

In Figure 2, each person is depicted as a circle and each unique handshake is depicted with a curve. The total handshakes is represented by $H(N) = \sum_{1}^{N-1} N$ or $H(N) = \frac{1}{2}N\,(N-1)$. Note, unlike the *Cowpens* task, the students were *not* given these representations and it was *not* anticipated that they would write their reasoning in algebraic notation or find a numeric answer to case 86.

Cowpens. Solving the *Cowpens* task hinged on how students deconstructed the given visual representation in order to predict its growth (see Figure 3). By evaluating the given visual pattern, students had the opportunity to develop multiplicative thinking, algebraic reasoning, and gain preliminary knowledge of variables and generalizations (English & Warren, 1998).

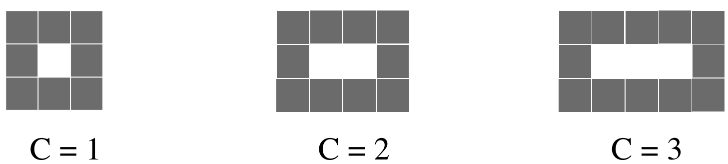

Figure3: Visual representation given to students for Cowpens

While there are multiple ways to deconstruct the images in Figure 3, Figure 4 represents the problem solving approach students used in this analysis.

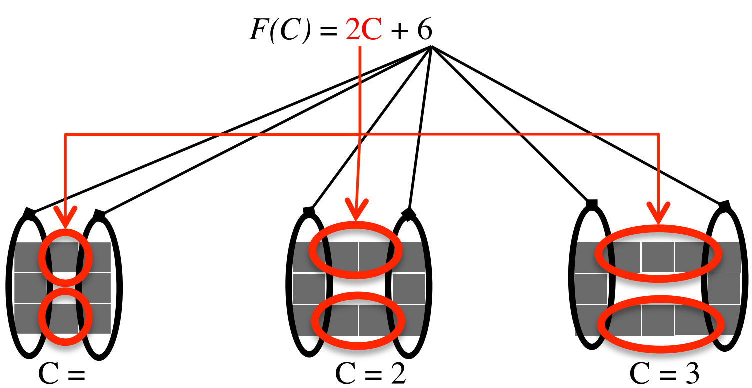

Figure 4: A deconstruction of Cowpens and corresponding algebraic function

Figure 4 shows a connection between the visual representations and terms in the resulting algebraic function $F(C) = 2C + 6$. In this form, *2C* represents the horizontal top and bottom fencing (not inclusive of corners), while *6* represents the two three-unit long vertical sides of fencing. Other forms of the solution include, for example, $F(C) = 2(C + 2) + 2$.

Data Sources

These data draw on video and students' written work from two of the six days of problem solving. With 55-minute classes and four focal groups, this resulted in approximately eight hours of video.

Data Analysis

The goal of this analysis was to progress the idea of perseverance as collective enterprise through empirical illustrations that demarcate the boundaries of the phenomenon, and show how it unfolds in relation to contextual factors like peer dynamics, the task, and teacher support. Our analyses generally followed what Powell, Francisco and Maher (2003) articulate as seven phases of video analysis. The first phase is *attentive viewing* of each video. Both authors and two undergraduate research assistants watched all of the videos (~24 hours) to becoming acquainted with the content of each and the study overall. The second phase (*describing the video*) yielded twenty four video content longs (Derry et all, 2010) that provided a time-stamped narrative of what transpired mathematically and interpersonally for each video. We then chose to operationalize "collectivity" at its maximum capacity: four-person group collaborations (rather than pairs). This led to a purposive sample of eight of the twenty four hours of video, which reflected all group-based (four-person) problem solving (*Cowpens* and *Handshake*). The third phase of video analysis, identifying *critical events*, involved identifying episodes of productive struggle in each of the eight video logs. As we did this, we realized that a chronological narrative of children's interactions was less useful for our analytic goal and we therefore recreated the logs by organizing them by episodes of struggle; that is, *struggle logs.* These logs involved systematically chronicling when a struggle emerged, notes on what led to the struggle, and a logging of the talk and activity that preceded and followed each episode. To note, in this phase of analysis we had not yet distinguished kinds of struggle – e.g., productive from unnecessary struggle. Thus, each row of the struggle log indicated the time, what talk and actions preceded the struggle, and how the group responded (e.g., peer-resolved, teacher intervention). This novel approach to video logging allowed the authors to discuss and agree on the kinds of struggles encountered in the videos, and to identify possible patterns in what talk and actions specifically indicated productive struggle. First working independently to categorize the kinds of indicators of productive struggle and then together, the authors agreed on five indicators of productive struggle in the logs:

1. Conflict in declared solution or strategy
2. Declaration of uncertainty about solution or strategy
3. Declaration of inelegant or inefficient strategy
4. Clarification of task expectations or features
5. Seeking expert support (TA or teacher)

These indicators are somewhat related to the coding scheme Goos and colleagues (2002) used to describe the conversational moves occurring in their groups. Specifically, their coding scheme had three elements: *self-disclosure*, which was self-oriented talk to clarify one's own thinking; *feedback request*, which were self-oriented questions inviting critique; and *other-monitoring*, which as other-oriented talk to engage with peers' thinking (p. 199). Our first and fourth indicators may be described as *other-monitoring*, where students declare their solutions or seek clarification on the task with one another as a precursor to engaging with each other's thinking. Our second and third indicators align to *feedback-request*, where declaring uncertainty or inefficiency invites others to modify one's thinking toward a more generative possibility. We reserve further discussion of our fifth and final indicator for later in this section.

We argue this preliminary list of *in situ* indicators is an analytic contribution of the study. We see this list as moving in the direction of assisting teachers or researchers in anticipating productive struggle, which in turn provides opportunities to support or advance students developing this capacity together. To further elaborate this analytic effort, we provide example struggle log entries for the primary four indicators (not seeking expert support) in Table 2.

Table 2

Example struggle log entries organized by indicators

Group: Task	**Time**	**Talk/Act**	**Description**
		1: Conflict in Declared Solutions	
Group 3: Cowpens	14:00	"We got 20." "…it's 26."	Eve and Layney (but mostly Eve) seem to have arrived at 26 by adding 14 to 12. The last thing they knew for sure was that 3 cows was 12 units. Not sure where 14 is coming from but I think she's thinking 7 cows and 2 per cow means 14…Xavier has drawn the case of 7 and literally counted the units of fencing to yield 20. Although Mason comes around declaring, "We drew it out!" Eve doesn't look at it.
Group 1: Cowpens	20:00	"Neil, it's 116." "No it's not. How much do you want to bet?"	The build up to this moment includes Neil hearing that the other pair is working on 103 cows. He calls out the answer as 212 and at first Sage disagrees (even though she's not done) and then concedes saying, "oh, if you count all the way around." Paola tells Neil she and Eve are still solving.
		2: Declaration of Uncertainty About Solution or Strategy	
Group 4: Handshake	18:00	"So I'm confused, does it work now?"	The group's initial reaction was that the proposition works. But then the TA comes and essentially gives away that for three people there are three handshakes by doing it with them using his hands. Aisha concludes, "so half doesn't work!" This question comes up shortly thereafter when Destiny asks to clarify what just happened…Aisha says it doesn't work if it's an odd number...Destiny asks again, "So *does* it work now?"
Group 3: Cowpens	16:00	"Wait, how did we do the first one again?"	As the girls go on to 32 cows, Eve tries to apply what they know from the case of 7 cows to the case of 32 cows. But in so doing, the fragility of her understanding of case 7 emerges again. She starts by asking how many times 7 goes into 32 and Layney starts to answer noting that there will be remainder.

			Eve then asks, "Wait, how did we do the first one again?"
		3: Declaration of Inelegant or Inefficient Strategy	
Group 1: Cowpens	16:00	"Girls, that's a lot of C's for me to count."	Sage and Paola are drawing out each of the 32 cows. When Teacher comes, Sage excitedly tells her that they found a pattern. Both girls explain the strategy to the teacher. The teacher sees their drawing and comments that's a lot of "Cs" to count. This prompts the girls to think of a strategy that applies to any given number of cows without having to draw out the cowpens and number of cows.
Group 2: Handshake	19:40	"There's always an easier way."	Conrad and Nathan have completed the calculation of 85 x 86 accurately but do not know if that's the same as the sum of 85 to 1. Conrad's intuitive sense of it, however, is that that is far too many handshakes. Nathan agrees and they both laugh a bit about this. Then, Nathan says, "I think we're going to have to do it the long way [add 1 to 85 like the girls]." Conrad, reluctant to do it says, "There's always an easier way." "There's always an easier way?" Nathan repeats/asks.
		4: : Clarifying Task Features Or Expectations	
Group 3: Handshake	14:40	"That's 6 in total, handshakes." "No it's not!"	The students are considering the case of three guests and Mason concludes it is 6 handshakes (he's counting each shake as two – two hand*s* shaking). Then Layney says it's 3 handshakes. This causes Eve and Layney to rise as the teacher says "try it, try it." In acting it out Mason agrees it's three shakes but he says the shake between Eve and Layney is "two in total, one for you and one for her" meaning he's counting it twice. This causes Layney to doubt herself and the teacher says to let Layney 'finish acting it out'…Eve then gestures to the worksheet, reads it, and says, "It says, this counts as one

			handshake."
Group 2: Cowpens	17:46	"The cows are not blocks." "I know but the cows are inside the blocks."	Jillian has drawn the case of 7 and is counting the cows in her total. Heather sees this as she looks over her shoulder and says you're not suppose to count the cows, the cows are not blocks. Jillian replies that that's true but they are inside of the blocks (this was clarified at the start of class when a student asked the same question).

Although at least one of these indicators preceded every episode of perseverance in problem solving identified in the videos, there were moments when the indicator was *not* followed by perseverance in problem solving. Consider for example, declared conflict in solutions. In some cases, students declared two different solutions to the same task and then a period of explaining, modeling, justifying, and critiquing each other's reasoning ensued, which evidenced engagement in productive struggle. At other times, students would agree to disagree and wait for the teacher to intervene, so there was no productive struggle. Once again, the individual-with-context perspective anticipates this possibility insofar as how one's peers react to a disagreement or whether or not the group has the (physical or conceptual) tools to resolve the disagreement, affords or constrains collective perseverance. The fifth indicator, seeking expert support, was the most inconsistent in identifying productive struggle. As one can imagine, students seek out the teacher or TA for a variety of reasons. And, if it was in relation to productive struggle, it often co-occurred with another indicator (e.g., telling the teacher that the group cannot determine which of two solutions is correct).

At this point our analytic approach somewhat departed from what Powell and colleagues (2003) describe as the remaining phases of analysis (i.e., transcription, coding, creating a storyline) which we reasoned appropriate given the goals of this analysis. We next selected particular cases for transcription that would demarcate the boundaries of perseverance as collective enterprise. There was no additional coding, though the extended case analysis that appears in the next section approximates what the Powell and colleagues call "creating a storyline" of perseverance as collective enterprise.

Analysis and Discussion

Our analyses are organized into two sections. The first section presents three brief cases of children's problem solving to demarcate problem solving from *persevering* in problem solving, and also highlights how contextual factors variedly constrain or afford perseverance as collective enterprise. The second section presents two cases of perseverance as collective enterprise. The first is relatively brief (approximately three minutes) while the latter unfolds over ~10 minutes. The purpose of providing two cases of varying lengths is to reassert the idea that perseverance or persistence is not defined by time on task but rather, by what transpires that suggests students grappling with important mathematical ideas, and how their collective efforts are advanced or constrained in context. To summarize what follows:

Section I

Case 1: Problem solving but *not* persevering because solution is immediately obvious

Case 2: Problem solving but *not* persevering because of shared misunderstanding

Case 3: Perseverance in problem solving but *not* as collective enterprise

Section II

Case 4: Perseverance in problem solving as collective enterprise (brief)

Case 5: Perseverance in problem solving as collective enterprise (extended)

Section I: What is Not Perseverance (as Collective Enterprise)

Case 1. This is a case of problem solving but *not* perseverance. Group 2 is working on the handshake problem, and the case begins shortly after the group has successfully determined the number of handshakes is *not* equal to half the number of guests. The group determined this when Conrad argued that the first person in a party of 100 would shake 99 hands (i.e., more than 50) and Nathan offered the example of 17 guests to argue, "you can't half a handshake" (discussed in Case 3). Now, the task asks for a strategy to solve for the number of handshakes for 86 guests.

13:53	Jillian	Okay, you guys. Look what I mean. There's 85 people. Okay, one person shakes (*gestures handshakes*) 'Hi, I'm,' [for] 85 people. Then, the next person.
	Nathan	No, no.
	Jillian	Then, the next person shakes hands with –
	Conrad	One person shakes hands with –
	Nathan	Yeah, 84 people.
	Jillian	Then the next person shakes hands with 83 people.
	Conrad	What's the problem?
	Nathan	[*To Jillian:*] That's what I mean. Subtract one each time. (*Extends arms to her*)
14:11	Jillian	Exactly, exactly! (*Nods and extends arms to Nathan; both smile triumphantly*)

Jillian and Nathan are the primary speakers during this episode of problem solving. When Conrad asks, "What's the problem?" he seeks to clarify the number of guests (he thought it was 84 and is confused that the first person shakes 84 hands) and subsequently reaches for the task handout before agreeing to what is being proposed as a solution. Figure 5 shows the group's physical positioning toward one another as Jillian and Nathan (top left and bottom right corners, respectively) agree on the strategy. Heather (bottom left) is attending to their talk with her gaze and body positioning as Conrad looks on.

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INSERT (PHOTO) FIGURE 5 HERE

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The group then agrees that the first person shakes 85 hands and each subsequent guest shakes one less hand (85+84+…+2+1). Having quickly reached consensus, they complete their problem solving efforts.

Case 1 analysis. The group does not engage in productive struggle – the mathematical relationship of guests to handshakes is immediately apparent to them because it follows from how they solved the first part of the task. Moreover, it follows from *how* they solved the first part of the task. Conrad previously explained that the first person of 100 shakes 99 hands. This laid a conceptual foundation for the students to extend in the current task, as Jillian does, by leveraging Conrad's explanation to say that the first person of 86 shakes 85 hands and, with Nathan, she extends the logic to each subsequent guest (i.e., one less). Thus the group completes the task with relative ease. Notably, had the group only considered Nathan's previous explanation (no half handshakes), the mathematical relationship between 86 guests (an even value) and handshakes may not have been so apparent and an opportunity for productive struggle would emerge. In these exchanges the students are problem solving: they are representing, justifying, and making sense of the problem and others' reasoning. Moreover, the direction of talk, body positioning, gaze, and reaching consensus, evidences problem solving collectively. However, no struggle renders this a case of problem solving but not perseverance.

Case 2. This is also a case of problem solving but *not* perseverance. This case, involving Group 1, is similar to the previous in that the group[3] quickly agrees on a solution for the handshake problem. What differs is that the group agrees on an *incorrect* relationship between guests and handshakes. The episode begins with the group's initial discussion of the handshake problem for which they are asked if the total number of handshakes is half the number of guests.

14:47	1	Sage	If every person shook everyone's hand it would be 99 shakes for that person.
	2	Neil	Because they can't shake each other's hand –
	3	Both	– You can't shake your *own* hand (*Neil and Sage's speech overlap as Sage shakes her own hand to illustrate).*
	4	Neil	Yeah.
	5	Sage	So then that means 99 times 100.
	6	Neil	No, 99 times 99.
	7	Sage	No, 100 people. 99 shakes per each. (*Sage gestures with her pencil left to right in a line as if showing 100 and then returns to the left and gestures as*

[3] The fourth member of the group was sharpening her pencil during the majority of these exchanges.

			if pointing to each separately. Cooper, Neil and Sage subsequently work on the arithmetic problem silently)
15:20	8	Neil	Nine nine zero zero. (*Neil and Sage continue solving)*
	9	Cooper	Nine thousand nine hundred.
	10	Neil	[*Peering over Neil's shoulder:*] What'd you get?
	11	Cooper	Nine thousand nine hundred.
	12	Neil	Yeah, that's what I got.
15:45	13	Sage	Yeah, I got nine thousand nine hundred.

At this point, having reached consensus, the group's problem solving efforts end. When Paola, the fourth member returns, the group tells her to try the problem as they look on.

Case 2 analysis. With relative ease, the group determines the proposed solution is wrong by virtue of agreeing on an alternate strategy. In fact, when the teacher approaches the group after these exchanges and asks if the proposed strategy is reasonable, Cooper, Neil and Sage immediately answer "no" in chorus. Their solution, which Sage initially offers (line 5), represents a misunderstanding of the mathematical relationship between *consecutive* guests shaking hands (i.e., one less than the previous), which is critical to the task. In fact, it is the group's certainty in the incorrect solution that constrains them from engaging in productive struggle. In their solution (99 x 100), guests shake everyone else's hand regardless of having shaken it before. Despite the inaccuracy of their solution, the students are problem solving in the way Sage and Neil agree on the logic of one person shaking 99 hands and clarifying that can be represented as 99 times 100 people, as Cooper listens attentively with his gaze and body positioned in their direction. Subsequently, all three proceed to solve for the answer and reach consensus easily. Once again, no struggle renders this a case of problem solving but not perseverance despite the opportunity for productive struggle *if,* for example, a peer or teacher questioned the solution.

Case 3. This is a case of perseverance but *not* as collective enterprise. Here, Group 1 is deliberating the first part of the handshake problem in which they must determine if the proposition of handshakes equaling one half the number of guests is correct. Conrad is the first in the group to say the proposition is false arguing, “If there’s 100 people in a party, and then I have to shake 99 people, that’s already more than 50.” As Conrad offers his reasoning and explanation, Jillian is looking to start with a smaller case to investigate the proposition. At first she thinks 50 guests then settles on 26 guests so that each guest can be represented by a letter, and the handshakes would be represented by coupled letters (e.g., AB, AC, AD….YZ). As Conrad beseeches Jillian to listen, Nathan agrees with Conrad but argues it’s false because “If you have 17 people…you can’t half a handshake.” Jillian and Heather, however, continue to pursue their own way of modeling the handshakes as coupled letters of the alphabet. As they do so, the students engage in the following exchanges:

10:37	1	Conrad	You don’t have to do that. Jillian, listen to my simple method of figuring it out. You like to draw. It’s a bad idea.
	2	Nathan	Oh my god! It’s very simple.
	3	Conrad	So say there’s 100 people. I have to shake 99 people already. So it’s wrong.
	4	Nathan	[*To Jillian and Heather:*] Yeah, because 99 people is way more than 50.
	5	Jillian	[*Smiling at Nathan*] Okay, but now – *(Looks at her notebook again)*
	6	Nathan	– And even if there’s 17 people, you can’t half a handshake.
	7	Jillian	Can I see the paper? I want to see the paper. [*Holding task she reads:*] Evaluate the strategy -
	8	Conrad	[*To Nathan:*] I’ll still have to shake 16 people, and another person has to shake more people.
	9	Nathan	[*To Conrad:*] I know. (*Nodding his head*)
11:05	10	Jillian	Okay, A, B.

Conrad and Nathan directly challenge Jillian’s pursuit of the alphabet strategy (“You don’t have to do that” and “Oh my god!”), and judge her preference for drawing as a bad choice (Line 1). Nathan and Conrad’s efforts to dissuade Jillian (and Heather) prompts them to connect their originally separate lines of reasoning (Lines 4, 6, 8). Despite their efforts, Jillian and Heather continue to pair letters as the others pepper them with critiques of inefficiency (e.g., “You don’t have to do everybody. How many letters are you doing, Jillian?”) and directives to stop (e.g., “Don’t you dare make more than three [letters]!”) until finally Conrad attempts to belittle Jillian and Heather’s mathematical capabilities by saying, “This is why they’re so slow Nathan, and we’re so fast.”

Case 3 analysis. Jillian and Heather are engaged in productive struggle to understand the relationship between the number of guests and handshakes. Their solution pathway – to model a smaller case of 26 guests – may not be most efficient, but is a reasonable approach to determining the proposition’s validity (i.e., if they make more then 13 pairs they will have proven the proposition false). Nathan and Conrad, in contrast, are *not* engaged in productive struggle as both saw the proposition as false almost immediately (though for different reasons). Since Nathan and Conrad are unable to convince Jillian or Heather of their approaches there is no opportunity to reach consensus and therefore this is not a case of perseverance as collective enterprise. Unlike previous cases, what is constraining the group is not the task *per se*, nor is it that they have reached consensus on an inaccurate solution. Rather, it is the peer dynamics that have grown up around the varied attempts at problem solving that constrains the group. Nathan and Conrad belittle and mock Heather and Jillian rather than, for example, attempting Jillian’s method and agreeing that creating more than 13 pairs of letters makes the proposition false. On the other hand, Jillian seems to be reacting to Nathan and Conrad dissuading her from pursuing her approach with increased resolve. As one pair experiences no struggle and another experiences productive struggle, neither pair changes their approach to problem solving, and the group fails to engage in perseverance as collective enterprise.

Summary of Section I. In all three cases the students were working on the handshake problem. The first two cases illustrate children’s collaborative problem solving but *not* perseverance as collective enterprise. In one case, the resulting arithmetic series was easily determined as the group extended its understanding of the first person shaking 99 hands to subsequent guests. In the other, using the same idea of the first person shaking 99 hands the group drew a different conclusion – everyone shakes 99 hands and no struggle emerged. Here we see how the mathematical rationale for one part of a task may have afforded one group to see the underlying mathematical idea of arithmetic series easily, but not another. The idiosyncratic nature of whether students engage in productive struggle *with same task and reasoning* reaffirms the idea that perseverance as an interactive accomplishment between opportunities for productive struggle, whether the group takes up such opportunities, and how their collective efforts at problem solving unfold. In the third case, Jillian and Heather were not afforded much space to persevere: Nathan and Conrad repeatedly interrupted their work and belittled their approach. On the other hand, despite Nathan and Conrad explaining their reasoning the others remained unconvinced or unwilling to change their approach. These interpersonal dynamics that grew up around the productive struggle of two (but not four) constrained what was possible for the group to accomplish. This is to say, the dynamics constrained perseverance as a *collective enterprise*.

Section II: A Brief and Extended Case of Perseverance as Collective Enterprise

Case 4. This is a case of perseverance as collective enterprise. Group 3 engages in productive struggle where they are grappling with how to prove that the total number of handshakes is not half the number of guests *for even values*. The group has already determined the strategy fails for odd values when Xavier argues there can be no half handshake. Shortly thereafter Ms. Munroe suggests the group stand up and investigate further. Eve, Layney, and Mason act out the case of 3 guests (which yields 3 handshakes), during which time they clarify for Mason that although each handshake involves two hands, it counts as one handshake (see Table 2, Indicator 4, Group 3). Eve initiates the current efforts by choosing a case of six guests to investigate further.

15:48	1	Eve	No, like if you have six people: one, two, three, four, five, six [*Eve draws circles to represent people*] these ones shake hands, these ones shakes hands, these ones shakes hands, those ones shakes hands, these ones handshake, ooh, these ones shakes hands, these ones handshake, these ones shakes hands [*she is connecting the circles with lines. Looking up at the teacher*:] It’s hard!

	2	Teacher	It is hard. But you have [unintelligible] here (*the teacher makes a circular gesture indicating her peers in the group and then moves on*)
16:07	3	Mason	And these two, and these two. Oh, and these two also! (*Mason is reaching across the table and gesturing to Eve's circles*)
	4	Layney	Look, Eve, you forgot a lot of things!
	5	Mason	And then, wait, hold on (*Eve smiles and relinquishes the paper to Mason who picks up the pencil and adds more connections*)
	6	Layney	[*Laughing jovially:*] Let's just say she forgot a lot. [*Leaning in toward Mason:*] And then you forgot that guy and, look this guy (*Layney then takes the pencil from Mason and adds more connections*)
	7	Xavier	[*As the three students work on a common paper, Xavier draws circles on his own paper and is doing something similar with connectors*]
	8	Mason	Oh, yeah! *(Adds more connections to Layney's work)*
16:25	9	Eve	Oh, wait! Let me see that. (*Eve stands up and reaches for the pencil and paper from Layney)*

At this point, Eve seems to realize they need to change their random approach to tracking handshakes and when she takes the paper, draws six new circles and begins to connect them in a more systematic fashion. This prompts Mason to suggest, "Just go up to four," meaning they could test a smaller case but Eve says "no," and continues pursuing the case of six. Over the next three minutes or so the students try a variety of approaches. For example, Xavier continues developing an image on his paper with two rows of three circles, connected vertically and diagonally to indicate ten handshakes among six guests. Eve meanwhile explains to Layney and Mason that she can make a list of paired numbers to represent handshakes between numerals: 12, 13, 14, 15, 16, 23, 24, 25, 26, and so on. Eve concludes there are 15 handshakes from this approach, which prompts Xavier to look up and then return to his work. We then see him add two more handshakes that connect the far corners of his representation. Soon thereafter he says, "Guys, I got 12. Guys, I got 12. Guys, I got 12. Eve, what did you get?" She answers 15 and he repeats he got 12. Eve then comes around the table and begins solving from scratch by recreating the list with him on Xavier's paper. As she does so, Layney looks on and Mason begins working on his paper. When Eve is done she returns to her seat at 18:40. Figures 6-10 depict moments of problem solving until the group reaches consensus that six guests engage in 15 handshakes and therefore, the proposed strategy is wrong.

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Case 4 analysis. The group is collectively persevering. They are representing their mathematical ideas with images and with their bodies (as in the case of 3 guests). They are also monitoring their progress, as when Layney and Mason identify missing handshakes on Eve's representation and later, when Eve starts over with a more systematic approach of pairing numbers in a list. Although Xavier seems to be working apart from the others, his actions unfold in parallel to them. For example, he also uses circles to represent the six guests after watching Eve's initial attempts. He adds additional handshakes when he hears her call out a total. And, he solicits Eve's attention when he is done and together they clarify his understanding. We cannot explain why Xavier remains in his seat while the others do not, although he says to Layney at one point, "I don't like talking across the table". Nonetheless, we interpret Xavier's parallel and corresponding actions as engagement in problem solving with the others.

There are several contextual factors that afford perseverance as collective enterprise. Their efforts are first advanced when the teacher suggests they act out the case of three guests. This allowed them to embody and visualize key features of the task (i.e., each handshake counts once; everyone shakes everyone else's hand but not themselves). This also allowed the group to reconfirm that the proposed strategy fails for odd cases, but left the issue of even cases unresolved. At that point, Eve's choice of six guests offered the group a common object of attention through which to further their problem solving. After Eve declares, "It's hard!" (Line 1), the teacher affirms her feelings but does not offer additional support, and by gesturing a reminder that she has a group to rely on, leaves. Taking up the opportunity to engage in productive struggle together, Layney and Mason monitor and contribute to Eve's work, which advances their problem solving efforts as they get closer to accounting for all of the handshakes. When Eve restarts their efforts with a more systematic (and trustworthy) approach, their efforts carry them to a solution. Concomitantly, Xavier leverages Eve's approach and pursues the problem in parallel, uses her utterances as impetus to revisit his own work, and by comparing answers later, works with Eve to secure his understanding. Taken together, this case elucidates how constituent factors of the context afforded perseverance as collective enterprise. In fact, Case 4 shows how sometimes the collective can engage in parallel mathematical activity, even critiquing one another (i.e., Eve being told she forgot a lot of handshakes), but by virtue of repeatedly engaging one another, monitoring one another's progress, assisting one another, perseverance as collective enterprise was achieved. Altogether, the group's efforts unfold in just over three minutes while the cases in Section I were resolved within a minute or so. Although we are not making claims about how much time "counts" as persisting, a comparison of time across cases make clear that the solution pathway was not as immediately apparent as it was in the previous section.

Case 5. This extended case of perseverance as collective enterprise involves Group 4 and the *Cowpens* task. The group will engage in multiple productive struggles, and often immediately following the pairs working in parallel before returning to the group to assess their reasoning and monitor their progress. Indicators of the struggle include declared conflict in solutions, declaration of inefficient strategy, and declared uncertainty about the strategy.

The underlying mathematical idea and relationship at the heart of this case (and the task) is finding a non-recursive relationship between the number of cows and units of fencing to predict any case of cows. The pairs determine two relationships, neither of which proves sufficient (center column, Table 3). Through discussion and support from the teacher, the group grapples with the mathematical relationship of cows to fences until eventually they reach a shared understanding of the correct function.

Table 3

What each student pair initially determines as the mathematical relationship versus the correct function

Pair	The Pair's Function Relating Fences (F) to Cows (C)	Correct Function
Aisha and Destiny	$F(C) = F(C-1) + 2$	$F(C) = 2C + 6$
Emilio and Daniel	$F(C) = 2C$	

In the opening moments, the students propose, evaluate, and justify an initial conjecture about the pattern's growth. As they speak, they point and count the units of fencing in the visual representations and eventually align them to their conjecture.

12:17	1	Destiny	No, cause you add 2 [fences] from 2 cows [to get fences for case 3].
	2	Emilio	One two three—one two three for five six seven eight. One two three four five six seven eight nine ten.
	3	Aisha	One two three four five—you add two.
	4	Emilio	Yeah, you add two to every one.

	5	Aisha	So three is twelve. Right? Yeah, three cows is 12. Four cows will be 14.
…			
	13	Aisha	[*To Destiny:*] Start from the bottom row, they [the top and bottom rows] each go up by one.
	14	Destiny	Yeah.
13:26	15	Emilio	Yeah that's true.

Facing the others, Destiny offers the first conjecture of how the pattern is growing (Line 1), which Aisha repeats (Line 3) and Emilio evaluates and endorses (Line 4). In Line 13, Aisha points to the images and links the agreed-upon verbal conjecture to rows of fencing in the visual representation and shortly thereafter Destiny and Emilio agree (Lines 13, 14, 15). Between their talk and gestures, these exchanges show the group problem solving though they have yet to engage in productive struggle. Moments after the last line, Ms. Munroe comments, "Good sharing, guys," in response to their joint deliberations.

Having established the fences will "grow by two," Emilio suggests a slightly different approach to determining the number of fencing units for 7, 32 and 103 cows: He'll count the case number aloud and Daniel should count up by two's. In what follows we see Aisha remind Emilio they are working as a group, which prompts him to ask permission to pursue his approach. They then reconfigure as pairs.

13:54	27	Aisha	[*Overhearing Emilio discuss his approach*] We're supposed to be working in our group.
	28	Emilio	Okay but first can we, like, try it out and then we'll tell you what we're doing?
	29	Aisha	Yeah okay go.
	30	Destiny	20, 20!
	31	Aisha	Are you sure?
	32	Destiny	I'm sure.
	33	Emilio	[*Daniel whispers numbers consecutively 1 through 7 as Emilio responds:]* 2, 4, 6, 8, 10, 12, 14. 7 cows is 14.

	34	Daniel	Oh, I get it.
	35	Emilio	Right? Cause 7 times 2 is 14
	36	Aisha	[*To Destiny*:] It says here you have to draw [the case of 7 cows].
	37	Daniel	So it's 64. 64, 206. Got it [*Raises hand to get the teacher*].
	38	Aisha	[*To Daniel:*] What?
	39	Emilio	[*Bangs the table*] We have to explain it to them.
14:41	40	Daniel	Fine. It's just—you multiply—it says 7 cows.

When problem solving apart, Emilio forgets that the first case begins with eight units of fence (Line 33) and inaccurately concludes 7 cows yield 14 units of fence (Line 35). This does not remain a simple counting mistake – this leads him to conclude that the units of fencing are *double the number of cows* so that 32 and 103 cows yield 64 and 206 units of fencing, respectively (Line 34). Meanwhile, Destiny and Aisha accurately determine 7 cows yield 20 units of fencing (Line 30). The conflict in solutions was related to problem solving apart – ostensibly, had Emilio and Daniel written out each successive number of fences as Destiny had, or had they been reminded by Aisha to draw case 7, they would have reached the same conclusion. Rather than remain apart, however, the students know they are accountable to the collective throughout (Line 27, 28, 39) and following these exchanges, they return to the group.

Together again, the group must grapple with two different solutions that represent two different relationships between cows and fences.

14:54	42	Aisha	So what's your answer?
	43	Daniel	14.
	44	Aisha	[*Hand on forehead*] That is so wrong!
	45	Daniel	So what did you get?
	46	Destiny	Okay so it's one two three four five six seven eight. Eight, you start at eight [units of fencing for case 1].
	47	Aisha	If all of these go by two, [case] four would have fourteen.

	48	Daniel	I have no idea what you’re saying.
15:29	49	Destiny	This is twelve [*gestures total fencing for case 3*]. This is fourteen [*gestures total fencing*]. This is four [cows: *gestures to same image*].

These exchanges demonstrate shared engagement in productive struggle. Daniel, in declaring he does not understand what’s being said, is grappling with a mathematical idea that is just beyond his current thinking. Destiny and Aisha offer a methodic and reasoned explanation and justification. And Emilio, moments after these exchanges, solicits a TA and asks, “Which one is right? Twenty or fourteen?” thereby evidencing he too is involved in the struggle. Before the TA responds (she will suggest Emilio and Daniel draw case 7), Destiny interjects with one last effort to explain. Pointing to case 3 (the last given image) she says: “Twenty. Oh my gosh! ‘Cause it’s 12 here...It’s right here. Four cows would *have to* be fourteen.” At that point, Daniel concedes, “Dude, it’s 20.” His tone, gaze, and use of “Dude,” strongly suggest Destiny convinced him and he is now joining her effort to convince Emilio.

Despite their collective efforts, Emilio is unconvinced and decides to follow the TA’s suggestion to draw case 7 as Daniel looks on. Destiny and Aisha move on to solve for cases 32 and 103 using the “grow by two” rule. In what follows, they encounter the productive struggle again and declare it by first declaring uncertainty in their solution (Table 2, Indicator 2) and then identifying the limits of a recursive rule for large numbers of cows (Table 2, Indicator 3).

16:23	68	Aisha	Thirty-two cows. Okay, so at seven it’s twenty.
	69	Destiny	Okay, can you count that by two?
	70	Aisha	Two, four. Okay let me tally. So twenty twenty-two. How about you tally. No, no, no, I’ll tally, you count.
17:10	71	Destiny	22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68. Okay, I think that’s enough.
	…		
	73	Aisha	24, 25, 26, 27, 28, 29, 30, 31, 32. No, this is 32.

	74	Destiny	Then why did we put 7?
	75	Aisha	7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31. This is 68.
	76	Destiny	So 70. But somehow that seems wrong.
	…		
	79	Aisha	70. Let’s just keep it for now. 70.
	80	Destiny	[*Looking ahead to 103 cows*] I don’t want to count all the way.
17:57	81	Aisha	We’re not gonna count ‘til 100. We need a new strategy for that.

Because they are using a recursive function, Aisha and Destiny must move sequentially from the last case they solved up to case 32. Through tallying and counting they arrive at 70 fencing units but Destiny declares uncertainty about the solution (“But somehow that seems wrong”) to which Aisha suggests they keep the answer as it is (Line 79). Soon thereafter they both admit their strategy is inefficient as Destiny declares she won’t continue in this fashion for 103 cows and Aisha responds with “We need a new strategy for that” (Lines 80, 81) These turns of talk indicate the opportunity for productive struggle – to grapple with finding a relationship between cows and fences that is not recursive – and yet it remains an idea beyond their immediate reach.

The next set of interactions begins when the teacher checks in as Aisha and Destiny continue grappling with case 32. When the teacher comes to the table, Emilio explains, “We’re still on number one,” (i.e., 7 cows). Ms. Munroe suggests Aisha talk to Emilio and Daniel about case 32. Aisha takes up the teacher’s move to support them collectively and explains to Emilio how she counted on from the fencing units for 7 cows and at one point suggests to him, “Let’s count together.” Emilio, unconvinced and unable to track how Aisha builds from 7 cows says repeatedly, “I’m sorry but I don’t get that.” Meanwhile, Destiny continues to express doubt that their solution of 70 fencing units for 32 cows is correct. Between Emilio being unconvinced and Aisha and Destiny’s wavering certainty, the students are engaged in productive struggle: How can we be convinced of the mathematical relationship between cows and fences?

For a few moments the group appears stalled and unable to persist. This changed through continued discussion within the collective. It begins when Daniel leverages Aisha's explanation of 70 fencing units for 32 cows. At first, Daniel seeks only to confirm her answer by asking, "Wait, you got 70?" Aisha, thinking Daniel is merely writing down her answer accuses him of copying and says, "You have to figure out how we got it." In keeping with previous interactions, Aisha holds Daniel accountable to the norms of collective problem solving. When Daniel then claims he understands, Aisha rebukes, "You have to write it down!" What Aisha does not realize is that Daniel has, in fact, stumbled upon the correct function by writing down the numbers he has heard so far. What he has gleaned from his own drawings and now the group conversation is: 1) 7 cows yield 20 fencing units; 2) 32 cows yield 70 fencing units. Recalling the inaccurate conjecture of doubling cows to determine fences, Daniel determines the correct function by manipulating the below information on his paper, and focusing on the last two columns:

Cows	If Cows Were Doubled	Fencing
7	14	20
32	64	70

As he continues working on his paper, all three students are gazing intently at his writing and press him on his thinking. Emilio asks, "Where'd you get the six from?" "Yeah, what?" Destiny asks. "You can't just find a six," Aisha cautions. Looking at his paper and then the group, he explains, "You times 7 by 2 and you put 6, it's 20. Times 32 by 2 plus 6 [and it] is 70." Destiny then says, "Okay, then we did get the correct answer," to which Daniel says excitedly, "Yeah, yeah, you guys did!" These exchanges are important. Destiny and Aisha determined a solution for 32 cows that they cannot confirm independently; Daniel uses their solution *as though* confirmed, which leads him to deduce the correct function; Destiny takes his deduction to be independent confirmation of her answer; and, Daniel confirms that she was right all along. By deducing the function $F = 2C + 6$ from what he overhears in the collective and what he and Emilio initially conjectured ($F = 2C$), Daniel effectively ends their problem solving efforts as a non-recursive relationship between cows and fences emerges. Together the students then agree that the case of 103 yields 212 fencing units and the task is complete. However, not having grappled with the relationship visually, and Aisha not being fully convinced of why the function works, has two immediate consequences for the group moving forward.

First, soon after Daniel declares his solution, he excitedly calls Ms. Munroe to show her his work, and a new productive struggle emerges: What is the relationship between the numeric pattern and the visual representation?

20:41	122	Teacher	So what did you do, darling?
	123	Daniel	So first 7 times 2 plus 6.
	124	Teacher	What's the 6? Plus 6?
	125	Daniel	I don't know. We just got it.
	126	Teacher	You just got 6 out of the air? Shoop, I'm gonna add six? (*gestures to grabbing something out of the air*)
	127	Daniel	No, no, because wee got these 2 answers [*referring to 20 and 70*].
	128	Teacher	Uh huh.
	129	Daniel	And then so we noticed that if we times 7 cows plus 2 equals 14 plus 6, it's 20. If you times 32 by 2, it equals 64 plus 6 is 70. So 103—
	130	Teacher	But do you know what 6 is representing? I do, but I want to know if you do. The pattern works.
	131	Emilio	Our 6 is representing the cows. No, wait.
21:33	132	Teacher	[*Walking away*] It's not the cows.

As this conversation reveals, Daniel deduced the correct mathematical relationship by having "noticed" (Line 129) a numeric pattern but cannot yet connect it to the visual representations. Without that connection, the task remains unaccomplished. When Ms. Munroe poignantly asks, "But do you know what six is representing?" she clarifies a new productive struggle for them to deliberate further. As she walked away, Emilio called out, “No, wait! The blocks!” and then high fives Daniel shouting, “Yes!” A few minutes later and just before the whole-class discussion, Daniel whispers to Emilio, “I don’t know how you found that out [what 6 represents]!” Emilio’s response shows how he relied on the collective:

> Dude, because I saw three plus three [is] six. Because you know how Aisha said those three don’t change? So I added them up. And when [Ms. Munroe] said that you grabbed six out of the air, I was like, they are the blocks on the side!

From the beginning, Aisha and Destiny have been problem solving through the representation – pointing and counting to images until the case of 32. During those deliberations, Emilio heard Aisha comment on the “six blocks” (referring to the two three-block columns that go unchanged). Here, in grappling with how to connect the numeric relationship to the visual representation, he invokes what he learned in the collective and uses it to advance their understanding. This is to say, Emilio’s recollection of *Aisha* connecting the visual representation to numeric values (three and three) was instrumental to grappling with the relationship between the image and his function. As Emilio explains where the six comes from to Daniel, Aisha and Destiny watch impassively. Though it is unclear during this task whether they too understand how to connect the function and visual representation, when they embark on the subsequent *Bullpens* task as a pair, their understanding of this relationship becomes apparent as they easily leverage what they learned for a more challenging task.

Summary of Section II. In this section we provided two illustrative cases of perseverance as collective enterprise. Both cases evidence students grappling with mathematical ideas or relationships that are within reach but as yet unformulated. Moreover, both cases evidence how interdependently perseverance was experienced by individuals (e.g., Xaiver), pairs (e.g., Aisha and Destiny), triads (e.g., Eve, Layney, Mason) and, as was operationalized for “collectivity” in this analysis, the table group. In both cases students were working in their zone of proximal development. They were formulating, evaluating, and leveraging one another’s mathematical reasoning when engaging in productive struggle over time. To note, despite problem solving apart from the collective at times, both groups reconvene, discuss their thinking and progress, and find new paths forward as a collective (see Figure 11). In this way, the students variedly positioned as more or less capable peers in relation to one another and required only modest support from the TA or teacher. Though modest, these more expert interventions served to sustain the group grappling with the mathematical relationship by drawing out a case, embodying the handshakes, or reminding students to use their peers as resources. With discussion and the freedom to pursue their own paths of reasoning with concurrent support and accountability to the group, both cases demonstrate perseverance as collective enterprise.

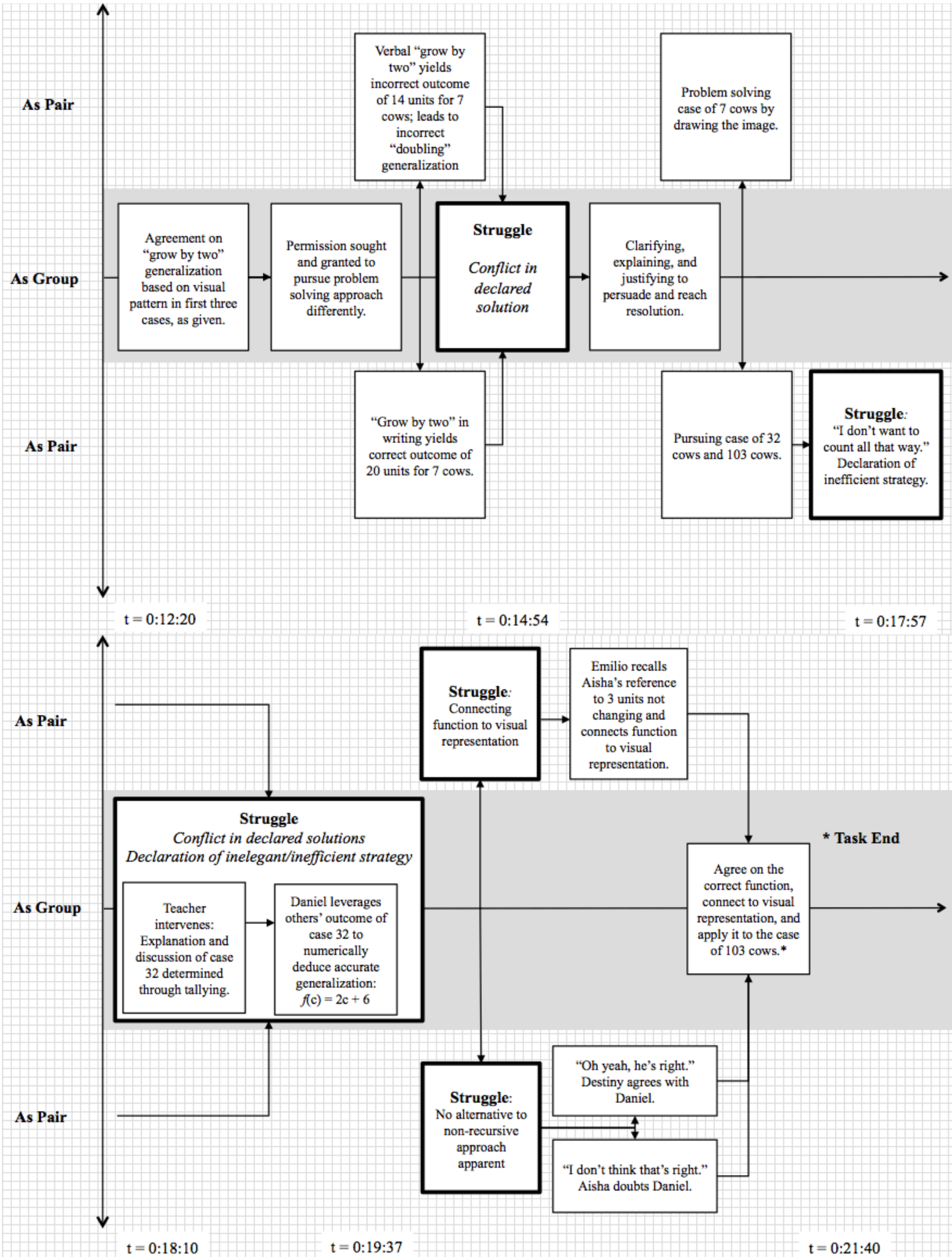

Figure 11: Case 5 timeline of productive struggle with in situ *indicators*

Conclusion

The goal of this analysis was to progress the idea of perseverance in collaborative learning environments as it increasingly permeates mathematics frameworks in and beyond the U.S. The choice of words such as "diligence," "steady effort," "productive struggle," or "persistence" in such frameworks reject that we simply expect children to "problem solve." Rather, such words laminate problem solving with the expectation that children encounter challenges and meet them with exceptional resolve. Engaging in empirical dialogue over how to identify children as perseverant or what actions are taken as evidence of persevering, is therefore a worthy object of attention. Else, the value of such a construct may be eroded by the subjectivities of those who *a priori* or mistakenly ascribe such positive descriptors to particular kinds of learners and not others (e.g., by gender, race, prior achievement, perceived talent) or particular actions and not others. The value of employing an individuals-with-context perspective (Gresalfi, 2009) on perseverance is that it resists this possibility; instead, it distributes attention to how the task, peers, the teacher, and other aspects of context afford or constrain children from persevering. This more expansive view allows teachers multiple ways to reorganize, shift, or change contextual circumstances to better promote perseverance – rewriting the task from procedural to conceptual, revising norms of collaboration, looking for indicators of perseverance and intervening to advance their efforts. To wit, this analysis offers a preliminary list of *in situ* indicators of perseverance that teachers (or researchers) can use to better support (or study) this phenomenon in classrooms.

Our analysis also places a premium on the generative and significant work that children can accomplish together, which compels considering perseverance a collective enterprise. Thus while a teacher should invest in understanding how any one child rises to the expectation of productive struggle, in collaborative contexts it is equally important to understand how children rise to the expectation *together*. This latter view grows from the idea that how children treat one another when collaborating in mathematics – whether belittling someone's efforts to represent her ideas differently, or sharing a pencil to help her in her efforts, or listening to and questioning each other's thinking – influences the creation of supportive, egalitarian learning communities (Sengupta-Irving, 2014; Barron, 2003; Boaler, 2008; Cohen

& Lotan, 2014; Langer-Osuna, 2011; Sfard & Kieran, 2001). Finally, our conceptual framing coordinates no struggle, productive struggle, and unnecessary struggle with the Vygotskian notion of ZPD as a leading window of learning, which we see as beginning a more considered discussion of the interrelations between persevering and learning both individually and collectively.

As a mathematics education research community, we see part of our charge as investigating how the expectations placed on children in schools shape what is possible for them in becoming confident, capable and engaged learners. From the vantage point of perseverance as collective enterprise, we see a renewed charge that children engage in conceptually rich learning opportunities. Indeed, while procedural understanding is integral to proficiency (NRC, 2001), engaging children regularly in procedural learning forecloses on opportunities for productive struggle. In other words, prescription can constrain grappling with important mathematical ideas and relationships. The tasks used in this study (Table 2) tilt toward opportunities for productive struggle *because* they were groupworthy – each had multiple entry points, solution pathways, and opportunities for interdependence and individual accountability (Lotan, 2003; also see Ball & Bass, 2015 for discussion of perseverance using the Trains & Ways task). Such tasks can compel children to engage in the very practices mathematics reforms prize: making sense of problems, modeling and reasoning about them, as well as monitoring or changing directions in their approach (CCSI, 2010).

From the vantage point of perseverance as collective enterprise, we also see a heightened charge to teachers in supporting peer collaboration and maintaining the cognitive demand of tasks throughout small group deliberations (Stein, 2000; Stein & Smith, 2011). Ms. Munroe, for example, was accustomed to monitoring students during small group discussions. Yet, she was newer to the idea of student-driven inquiry (Sengupta-Irving, Redman & Enyedy, 2013) and was learning to intervene in ways that returned children to the collective and not, for example, offering her own ideas or hints to further the group along. Whether responding to Eve's declaration, "It's hard," by gesturing her peers as resources, or walking away from Emilio to deliberate on where the six came from, or asking Aisha and Destiny to engage their peers in a discussion of case 32, Ms. Munroe's active support of *collectively* persevering was significant.

In contrast, there were multiple occasions when the Teaching Assistants inserted themselves in ways that foreclosed on aspects of productive struggle. For example, soon after the TA gave away that three guests yield three handshakes by using his own body to model the task (see Table 2, Indicator 2, Group 4) Destiny is left asking, “So I’m confused, does it work now?” In contrast, when Ms. Munroe inserted herself into the same group for the same task, she encouraged the students to try the task for themselves and looked on as they did. When they finished she simply noted, “This is a strategy,” gesturing with her hands the idea of physically shaking hands with one another. From there, the students were left to continue grappling with how to leverage that strategy moving forward.

In the next wave of reforms, it may be that educators, researchers, and policy makers treat perseverance in problem solving as subsumed in the use of conceptually rich, cognitively demanding, expansive learning opportunities, and the phrase itself disappears. Losing it as an object of attention would mean that American mathematics classrooms have uniformly rewritten the mistaken belief that excellence is a function of innate talent (Stevenson & Stigler, 1992) and have instead proven the significance of steady effort and fortitude. Even if we were to concede that future possibility, the rationale for perseverance *as collective enterprise* would remain because it centers not only the idea of steady effort and fortitude, but that learners are stronger together. It would insist on a more aspirational kind of mathematics learning: creating opportunities to learn that also cultivate a sense of community that proves itself supportive in the way students grapple with important mathematical ideas and relationships together, and without constraint.

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