2016 NCTM RESEARCH CONFERENCE
April 11-13 • San Francisco

Paper Title: Middle School Students' Development of Algebraic Reasoning: Comparing Effects of Three Instructional Approaches (Visual, Structural, and Modeling)

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Session Title: IPS6

Session Type: Interactive Paper Session

Presentation Date: Tuesday, April 12, 2016: 10:00 AM - 11:15 AM

Presentation Location: San Francisco, California

Middle School Students' Development of Algebraic Reasoning: Comparing Effects of Three Instructional Approaches (Visual, Structural, and Modeling)

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Keywords: Algebra readiness, algebraic structure, modeling, visual-number pattern

Abstract

This paper describes a teaching-intervention study involving three seventh-grade classes in a mid-West middle-school, and discusses analyses of data arising from the interventions. Three introductory algebra approaches—a "visual-number" approach, a "modeling" approach, and a "structural" approach—were employed, with just one approach being taught to each of three participating classes. Pre-teaching and post-teaching pencil-and-paper data as well as interview data were collected, the pencil-and-paper instruments being an *Algebra Readiness* test, a *Visual-Numbe*r test, a *Modeling test,* and a *Structure* test. In addition to data gathered from responses to the pencil-and-paper instruments, data from 36 one-on-one interviews with students (18 pre-teaching and 18 post-teaching) were also collected and analyzed. Initial findings indicated that whereas the *modeling* class's mean gain score was significantly different from zero, the mean gain scores for the other two groups increased only slightly.

Introduction

Kaput (1999) summarized three different forms of algebraic reasoning: algebra as the study of structures arising in arithmetic and in quantitative reasoning, algebra as the study of functions, and algebra in the process of modeling. Kieran (2006) seemed to concur with Kaput's view when she recommended that algebra education research should concentrate on three main themes: (a) algebra word problems, (b) developing rules for geometric and numerical patterns; and (c) patterns within numerical relationships.

According to Kieran (2006), much of the pre-1990s research on school algebra conducted focused on the teaching and learning of children aged from about 13 to 14. But, since the mid-1990s the most dominant emphases have related to the need to develop algebraic thinking among elementary-school students, among prospective teachers and among in-service teachers of algebra.

Kieran also pointed to the need to reach agreement on what constitutes effective algebra teaching, and on how students can be helped to use algebra to model physical situations.

Main Components of Introductory Algebra

Researchers (e.g., Cai & Knuth, 2011; Kieran, 2006, 2007) have drawn attention to the following three forms of algebraic reasoning:

1. The recognition, formalization, and use of structures within sets of numbers. This can involve reasoning about operations and structural properties with respect to sets of numbers—like, for example, reflecting on whether the associative or commutative properties hold for different operations when they are applied to different sets of numbers.

2. Generalizing numerical and visual patterns to describe relationships between variables. This form involves exploring and expressing regularities in, for example, growth patterns concerned with the sum of odd natural numbers.

3. Modeling real-life situations using algebra (e.g., developing appropriate equations and inequalities).

Approach to Introducing Algebra: Structure

Kieran (2011) defined elementary algebra as "the structural face of arithmetic operations viewed not just as procedures for calculations but also as relational objects" (p. 587). She stated that "recognition and use of structure" are at the core of school algebra. Research has indicated that elementary and middle-school students only gradually tend to become aware of structural properties associated with the real-number system. As would be expected, most middle-school students do not fully understand such properties, and even among those that do, there are many who cannot generalize those properties to letter-rich algebra education environments (Booth, 1988). The need to become aware of numerical relationships offers a context for fostering structural awareness and algebraic thinking (Campbell & Zazkis, 2001). MacGregor (1991) defined the core aspects of algebra as language-based—reading skills, including recognition and comprehension of words; concepts associated with the operations of arithmetic; meanings of algebraic letters and modeling; equations; and structures and generalizations.

Two of the most challenging aspects of research into early algebra learning are being able to map student transitions from recognition of numerical properties to algebraic structures, and to define levels of algebraic abstraction. *Standards for Mathematics Practice* 7, which is part of common-core mathematics (hereafter referred to as "CCSSM") encourages students to look for and make use of structure. The aim is for students not only to learn arithmetic processes but also to come to see these as abstract objects in their own right.

Introducing Algebra through Visual and Numerical Patterns

Pattern generalization in the context of algebraic reasoning can provide a valuable perspective on the process of generalization. It has been claimed that pattern recognition and generalization tasks provide students with the opportunity to capture and generalize regularities by using the mathematical language of formulas, and that that will support their development in reasoning and proof (Kaput, 1999; Kieran, 2006).

The difficulties which many students have in learning to use algebraic notation to express general forms of visual and numerical patterns has been well documented (see, e.g., MacGregor & Stacey, 1997; Radford, 2000). In spite of such difficulties, some researchers (e.g., Herscovics & Linchevski, 1994; Warren, 2000) have claimed that engaging students in visual and non-visual patterning activities can provide strong support for their development with respect to symbolic and functional thinking.

Several patterning studies conducted at the upper-elementary and middle-school levels have provided evidence that students' generalizations shift from the recursive to more explicit forms. The examination of pictorial growth patterns in the context of exploring generalizations can help children to develop their algebraic reasoning. Some researchers (e.g., Carraher, Martinez, & Schliemann, 2008; Carraher, Schliemann & Brizuela, 2001; Walkowiak, 2014) have found that although elementary students can use their own intuitive representations and pictorial patterns as valuable and promising tools for their development of algebraic thinking, many elementary and middle-school students still find it difficult to bridge the gap between intuitive knowledge and conventional representations

Instead of arithmetical generalization through numerical counting, algebraic strategies can provide more efficient explicit modes of counting, especially if they emphasize and focus on the structure of a pattern and generalizations. Kieran (2011) argued that generalizing is considered as both a path to, and a characteristic of, algebraic thinking. Pattern generalization in the context of acquiring algebraic reasoning provides a valuable perspective on the process of generalization.

Another way of generalizing is to recognize patterns and structures in problems and in real-life situations. It has long been assumed that students can capture and generalize regularities by using the mathematical language of formulas. Formulas and rules can be used for expressing generalities and, therefore algebraic language and symbolic notation are needed to support reasoning and proof. One of the sophisticated levels of algebra learning is to realize higher levels of abstraction and be able to complete transition to algebra. In this process, students can attain higher levels of structural abstraction and structural awareness.

Approach to Introducing Algebra: Modeling

Arzarello and Robutti (2001) stated that modeling provides a "bridge to algebra," and many researchers have argued that engagement in modeling tasks can help students develop their symbolic representation skills. Elsewhere, Kieran (2011) has argued that modeling is considered as both a path to, and a characteristic of, algebraic thinking. According to the National Council of Teachers of Mathematics (2000), an understanding of the meanings and uses of the concept of a variable can help learners deal with numerical examples for which multiple representations (especially verbal, tabular, and graphical representations) might be needed. Over the past few decades there has been an increasing emphasis on the need for modeling activities in school algebra but, according to Kieran (2007), the meaning of the term "modeling" has not always been clear for it has been interpreted in a broad range of situations (e.g., with respect to word problems, physical situations, concrete models, etc.). By engaging in modeling activities, students can capture and generalize regularities by using the mathematical language of formulas. Seen from that vantage point, algebraic language and symbolic notations can become necessary and powerful tools to support reasoning and proof,

One of eight Standards for Mathematics Practices (SMP) of common-core mathematics, specifically SMP4 ("model with mathematics"), focuses directly on modeling, and it has been claimed that modeling draws on and develops all eight standards for regular mathematics classroom practice. It is claimed that the common-core modeling standard will help students develop their reasoning capabilities, both in mathematics classes and in beyond-classroom situations in the real world.

Meyer (2015) studied modeling tasks in two common-core-aligned textbooks and identified only a few tasks out of 83 which encouraged students to engage in all five modeling steps—ask the question, select the modeling approach, formulate the model, apply the model, answer the question (Meerschaert, 2013). Meyer urged teachers to offer students opportunities to model.

Aims and Rationale for the Research

Although it is well recognized that most middle-school students typically receive their first formal instruction in algebra and symbolic representation in the sixth and seventh grades (CCSSM, 2010), a review of the literature found very few research instruments that had been validated for measuring middle-school students' algebraic thinking skills. Although research has pointed to structure, visual-number pattering, and modeling being important components of children's algebraic reasoning, very little research has investigated how, and the extent to which, interventions based on these components affect students' readiness to learn secondary-school algebra.

The concept of a "variable" can arise in many different contexts in middle-school mathematics, including statements and applications of structural number properties. But has enough careful thought to curriculum development issues? In a statement like "for all real numbers a, b, c it must be true that $a \times (b + c) = a \times b + a \times \mathrm{c}$)," for example, the letters a, b, c are, in fact being used as variables despite the fact that in early secondary-school algebra early-alphabet letters lime a, b, c are usually used for "constants" and later-alphabet letters like x, y, z are used to denote variables. Does such inconsistency hinder students' development of their generalization powers? Furthermore, how well do middle-school students who are asked to create models for patterns that involve the idea of a sequence in fact improve their ability to create algebraic formulas which correspond to the patterns?

Methodology

Considerations associated with the points made in the above review resulted in the formation of the following four research questions which are examined in this study in which three different seventh-grade classes in the same middle-school were involved in interventions. One intervention aimed to improved students' awareness of algebra structures; a second, aimed to assist them to develop formulas which modeled given numerical and visual patterns; the third was aimed at helping students to solve modeling problems.

1. Immediately after all three lessons were completed were there statistically significant differences between the three groups' mean gain scores on pencil-and-paper tests of structure, visual-number patterns, and modeling?
2. What were the overall effects of each of the three interventions with respect to mean gain scores on the Algebra Readiness Test (ART)?
3. Immediately after all three lessons were completed, were there educationally noticeable differences between the concept images of the students, with respect to the concept of a variable, in comparison with the concept images that the students had before the intervention began?
4. To what extent was Ms. X's classroom teaching similar for the three interventions, and is it possible to link the teaching to changes in students' concept images?

Theoretical Background

The theoretical base for the study was Sfard and Linchevsky's (1994) concept of "reification." Researchers distinguished between (a) operational conceptions, (b) transitional conceptions, and (c) structural conceptions. Sfard and Linchevsky proposed that children's early operational conceptions tend to be related to concrete or familiar situations, but as they grow older there is movement from the concrete to the abstract, with students learning to build "ideas on ideas," and there is movement in the direct of reification. The second theoretical lens that was employed in this study was that of a *concept image*—which refers to the total cognitive structure that is associated with a concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). The expectation was that as the students began to work in situations involving variables, they would begin to reify the concept of a variable, and that would lead them continually to revise their concept images of a variable. Variables would begin to be seen as abstractions rather than as symbols completely tied to particular activities. From this perspective, the researcher was interested to investigate the idea that the acquisition of a reified concept of a variable would depend more on a learner's developing concept image than merely on the set of words which might be regarded as that learner's concept definition.

The combination of these two theoretical lenses meant that the investigation was concerned with studying the students' development from process level to object level, from visual-concrete imaginary to abstract structure, and from modeling language to general algebraic language.

Professional Development Focus on Middle School Students' Algebraic Reasoning

Each of the three participating classes was involved, over a six-week period, during the Spring semester 2015, in carefully-planned lessons consistent with a professional development program led by two senior mathematics education professors with strong backgrounds in algebra education. The professional development sessions had occupied 12 hours over a period of three weeks, and in those sessions the only persons present were the seventh-grade mathematics teacher ("Ms. *X*"), the two experienced mathematics education professors, and the present writer.

Following the period of professional development Ms *X* then taught all the intervention lessons to her three classes. Lessons were mostly problem-based (task-based) and the main goal in each of the classes was for students to gain relational understanding Skemp, 1977) of what they were doing as they explored how algebraic notations might be used advantageously in problem-based situations.

Instruments and Data Collection

Over the spring and summer semesters of 2014, the present writer created four pencil-and-paper instruments. One of the tests concentrated on structural aspects of early algebra, another on modeling, and a third on visual or number patterns. Draft versions of these three instruments were administered to eighth-grade students and after analysis, three separate pencil-and-paper instruments were created for the study described in this paper.

The fourth test was an "Algebra Readiness Test" (hereafter referred to as "ART"), which was also developed by the writer. When developing the 20-item ART, five major "big ideas" associated with elementary algebraic reasoning were consciously taken into account: (a) expressions and equations, (b) functional thinking, (c) generalized arithmetic, (d) the use of letters as variables, and (e) proportional reasoning. When data from the first version of ART generated an unsatisfactory Cronbach alpha reliability coefficient, further trials were conducted until a satisfactory version of ART, with Cronbach alpha reliability of 0.84, was obtained.

In order to complement data gained from analysis of students' responses to questions on the four pencil-and-paper instruments, 18 students were selected based on their group's modeling, structure, and visual pre-teaching scores. Six interviewees, high and low scorers, from each group, were selected. The study participants were from three different classes (see Table 1). They were interviewed on a 1-1 basis according to interview protocols recommended by Newman (1983). The purpose of the interviews was to ascertain how the interviewees were thinking about carefully selected pencil-and-paper tasks.

Table 1
Summary of the Study's Participating Students

Groups of Participating Seventh-Grade Students	Number of Participants
Class 1 (Pre-algebra class)	24
Class 2 (Pre-algebra class)	24
Class 3 (Algebra class)	23
All Seventh-graders	71

Results and Discussion

Research Question 1

The question was: "Immediately after all three lessons were completed were there statistically significant differences between the three groups' mean gain scores on pencil-and-paper tests of structure, visual-number patterns, and modeling?" The three groups' pre-teaching and post-teaching mean test scores on pencil-and-paper tests (designed specifically for each group) are shown in Figures 1, 2, and 3. Analysis of the mean gain scores indicated that whereas the Modeling and Visual groups' mean gain scores were significantly different from zero, the mean gain scores for the Structure group increased only slightly.

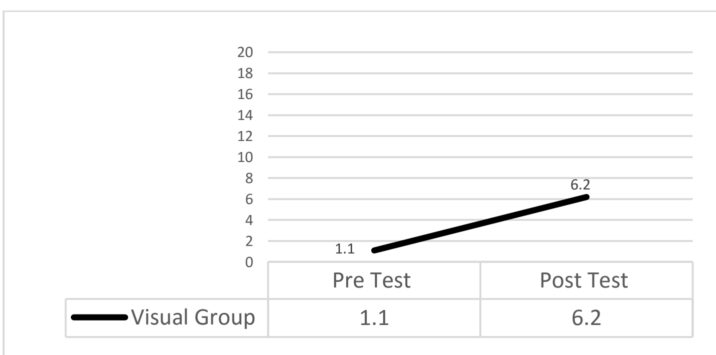

Figure 1: Visual group's mean scores on pre-teaching and post-teaching (maximum possible score was 20).

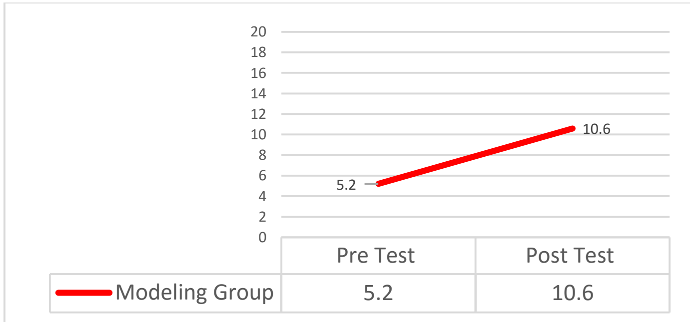

Figure 2: Modeling group's mean scores on pre-teaching and post-teaching (maximum possible score was 20).

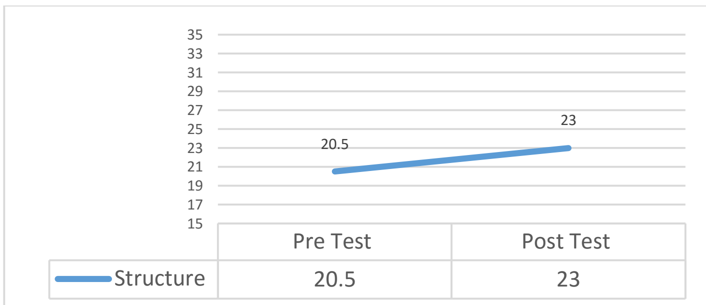

Figure 3: Structure group's mean scores on pre-teaching and post-teaching (maximum possible score was 40.

Of the three groups, it can be seen that the most impressive gains were obtained by the Visual group (from 1.1 to 6.2). The ratio was more than 5, whereas for the Modeling group (from 5.2 to 10.6) the ratio was only slightly more than 2. The Structure group (from 20.5 to 23.0) would appear to have had the least impressive gains

Research Question 2

The question was: "What were the overall effects of each of the three interventions with respect to mean gain scores on the Algebra Readiness Test (ART)?" Mean scores for all three groups on the pre-teaching and post-teaching ART administrations are shown in Figure 4.

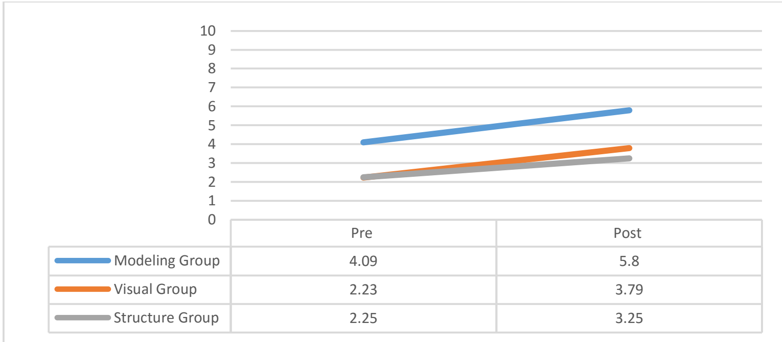

Figure 4: All groups' ART mean scores on pre-teaching and post-teaching.

Although analysis revealed that the means of all three groups increased slightly, quantitative analysis revealed that a large mean gain scores and a medium Cohen's d effect size was obtained for the Modeling group (calculated Cohen's *d* effect size is 0.49); by contrast, the Visual group's effect size was 0.38) and the Structure group's effect size was 0.33.

Research Question 3

1. The question was: "Immediately after all three lessons were completed, were there educationally noticeable differences between the concept images of the students, with respect to the concept of a variable, in comparison with the concept images that the students had before the intervention began?"

In order to answer this question, pre- and post-teaching pencil-and-paper test responses, and pre- and post-teaching interview responses were analyzed. Analysis of the interview data, in particular, indicated that there was a change in students' concept images of a variable and in their understanding related to the meaning of formulating and making generalizations. Two conclusions seemed to be warranted:

1. Conventions associated with relating any input (*n*) to its corresponding output (expressed in terms of *n*) were not initially understood by many seventh-graders. Figure 5 shows a most common pre-teaching interview response to the fun-park task, which was one of the

interview questions. Figure 6 shows the post-teaching response to the same task by the same student. For this fun-park question interviewees were asked:

> Suppose you went to a Fun Park, and it cost $10 to get in. Then, it cost $3 for every ride that you had. That information can be seen in this table:

Number of Rides You Have	0	1	2	3	4	…	*n*
Total Cost	$10	$13	$16	$19	?		??

Then they were asked: What number should we place under the 4 in the table? And, finally, they were asked: What do you think we should we put under the *n*?

Number of Rides You Have	0	1	2	3	4	… 5	*n* 6
Total Cost	$10	$13	$16	$19	$22	$ … 25	$28

Figure 5: A common pre-teaching interview response sample relation to the fun-park task.

Number of Rides You Have	0	1	2	3	4	…	*n*
Total Cost	$10	$13	$16	$19	22	…	Sn = 3n + 10

Figure 5: Changed concept image post-teaching interview response sample relation to the same generalization task.

2. The subscript notation (t_n or S_n) is very difficult notation for middle-school algebra students, and probably should not be introduced until the ninth-grade (or even later than that).

Research Question 4

The question was: To what extent was Ms. X's classroom teaching similar for the three interventions, and is it possible to link the teaching to changes in students' concept images? In this investigation, students worked in whole-class environments and the teacher's instruction was mostly direct and not dialogic. The researcher and the two senior professors who observed all the lessons believe that the students would have learned more if there had been more interaction between the students and the teacher. The work covered in the sessions was not easy, and was probably ahead of what might reasonably be expected from common-core algebra considerations.

The three observers noted the following aspects that were present in each lesson:

1. Ms. *X* tended to ask leading questions, and to push her students toward correct answers. The students expected, and respected, this, and were pleased to respond if they felt they could.

2. Ms. *X* and her students tended to prefer to think in terms of numbers rather than in terms of generalized patterns. Even when a question invited visual thinking, Ms. *X* and her students usually (though not always) preferred to consider numerical patterns rather than visual patterns.

3. Students in each of the three classes found it difficult to identify explicit rules which generalized to "the *n*th case." This was not surprising, considering these were seventh-grade classes. Although, in several instances a few students did attempt to generalize, more often than not exercises which invited students to use spatial reasoning in order to generalize were converted to numerical checking sessions.

4. In each class there were a few students who became, and stayed, "lost." This was confirmed by the observers who took the opportunity to speak with individual students from time to time.

5. On the whole, the observers felt that although some students had definitely gained a better idea of elementary algebra as a result of their participation in the intervention lessons, this was not true of all students, and it may not have been true of a majority of the students. In order to find out

whether that was indeed the case it will be necessary to wait until results on a retention test become known.

Discussion

The findings from the present research underscore the importance of, and need for, a formative algebra readiness instrument. An NCTM Research Committee (2013) stated that: “it is not clear how these assessments will influence the teaching and learning of mathematics, yet the investment in time and resources to develop these assessments at a large scale suggests that their influence may be profound” (p. 341). Since the effect sizes that were found were small, a tentative conclusion of the present study might be that any pre-algebra program based entirely on geometrical patterns, or based entirely on structural aspects, would be inadequate. It could be that case that a holistic approach to pre-algebra is needed

This study contributes to an under-represented area in the mathematics education research literature. As educators we need to ask some practical questions: First, are all 7th grade or 8th grade students ready for algebra? Secondly, are current 5th, 6th, and 7th grade curricula sufficiently rigorous to prepare students well for algebra? Stephens, Isler, Marum, Blanton, Knuth, and Gardiner (2011) reported that after a sustained early algebra intervention, students grew in their abilities to shift from recursive to conversational thinking about linear functions and to represent correspondence rules in both words and in variable form. Cai and Knuth (2011), who looked across studies completed by 2011, noticed two dominant ideas emerging from the research. The first was concerned with the need to develop students' algebraic thinking and to integrate rich and relevant tasks that into the school mathematics curriculum. This step includes both the designing of tasks and curricula and making connections between arithmetic and algebra. The second dominant idea related to the importance of supporting teachers’ efforts to implement promising practices that foster the development of students' algebraic thinking. Further research should provide insight into how students' modeling and generalization activities assist to develop their concept of a variable.

Although this study generated informative analyses of data, it was clear from the analyses that a different teaching approach, and a new approach to professional development, was needed for the further studies. The method of professional development for the teacher in this study was one-on-one instruction based on notes that had been especially prepared for the occasion. The volume and the complexity of the material tended to be overwhelming for the teacher. It was recognized that in future professional-development studies an approach which engaged teachers more actively than had been the case in this student was desirable. With such approach the participating teachers might observe experienced mathematics educators teaching actual classroom setting in the same school and then teachers would observe finally when it comes to real research setting they would be more ready and confident.

The fact that the student participants in this study were not randomly allocated to groups meant that the three groups differed in their initial knowledge and understandings of algebraic notations and concepts, and this could have had an impact on the students' reactions to the intervention lessons. This realization pointed to the need for random allocation to intervention groups in the future studies, and this was done in a new study, not reported here, carried out by the present writer. Finally, it should not be assumed that seventh-grade teachers and students are ready to use CCSSM sixth and seventh-grade content standards with respect to algebra.

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