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Theoretical Framework

Elementary children labeled as having learning disabilities or difficulties (MD) begin their study of fractions with similar yet more diminished conceptual understandings of fractions as quantities then what is documented among their peers (Hecht, Vagi, & Torgeson, 2007). An incomplete understanding of fraction concepts impacts children's ability to operate or apply computational procedures with fractions in higher-level mathematics (National Mathematics Advisory Panel, 2008). These findings suggest a continued instructional focus on the development of conceptual understanding of fractions as quantities for these children is critical.

A review of literature on fraction understandings in special education revealed deficit depictions of children with MD evident in assessment and instructional procedures (e.g., needs for explicit demonstration of efficient strategies within instruction). Such a depiction may stem from views of children with MD as possessing innate cognitive factors that render them unable to hold understanding of concepts (Davis, Cannistraci, Rogers, Gatenby, Fuchs, Anderson, et al., 2009) or perhaps hold qualitatively different understandings (Mazzocco & Devlin, 2008).

The deficit view of children with MD seems to yield a conceptualization of "understanding" for these children as their responsiveness to static, teacher-led instruction (e.g., Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Test & Ellis, 2005). Yet, understanding cannot be imposed onto children (Baroody, Cibulskis, Lai, & Li, 2004). Instead, supporting children's conceptual growth involves *children* "[solving] problems within [their] reach [while] grappling with key mathematical ideas that are comprehendible but not yet well formed" (Hiebert & Grouws, 2007, pp. 387). If teachers wish to support children's growing understandings in instruction, then a characterization of the understandings children with MDs do hold of fraction concepts provides a indispensable foundation (Daro, Mosher, & Corcoran, 2011). This study extends current literature by presenting key understandings of unit fractions evidenced by 44 children with MD in semi-structured clinical interviews. The research question was as follows:

What initial understandings of fractions do children with MD evidence through employed problem solving strategies and observable operations?

Methods

We completed 44 interviews with children in the second, third, fourth, and fifth grades who assented and whose parents provided informed consent (see Table 1). Twenty-one of the 44 children have documented, cognitively defined learning disability (LD). Twenty-three of the children were labeled "Tier 2", indicating documented, pervasive performance issues in mathematics not necessarily cognitive in origin.

Table 1. *Characteristics of Students*

	LD (%) N = 21	Tier 2 (%) N = 23
Grade		
2	5%	13%
3	24%	13%
4	48%	31%
5	23%	43%
Gender		
Male	76%	57%
Female	24%	43%
Ethnicity		
Caucasian	10%	9%
Black	14%	30%
Hispanic	71%	61%
Burmese	5%	0%
Disability*		
LD, Working Memory	29%	0%
LD, Processing	10%	0%
LD, LTM	5%	0%
LD, Fluid Reasoning	5%	0%
LD, Comorbid	51%	0%

**Predominant cognitive difficulty at or below 15th percentile*

Researchers designed a set of seven problem situations for use in the study based on a synopsis of prior research. Problem situations were based in the context of equal sharing; the number of sharers ranged from two to four and the number of objects shared ranged from one to nine. Table 2 lists the tasks.

Three situations were designed to elicit unit fractional values less than one (i.e., one-half, one-third) through the context of sharing a “French fry” (Tzur, 2007) These tasks included sharing one whole item between two and three people. Another task asked children to rebuild the whole when given a share, or part (i.e., one fifth or one eighth) (Wilkins et al., 2013). These situations were used to elicit and assess each student's propensity to utilize partitioning and iterating operations to conceive of the whole as so many same sized copies of a fractional unit and coordinate unit fractions with respect to one whole. Throughout all tasks, children's representational levels (i.e., tangible, figurative, or symbolic) and naming of the created quantity (in activity and in answer to "how much?" or "what do we call that?") were recorded.

Another four situations were designed elicit fractional values greater than one (i.e. number of items > number of sharers) and less than one (i.e. number of items < number of sharers) in a story context (e.g., Four friends share three ice cream bars so that each friend got the same amount to eat. How much of an ice cream bar did each friend receive?). These tasks included sharing five wholes between two people, nine wholes between four people, three wholes between four people, and operating with an established fractional quantity to reason about a total (i.e., $\frac{2}{3} \times \blacksquare = 6$). These situations were used to elicit and assess children's propensity to use a mentally planned partitioning of each whole item coordinated with the number of sharers, combine the created unit fractions, and quantify it as an equal share with respect to one whole [i.e., three items shared by four people as (1 ÷ 4) + (1 ÷ 4) + (1 ÷ 4)]. Throughout the tasks, children's representational level (i.e., tangible, figurative, or symbolic) and naming of the created quantity in answer to "how much of a whole does each person get?" were recorded.

Table 2.
Interview Tasks

Problem Tasks
1. Please share this fry equally between the two of us. What do you call each part of the French fry? How do you know?
2. Suppose we want to share the Fry among three people now. This time, you cannot fold or use a ruler. Show me the size of the share. How do you know it is the correct size? What do you call each part of the French fry? How do you know?
3. Here is one whole French fry. Here is one person's share. What part of the whole is the share? How do you know?
4. Harry and Larry order 5 sandwiches to share equally between them. How many sandwiches does each of them receive?
5. 4 children share 3 large candy bars. Each child eats the same amount and they finish all 3 candy bars. How much of a candy bar does each child eat?
6. 4 children share 9 sticks of clay for a project. How many sticks of clay does each child use?
7. Each student uses 2/3 of a sandwich bun for his or her lunch. How many students were there if 6 sandwich buns were used?

Semi-structured clinical interviews (Ginsburg, 1997) were done with each child individually and were audio and video recorded. Children were encouraged to solve each task in a way that made sense to them – they could use the manipulative materials, paper and pencil, or no materials to aid them in reaching a solution. Tasks were administered to children until it was evident that (a) no new insights into how each child conceived of unit and non-unit fractions emerged or (b) the child could no longer provide a solution to the problems on his/her own or with prompting.

Data analysis was done on three levels. First, constant comparison method to delineate elements of children's conceptions of unit fractions (Strauss & Corbin, 1997; Leech & Onwuegbuzie, 2007). Three researchers reviewed the first four videotaped interviews as a team, inspecting each task. Researchers examined (a) the way in which the child solved the problem (nuances within evidenced strategies along with the nature of associated conceptions led us to ultimately refer to ways of solving problems by levels) and observable operations he or she employed. We then gave each element of children's thinking an initial code. Researchers also informally noted indicators of fraction understanding that began to emerge in the data (discussed further below). As more tasks and interviews were coded, we carefully compared each new chunk of data (i.e., each problem solution) with data coded previously and searched for confirming and disconfirming evidence to ensure consistency and validity (Leech & Onwuegbuzie, 2007). The iterative process of coding, comparing, and refining continued through three additional rounds of independent coding until all tasks in all interviews were coded.

Next, emergent coding (Grbich, 2012) was used to uncover indicators of understanding for each interview across the tasks as they were evident in children's problem solving and operations. Researchers identified major categories of indicators evidenced *across the tasks* within each interview by each child. The indicators were considered in terms of development reflected in how they emerged in the data. Indicators were then placed in a framework, which was used by two researchers to code all remaining interviews to establish each child's overall understandings (i.e., level).

Finally, data visualization techniques (Ward, Grinsteind, & Keim, 2010) were used to visually examine trends in reference to the indicators across all children interviewed. A heat map of coded indicators (lowest level –orange, highest level- yellow) for each child. The maps were analyzed to examine which indicators seemed to lead development at various levels and further refine the framework.

Results

Our aim in this study was to delineate the initial understandings of fractions held by children with MD. The research question addressed the initial understandings of fractions students MD evidenced through their employed problem solving, language, representations, and observable operations as they engage in equal sharing tasks along with any key understandings that seemed to emerge in their work.

Analyzed data are presented in two parts. First, results of the constant comparison analysis are presented in terms of level for problem solving, language, operations, representations found in the analysis. Second, results of emergent coding in terms of the key understandings are defined and quantified by level, a framework of key understandings is delineated, and results of trend analysis are illuminated using a heat map.

Constant Comparison Analysis

Problem solving levels. Qualitative analysis resulted in four levels of problem solving activity: (a) *No fractions*, (b) *Emergent sharing*, (c) *Half*, and (d) *Emergent relations/coordination*. Examples and descriptions follow.

No fractions. In Level 0, or *No Fractions*, children did not create fractions. In the first three tasks (hereafter referred to as "Fry tasks"), children did not engage with sharing the item and tried to add more items in order to create whole number shares. In the last four tasks (hereafter referred to as story problems), children created unequal shares (e.g. five items shared by two people would result in one person receiving two and one person receiving three) or added/took away items to create whole number shares. Five percent of children with LD and 0% of students in Tier 2 evidenced *No Fractions* as their dominant problem solving level.

Emergent sharing. In Level 1, or *Emergent Sharing*, children utilized guess and check or a whole number based "build up" to share. For instance, in the Fry tasks, children would count "one, two, three", from the left, making unequal parts. Children would partition but not iterate a part and did not exhaust the whole. Children sometimes extended their counting across the whole until they created a number of unequal pieces. In the story problems, children dealt out whole number objects, counting by ones, until they encountered a leftover. At that point, children either continued their whole number count onto the object or used a rudimentary halving mechanism to partition the item. Fifty percent of children with LD and 30% of children in Tier 2 evidenced *Emergent Sharing* as their dominant problem solving level.

Half. In Level 2, or *Half*, children began to coordinate making equal shares with exhausting the whole "after the fact". Children's in-activity plan usually became apparent in subsequent attempts to coordinate making same-sized parts and use up the whole (i.e., Fry tasks) or when dealing with "leftovers" (i.e., story problems). For example, in the Fry tasks, children would work from the middle or from the end points, trying to keep equal parts. They eventually exhausted the whole, visually working from the endpoints or midpoint, adjusting the part, until they were satisfied the parts were equal within the whole. In the story problem, children distributed halves until they encountered a leftover(s). At that point, students distributed wholes, halved the leftovers, and finally partitioned the last piece by the number of sharers. Thirty percent of children with LD and 35% of children in Tier 2 evidenced *Half* as their dominant problem solving level.

Emergent relation or coordination. In Level 3, or *Emergent Relation or Coordination*, children coordinated making equal shares with exhausting the whole and used it a plan coming into activity; children link partitioning to the number of sharers. In the Fry tasks, children used an object to stand in for a part and tested it against the length of the whole. Children would eventually exhaust the whole, usually by adjusting the created part in some manner across the whole until it was the correct length. In the story problems, children either (a) partitioned each item by the number of sharers or (b) created a number of parts equal to the number of sharers, sometimes utilizing multiplication facts. Fifteen percent of children with LD and 35% of children in Tier 2 evidenced *Emergent Relation or Coordination* as their dominant problem solving level.

Operations. Qualitative analysis resulted in three levels of partitioning operations; an iterating operation was also uncovered yet considered separately: (a) Partitioning with no regard to equal parts, (b) Partitioning with regard to equal "halves", and (c) Partitioning with regard to equal parts (all cases) and (d) Iteration (present/absent). Examples and descriptions follow.

Partitioning with no regard to equal parts. At this level, children partitioned objects yet did so somewhat "haphazardly", with little regard to equality of parts and no regard to exhausting the whole. When asked if it mattered that the parts were equal or if it was fair if the child produced unequal parts, the child replied, "No" or "It's OK if they are different". Ten percent of children with LD and 0% of children in Tier 2 evidenced this form of partitioning as their dominant operation.

Partitioning with regard to equal "halves". At this level, children partitioned objects and expressed specific regard to equality with respect to one half. With partitions other than one-half, children seemed to lose equality of the parts. Twenty percent of children with LD and 0% of children in Tier 2 evidenced this form of partitioning as their dominant operation.

Partitioning with regard to equal "parts". At this level, children partitioned objects and expressed specific regard to equality with respect to all parts created. Children's attention to equality at this point is explicit and is beginning to be linked to an exhaustion of one whole. Seventy percent of children with LD and 100% of children in Tier 2 evidenced this form of partitioning as their dominant operation.

Iteration. *Iteration*, as we defined it, involved a partition yet children independently disembeded and used one piece as a stand in for all pieces. Often, children tested the piece for "correctness" against the whole in activity. Iteration was coded as present or absent holistically across the tasks. Ten percent of children with LD evidenced iteration within their problem solving activity, while 55% of Tier 2 students used iteration.

Language and representations. The analysis resulted in three levels of *language* students used to quantify the equal share: (a) *Pieces* (i.e, named share as a number of pieces), (b) *Half* (i.e., used "half" to name all unit fractions), (c) *Developing* (i.e., quantified unit fractions in activity; did not transfer this naming over to quantify amount), and (d) *Solidified* (i.e., quantified unit and non-unit fractions dependent and/or independent of activity). Three representations, (a) *Tangible*, (b) *Figurative*, and (c) *Symbolic*, were also documented. Tangible representations included concrete items, like cubes or paper. Figurative representations included drawings or the use of fingers. Symbolic representations included numeric recordings of work or verbal descriptions of symbolic representations.

Table 3 demarcates results of the content analysis for students' dominant ways of problem solving, representation, operations, and representations as evidenced across the tasks.

Table 3.
Content Analysis Frequencies

	LD (%)	Tier 2 (%)
Problem Solving Strategy		
Level 0	5%	0%
Level 1	50%	30%
Level 2	30%	35%
Level 3	15%	35%
Operations		
Partitioning No Equality	10%	0%
Partitioning Equality to Half	20%	0%
Partitioning Equality	70%	100%
No Iteration	90%	45%
Iteration	10%	55%
Language		
Pieces/Whole Number	70%	55%
Halves	10%	20%
Developing	20%	15%
Solidified	0%	10%
Representations		
Tangible	50%	40%
Figurative	50%	55%
Symbolic	0%	5%

Emergent Analysis

Key understandings. The next paragraphs describe key understandings (i.e., divisible whole, partitioning plan, notions of equality within the whole) that emerged as a result of analysis in terms of level. Then, we present the framework and the numbers of students falling at each level.

Divisible whole. Emergent analysis resulted in three levels of observed activity: (a) *No fractions (coded as 0)*, (b) *Developing (coded as 0.5)*, and (c) *Solidified (coded as 1)*. *No Fractions* indicated children propensity to only deal in terms of whole units- to these children, the whole is not yet conceived of as divisible, so fractions are not created. *Developing* notions of a divisible whole were viewed as a reluctant, after-the-fact notion of the whole as divisible. Children did not want to create fractional shares initially, but seemed to do so "begrudgingly". A *Solidified* notion of a divisible whole was evidenced when children readily cut apart a whole or wholes without hesitation.

Partitioning plan. Emergent analysis resulted in four levels of observed activity: (a) *No fractions (coded as 0)*, (b) *Developing (coded as 0.5)*, and (c) *A-priori Link to Sharers (coded as 1)*. *No Fractions* codes indicated students did not act on the whole in terms of partitioning. *Developing codes* meant that a plan for creating a seemingly known number of total pieces across the whole(s) is not yet anticipated or carried out in activity. Yet, children used whole numbers as a rudimentary, activity-based plan for creating fractional units within one whole (i.e., the Fry tasks) or the "leftover" (i.e., the story problems). *A-priori Link to Sharers* codes indicated that children used the number of sharers to create a known number of parts. A use of multiplication, and at times division, seemed to accompany children's thinking.

Relation between partitions and parts created. Emergent analysis uncovered children's notions of a relation between partitions and parts created as *Absent* (coded as 0) or *Present (coded as 1)*. Researchers coded *Absent* if children (a) seemed to confuse cuts with parts created (e.g., to make fourths, the child folds a paper four times) or (b) made explicit statements in their activity that indicated a lack of association between parts and cuts (e.g., "three cuts for three parts"). *Present* was coded as the lack of any indication to the contrary; children's partitioning seemed to align to the number of parts they made in activity.

Exhausting the whole. Emergent analysis resulted in three levels of observed activity in terms of exhausting the whole: (a) *Early (coded as 0)*, (b) *Developing (coded as 0.5)*, and (c) *Solidified (coded as 1). Early* codes indicated children's attention was solely on making equal parts. *Developing* codes indicated the child's attention shifted to the whole and was beginning to attend to sizes of equal parts within the whole, but children had difficulty making equal parts and exhausting the whole concomitantly. *Solidified codes* indicated that the child, in activity, coordinated making equal parts within the whole.

Notions of equality of the parts. Emergent analysis resulted in three levels of problem solving: (a) *Early (coded as 0)*, (b) *Developing (coded as 0.5)*, and (c) *Solidified (coded as 1)*. *Early* was defined as created parts unequal in size; children were not bothered by their inequality (i.e., when asked, children said that it is OK that the parts are different sizes or given a justification based on informal notions of "fair" to give more to one person because they are bigger). *Developing* was defined as children stating or explaining in their activity that the parts should be equal. Children paid close attention to making equal parts yet has difficulty because they started to pay attention to the parts with respect to the whole. Yet, children had yet to coordinate the parts and the exhaustion of the whole. *Solidified* was defined as students paying close attention to equality of parts with respect to the whole.

A framework of key understandings. The key understandings evidenced a framework in terms of all children's initial fractional knowledge[1] (see Table 4). Forty percent of children with LD were coded as a Level One; 20% of children in Tier 2 were coded as Level One. Fifty percent of children with LD were coded as a Level Two; 45% of children in Tier 2 were coded as Level Two. Ten percent of children with LD were coded as a Level Three; 35% of children in Tier 2 were coded as a Level Three.

Trend Analysis

In this final section of analysis, we present a mapping of the key understandings as evidenced by each child holistically to illuminate indicators that lead development at each level and key understandings that seemed to emerge together.

Heat map. Figure 1 illustrates a visualization of the key understandings for 39 of the 43 children[2] across the interview tasks against their overall trajectory level code. The following paragraphs describe which key understandings led development at each level holistically for all children interviewed. As illustrated in the heat map, seven out of 15 children who were holistically coded as Level One fully conceived of the whole as divisible (42%), while the remaining eight children's conception of a divisible whole was developing. An understanding of the need for equality of the parts was coded as developing in all but one of the children. Eleven children in Level One showed an early understanding of exhausting the whole when sharing. A plan for partitioning going into modeled activity was coded as early or developing in all children. Thirteen children coded at Level One seemed to confuse partition lines with parts in their activity. Independent iteration was not used. From the evident trends, what leads development at Level One include children's developing (a) notion that the whole is divisible and (b) recognition of the need for equal parts; indicators showed as "1" or "0.5" in most students.

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[1] We combined equality of parts and exhausting the whole into one category as their level descriptions were extremely connected. Moreover, Piaget et al. (1960) discussed equality of parts and exhausting the whole as complementary understandings.
[2] Elements of data were missing for one of the five key understandings for four of the students, so they were excluded from the heat map trend analysis.

Table 4. *Framework of key understandings.*

<table>
<tr><th>Level</th><th>Divisibility of the Whole</th><th>Partitioning Plan</th><th>Coordination of Equal Units Within the Whole</th></tr>
<tr><td>0</td><td colspan="3">Will only share/deal out wholes.
Whole not yet conceived as divisible.
Does not act on the whole or create fractions.</td></tr>
<tr><td>1</td><td>Seems to reluctantly cut into pieces.</td><td>Trial and Error based in whole number in activity.
• May begin to use “half” in activity, but it is not meaningful to the child as a quantity. It is a rudimentary sharing strategy based on two people breaking apart an item or items.</td><td>Child’s attention is on making a number of parts.
• Parts created are not equal in size, and the child is not bothered.</td></tr>
<tr><td>2</td><td rowspan="2">Readily divides whole without hesitation.</td><td>Plan becomes evident in how students attend to sharing leftover parts or wholes after dealing out wholes or halves.
• “Half” represents a meaningful quantity that the child uses to create pieces.</td><td>Begins to coordinate equal parts with exhausting the whole after they see the equal part(s) he/she created do not exhaust the whole.</td></tr>
<tr><td>3</td><td>Plans to create number of parts equal to number of sharers prior to activity.
• May use knowledge of multiplication/division.</td><td>Creates equal parts while exhausting the whole by using iteration to test the part against the whole.</td></tr>
</table>

Child #	Divisible Whole	Partitioning Plan	Exhaust Whole	Equality of Parts	Part/Cut Relation	Iterating	Trajectory Level
1	0.5	0	0	0.5	0	0	1
4	0.5	0	0	0.5	0	0	1
5	0.5	0	0	0.5	0	0	1
6	0.5	0.5	0	0.5	0	0	1
7	0.5	0	0	0.5	0	0	1
8	0.5	0	0	0.5	0	0	1
10	0.5	0.5	0.5	0.5	0	0	1
15	1	0	0	0.5	0	0	1
21	0.5	0	0	0.5	0	0	1
22	1	0.5	0	0.5	0	0	1
23	1	0.5	0	0.5	0	0	1
24	1	0.5	0.5	0.5	0	0	1
25	1	0	0	0.5	1	0	1
29	1	0.5	0.5	1	1	0	1
30	1	0.5	0.5	0.5	0	0	1
2	1	0.5	0.5	1	1	0	2
3	1	0.5	0.5	1	1	0	2
9	1	0.5	1	1	1	0	2
11	1	0.5	0.5	1	1	0	2
12	1	0.5	0.5	1	1	0	2
13	1	0.5	0.5	1	1	0	2
18	1	0.5	0.5	1	1	1	2
19	1	0.5	0.5	1	1	0	2
20	1	1	1	1	1	0	2
32	1	0.5	0.5	1	1	1	2
33	1	0.5	0.5	1	1	0	2
34	1	0.5	0.5	1	1	1	2
35	1	0.5	1	1	1	1	2
37	1	0.5	1	1	1	1	2
38	1	0.5	0.5	1	1	1	2
40	1	0.5	0.5	1	1	1	2
41	1	0.5	1	1	1	1	2
16	1	1	1	1	1	0	2
14	1	1	1	1	1	1	3
17	1	1	1	1	1	1	3
26	1	1	1	1	1	1	3
27	1	1	1	1	1	1	3
28	1	1	1	1	1	1	3
31	1	1	1	1	1	1	3
36	1	1	1	1	1	1	3
39	1	1	1	1	1	1	3

Figure 1. Heat Map.

All 18 children holistically coded at Level Two conceived of the whole as divisible. A notion that parts need to be equal becomes solid at this level of development, with all of the children evidencing recognition of the need for equal parts. The need to make equal parts, however, did not always reconcile with exhausting the whole, as children's propensity to exhaust the whole in their activity was coded as solidified in only six children. A partitioning plan ahead of modeled activity is coded as developing in all but one child. Parts created are now related to partitioning. Iteration also emerges among eight of the children. From the evident trends, it appears that what leads development at this level is not only children's evolving (a) coordination of created equal parts with exhausting the whole but also (b) an a-priori plan for creating the parts.

For the eight children coded as Level Three, most key understandings were coded as solidified in activity. The whole is divisible for 100% of the children; parts are related to cuts for 100% of the children. A partitioning plan and exhaustion of the whole were also solidified. Moreover, Iteration occurred independently. From the evident trends, what seems to lead development in Level Three is the testing of implicit or explicitly equal parts against the whole, seemingly in an effort to quantify the share in terms of the whole and/or to rectify creation of fractions in activity.

Conclusions, Implications, and Future Research

Many researchers define and document initial notions of fractions through equal sharing (Charles & Nason, 2000; Empson et al., 2006; Empson & Levi, 2011; Kieran, 1976; Pothier & Sawanda, 1983; Steffe & Olive, 2010; Streefland, 1993; Tzur, 1999, 2007; Tzur & Simon, 2004). Accordingly, we utilized this literature base to design equal sharing situations to uncover initial conceptions of fractions for children with MD. Results of the current study revealed a variety of problem solving strategies, language, operations, and representations evidenced by students with MD (i.e., LD and students in Tier 2 settings) representative of their initial notions of key fractional understandings. The majority of students with MD evidenced partitioning in their activity. Yet, many children used a rudimentary trail and error based partitioning plan or informal notions of "halving" to partition in the equal sharing situations. Anticipations of a plan for partitioning linking the number of sharers to the whole(s) prior to activity were absent in a majority of children in the study. Moreover, the children evidenced early notions of coordinating parts with respect to the whole, with many children attending to either the parts they created or the whole to be shared, but not both at once. Some children seemed to begin to realize the necessity of this coordination in their activity, yet few used an iterating operation to verify the coordination of the parts to the whole. Thus, our findings showed a predominance of children in the study evidenced early conceptions of fractions as quantities, both within the confines of our framework and when compared to existing research that documents previous findings of students' conceptions (Empson et al., 2006; Tzur, 2007; Steffe and Olive, 2010).

In the remaining sections, we discuss two key implications of the study for future practice: (a) a continued focus on conceptual understanding of fractions for students with LD and mathematics difficulties and (b) possible alternative starting points for intervention and instruction in fraction concepts then what are currently implemented in classrooms.

Diminished understandings of fractions as quantities can impact children's ability to operate with fractions in higher-level mathematical contexts (NMAP, 2008). In the current study, many of the children were able to partition, seemed to be developing plans for partitioning in their problem solving activity, and evidenced a nascent understanding of the magnitude of parts coordinated with respect to a whole. Yet, the operational aspects involved with more advanced notions of fractions (e.g., disembedding and iterating a part to confirm it as 1/n) seemed absent or at a rudimentary level of development in many of the children we interviewed. Thus, continuing to document how this knowledge might be extended such that children may develop their conceptions of fractional quantities seems critical (Vukovik, 2012).

Frameworks such as that documented in the current study may serve as a useful tool for practitioners and researchers wishing to gauge students' initial knowledge. It is important to note that few children, MD or otherwise, develop rich conceptions of fractions as quantities in the absence of instruction that promotes its construction (Gould, 2012). For instance, Steffe (2007) estimated many students who complete the fifth grade do not have a command of the operational or mathematical underpinnings of conceiving of fractional quantities- as much as 30%, a figure that includes children without LD. Thus, it is important to focus research and practice on the development of instructional experiences that support the advancement of students' conceptions.

What might be the start of such instructional experiences? To begin, we assert that instructional interventions must move away from explicit, part-of-whole based approaches found in much of the curriculum used in schools (e.g., shading parts of circular or linear wholes, write fraction names for the parts, and follow demonstrated procedures to find equivalent fractions) and move toward experiences that would build conceptions that were found to be underdeveloped in the study. Most children we interviewed in the current study were coded as a Level One or a Level Two in our framework, which means that their difficulties conceptualizing fractions centered on coordinating parts with respect to a referent whole and using notions of composites units and, later, multiplicative reasoning as a means to conceive of fractions (Olive & Vomvordi, 2006; Hackenberg, 2013). Prior research suggests many children with MD have underdeveloped numerical and multiplicative concepts and have benefited from instruction that immersed them in student-centered experiences that nurtured this reasoning (Tzur & Lambert, 2011; Tzur et al., 2012).

Most students involved in the current study evidenced varying propensities to partition tangible or figurative representations of equal sharing and quantified the result as a number of pieces, halves, or fractional names for the unit and non unit fractions they created. Lewis (2010, 2014) identified atypical ways in which two adults with LD understood fractional representations that led the participants away from understanding fractions as quantities. For example, the adults thought of shaded areas as "taken away" instead of as a part of a whole, and of one-half *not as a result* of, say, marking the whole into two equal parts but rather as the *action of marking* in and of itself. Lewis (2014) argued that their understandings, seemingly resistant to instructional intervention, reflected an "incompatibility between the student's cognitive processing and the mediated tools intended to support an understanding of fractional quantity (p. 380)".

In contrast, we did not see any evidence to suggest that students with LD conceived of partitioning the mathematical representations they utilized to solve *equal sharing problems* in atypical ways. Rather, due to the absence of disembedding and iterating operations, it is possible that many of these children hold a *parts within whole* idea of fractions as opposed a *parts to* *whole* idea (Hackenberg, 2013; Olive & Vomvordi, 2006); this may be linked to what Lewis (2014) calls an atypical use of representation. Yet, from the standpoint of some researchers, a parts within whole conception is likely the result of underdeveloped yet malleable operational schemes (Hackenberg, 2013; Olive & Vomvordi, 2006; Steffe & Olive, 2010; Tzur, 2007) and not the disability (Hunt & Empson, 2014). Arguably, the conception is likely the result of a curriculum that focuses on shading and vocabulary part-to-whole interpretations for fractions (Olive & Vomvordi, 2006) and can indeed be difficult to overcome. Alternate starting points for fractions concepts, such as those that begin with a ratio or relation notion of fractional units are related to yet already disembeded from the whole (Frudenthal, 1983) or building composite unit (Tzur & Lambert, 2011) and part whole reasoning in whole number, may warrant further examination.

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