Algebraic Knowledge for Teaching: An Analysis of US Experts' Lessons on Inverse Relations

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The learning and understanding of fundamental mathematical ideas such as inverse relations and the basic properties of operations (commutative, associative, and distributive) has drawn increasing attention (The Common Core State Standards Initiatives, CCSSI, 2010). These fundamental ideas are principled knowledge targeting mathematical relationships and structures. A mastery of such ideas can not only deepen students' understanding of arithmetic but also lay a foundation for students' future learning of algebra. Thus, researchers classified these ideas into early algebra topics (Carpenter, Franke, & Levi, 2003; Kaput, 2008; Schifter, 2011). Although the principled knowledge is powerful, it is abstract in nature, causing learning difficulties for students (Goldstone & Son, 2005). Students' learning difficulties also stem from classroom instruction . Often, classroom teachers do not notice algebraic opportunities that could foster students' awareness of these ideas. Instead, they simply discuss computation strategies and procedures, which does not enable students to learn the underlying principles and develop their algebraic thinking (Schifter, 2011). As such, while there is an increasing need to develop elementary students' algebraic thinking, there is a pressing need to first equip teachers with the necessary *algebraic knowledge for teaching* (AKT).

Prior research on early algebra is fruitful. However, these studies have often involved a wide range of early algebraic topics such as missing numbers, patterns and functions, with few focused on fundamental mathematical ideas such as inverse relations (if $a + b = c$, then $c – b = a$; or $a + b – b=a$, Baroody et al., 1999). Among the few studies that have explored the learning and teaching of fundamental principles, most of them focused on students' learning capabilities or challenges and not on the role of classroom instruction in shaping students' learning. A few case studies did report classroom episodes illustrating how teachers grasped or missed opportunities to facilitate students' learning (e.g., Schifter et al., 2011); yet, insights from case studies lack generality. In brief, the existing field lacks a systematic exploration on ways to teach fundamental mathematical ideas. Without a systematic investigation, it remains unclear how US elementary teachers may teach fundamental mathematical ideas such as inverse relations in existing classrooms. The purpose of this study is to study US expert teachers' lessons through a systematic analysis of classroom videos on the topic of inverse relations, which is expected to glean instructional insights (both successes and challenges) embedded in classroom instruction so as to contributing to the knowledge base of AKT.

Our inspection is guided by a three-component construct suggested by the IES recommendations for improving instruction: the use of worked examples, representations, and deep questions (Pashler et al., 2007). These recommendations were drawn from numerous high-quality studies in the fields of cognitive science, education and experimental psychology. They were intended to guide teachers at all grade levels in all disciplines to teach students' fundamental concepts. However, simply presenting teachers with these general instructional principles is not helpful with even relatively straightforward activities such as lesson planning (Ding & Calson, 2013); no mention how these principles may be implemented in complex classroom settings. This study takes a step further by using these recommendations as a cognitive construct to explore AKT-related insights based on expert teachers' actual classroom practices. In particular, we ask three questions: (1) How do the sampled US expert teachers use worked examples to teach inverse relations in elementary classrooms? (2) How do the sampled US expert teachers use representations to teach inverse relations in elementary classrooms? And (3) how do the sampled US expert teachers use deep questions to teach inverse relations in elementary classrooms? It is expected that findings from this study will inform the mathematics education field regarding the necessary AKT to teach fundamental mathematical ideas. These empirical findings may also enrich future research agendas involving the cognitive sciences and educational psychology.

Literature Review

The notion of "algebraic knowledge for teaching" (AKT) is consistent with the theory of "*mathematical knowledge for teaching* (MKT)" (e.g., Ball, Thames, & Phelps, 2008; Hill et al., 2008), focusing primarily on mathematical knowledge needed for the work of classroom teaching. Such professional knowledge can be drawn from teaching practices (Ball & Bass, 2003; Hiebert, Gallimore, & Stigler, 2002). Sound MKT is associated with high-quality teaching (Hill et al., 2008) and high levels of student achievement (Hill, Rowan, & Ball, 2005). In this study, AKT is a type of MKT needed for teaching early algebra. Early algebra does not simply mean adding algebra topics to elementary curricula (Carraher & Schielman, 2007; National Council of Teachers of Mathematics [NCTM], 2000). Rather, it refers to an infusion of algebraic thinking in early grades and takes a broad view of symbolic reasoning (Kaput, Carraher, & Blanton, 2008). Mainly through generalization, elementary students may be guided to see fundamental mathematical ideas or the "deeper underlying structure of mathematics" (Blanton & Kaput, 2005, p. 412), which not only deepens students' understanding of arithmetic but also lays a foundation for their future learning of algebra (Carpenter et al., 2003; Howe, 2009; Schoefeld, 2008).

Exploring AKT: The Case of Inverse Relations

There are various early algebra topics that demand AKT. This study focuses on one fundamental mathematical idea, inverse relations. Among the four basic operations, by definition, addition and subtraction are inverses while multiplication and division are also inverses (Vergnaud, 1988). "Inverse relations" in the current study refer to the *complement principle* (e.g., if $a+b=c$, then $c-b=a$; if $a\times b=c$, then $c\div b=a$), which can be initially learned through (a) fact family (e.g., 7+5=12, 5+7=12, 12-7=5, and 12-5=7), and (b) inverse word problems (the solutions form a fact family; Carpenter et al., 2003; Howe, 2009). An understanding of inverse relations contributes to one's full comprehension of the four operations and algebraic thinking (Carpenter et al., 2003; Stern, 2005) as well as mathematical flexibility (Nunes, Bryant, & Watson, 2009). Indeed, many children prior to elementary school (from 3 to 6 years old) have already demonstrated a degree of informal understanding of inverse relations (Gilmore & Spelke, 2008; Klein & Bisanz, 2000; Sherman & Bisanz, 2007; Sophian & Vong, 1995). However, elementary students were often found to lack formal understanding of this relation. For example, even after instruction, some students could not spontaneously use addition to solve subtraction problems (Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009a, b) or use addition to check subtraction (Baroody, 1987). In addition, word problems that demand inverse understanding (e.g., start-unknown problems) presented incredible challenges for students (Nunes et al., 2009). With regard to multiplicative inverse, existing studies demonstrated a parallel lack of understanding in students (Greer, 1994; Thompson, 1994; Vergnaud, 1988). For instance, students who used $P \div M = N$ to check whether P is a multiple of M, failed to recognize that obtaining $M \cdot N = P$ should produce the same conclusion (Vergnaud, 1988).

Why are inverse relations, a ubiquitous mathematical concept, so hard to learn? A review of literature reveals two major limitations of existing instruction during students' initial learning, which may cause students' later learning difficulties. First, existing instruction mainly focuses on number manipulations without using students' informal knowledge for sense-making. For example, although some textbooks present a fact family in concrete contexts, teachers rarely make use of them (Ding & Carlson, 2013). Other textbooks (e.g., *Everyday Math*) directly introduce a "fact triangle" (a type of flashcard) and expect students to generate a fact family based on the location of the numbers on this triangle (Ding, 2016). Without sense-making, students made mistakes such as 7÷35=5, and 5÷35=7 (Ding & Carlson, 2013). The second limitation of existing instruction is to teach inverse-based strategies rather than the underlying relation. For example, students in Baroody's study (1999) were led to think in the following way: "5 take away 3 makes what?" can be understood as "3 added to what makes 5?" It was not clear to students why this strategy worked. The textbooks in Torbeyns et al. (2009b) taught a strategy named *indirect addition* (e.g., using 79+2=81 to solve 81–79) by drawing a little arrow from the subtrahend to the minuend. Focusing on procedural strategies rather than the underlying relations does not contribute to students' deep initial learning and thus later transfer (Chi & VanLehn, 2012). These findings raise an important question: What kinds of AKT do teachers needed in order to better develop students' understanding of inverse relations so as to ease their algebraic learning?

Exploring AKT: A Cognitive Construct

To explore AKT, we focus on a three dimension cognitive construct – the use of worked examples, representations, and deep questions (Pashler et al., 2007). These dimensions are consistent with the key aspects of MKT (Ball et al., 2008). In fact, representations and deep questions relate to "tasks and discourse," which are robust factors of classroom instruction (Hiebert &Wearne, 1993). These dimensions form a cognitive construct that serves as a conceptual framework for video analysis in this study. Elaboration follows.

Interweaving worked examples with practice problems. Worked examples (problems with solutions given) help students acquire necessary schemas to solve new problems (Sweller & Cooper, 1985). Classroom experiments have reported that the use of worked examples is more effective than simply asking students to solve problems (Zhu & Simon, 1987). In addition, fading examples into practice is beneficial (Renkl, Atkinson, & Grobe, 2004). However, it has been found that U.S. teachers often spend little time discussing one example before rushing to practice problems (Stigler & Hiebert, 1999).

Connecting concrete and abstract representations. Concrete representations, such as graphs or word problems, support initial learning because they provide familiar situations that facilitate students' sense-making (Resnick, Cauzinille-Marmeche, & Mathieu, 1987). However, overexposing students to concrete representations may hinder their transfer of the learned knowledge because these representations contain irrelevant and distracting information (Kaminski, Sloutsky, & Heckler, 2008; Uttal, Liu, & Deloache, 1999). Thus, some researchers suggest fading the concreteness into abstract representations to promote generalization and transfer in new contexts (Goldstone & Son, 2005).

Asking deep questions to elicit students' self-explanations. Students can effectively learn new concepts and ideas through self-explanations (Chi, 2000; Chi et al., 1989). However, they themselves usually have little motivation or ability to generate high-quality explanations. It is necessary for teachers to ask deep questions to elicit students' explanations of the underlying principles, causal relationships, and structural knowledge (Craig, Sullins, Witherspoon, & Gholson, 2006).

The above recommendations provide general directions for organizing instruction to improve learning. However, without explicit illustrations, teachers may have challenges in incorporating them into practice (Ding & Carlson, 2013). For example, even if a textbook provides a worked example to teach inverse relations, instead of simply asking students to study the example, how can a teacher unpack it to help students make sense of inverse relations? How can this teacher guide students to purposefully represent and solve this problem to illustrate inverse relations? What types of deep questions can this teacher ask to make inverse relations explicit to students? These questions call for an exploration of detailed and transferable AKT that promotes students' algebraic readiness.

Exploring AKT: The study of experts' performance

This study explores AKT through an analysis of expert teachers' classroom performance. The analysis of classroom performance itself is a useful way to develop professional knowledge from practitioner's knowledge. For instance, Ball and colleagues developed the theory of MKT mainly based on empirical data of classroom practices. The current research particularly studies expert teachers' practices. The study of experts' domain-specific knowledge in their performance has a long tradition in cognitive psychology, which is significant in the development of expert systems (Leinhardt & Smith, 1985). According to Chi (2011), "An expert is someone who is relatively more advanced, as measured in a number of ways, such as academic qualifications, years of experience on the job, consensus among peers, assessment based on some external independent task, or assessment of domain-relevant content knowledge" (p.18). It is worthwhile to study the expert teachers' classroom performance due to several key principles of experts' knowledge: Experts notice meaningful patterns of information, their knowledge is well-organized and conditionalized, and the important aspects of knowledge can be flexibly retrieved and flexibly applied to new situations (Bransford, Brown, & Cocking, 1999). However, expertise does not guarantee that experts themselves are able to teach others (Bransford et al., 1999). As such, it is necessary to identify storable knowledge from experts' classroom performance.

While we anticipate that lessons of US expert teachers will contribute instructional insights to the knowledge base of AKT, we are cognizant of possible challenges due to cultural factors. First, U.S. elementary teachers are not specialized in teaching mathematics. They generally teach all subjects every day. An elementary teacher who is identified as an expert may have expertise in general teaching ability but not necessarily in teaching mathematics. As such, it is possible that expert teachers face challenges when teaching mathematics, especially early algebra. Second, teaching is a cultural activity (Stigler & Hiebert, 1999). Therefore, expert teachers' classroom instruction likely manifests cultural-based teaching styles, which may be attributed to their cultural beliefs about teaching and learning (Cai et al., 2014; Stigler & Hiebert, 1999). Cai et al. (2014) shows that sixteen U.S. expert teachers who had received presidential awardees for teaching excellence in mathematics commonly believed that instructional coherence refers to smooth teaching flow or the connections between teaching activities. Such an emphasis on surface connections is sharply different from their Chinese counterparts who emphasized the interconnected nature of mathematical knowledge and therefore valued emerged events (e.g., unexpected student questions) that may disrupt the teaching flow. Previous studies also revealed that US classrooms tend to focus on procedures (e.g., teaching key words) and computational strategies (Stigler et al., 1999). It is necessary to explore whether procedure-based teaching style continues in current expert teachers' classrooms and if so, whether such teaching style will bring challenges for the teaching and learning of fundamental mathematical ideas. In summary, with a focus on the instruction of inverse relations by U.S. expert teachers, it is expected that this study will identify insights for the use of worked examples, representations, and deep questions when teaching fundamental mathematical ideas so as to develop students' algebraic thinking.

Methods

This qualitative research is part of a five-year NSF project aimed at identifying AKT based on a cross-cultural analysis of US and Chinese expert teachers' videotaped lessons on two early algebra topics, the inverse relations and the basic properties of operations. For the current study, we only studied US expert teachers' classroom performance on inverse relations.

Participants

A total of eight US expert teachers who taught grades 1-4 participated in this study. The selection of teacher participants was based on three criterions: (a) teacher reputation and experience. All of the selected eight teachers had taught more than 10 years. Three of them were National Board Certified Teachers (NBCT), one was a NBCT candidate, and the rest were highly recommended by the school district and principals; (b) teacher AKT-based survey. In this survey, we asked in order to teach inverse relations (either additive or multiplicative), what example tasks, representations, and deep questions a teacher may use for instruction; and (c) teacher attitude survey. Teacher responses to both surveys were analyzed and compared by two project researchers. As a result, a total of eight teachers were selected with two for each grade. In this study, we named the two first grade (G1) teachers as T1 and T2, the second grade (G2) teachers as T3 and T4, the third grade (G3) teachers as T5 and T6, and finally, the fourth grade (G4) teachers as T7 and T8.

Instructional Tasks

Each teacher taught four lessons that were part of an existing curriculum, resulting in 32 US videotaped lessons. These lessons were selected by the project investigator based on a structure suggested by the literature, in consultation with the textbooks and the teacher participants. Table 1 shows the structure that guided lesson selection.

Table 1. *The Overall Structure that Guided the Lesson Design*

	Additive Topics		Multiplicative Topics
G1	• Fact family (or related facts) • Find the missing number • Using addition to compute subtraction • Initial unknown problem (to find how many initially)	G3	• Fact family (1) • Fact family (2) • Using multiplication to compute division • A topic suggested by teachers
G2	• Comparison word problem (1) - find the difference • Comparison word problem (2) - find the large or small quantity • Using addition to check for subtraction • A topic suggested by teachers	G4	• Comparison word problem (1) - find how many times • Comparison word problem (2) - find the small or large quantity • Using multiplication to check for division • Two-step word problems

As indicated by Table 1, all lessons involved the idea of inverse relations to a different degree. Topics included were the ones frequently discussed in the literature including fact families (also called related facts), using inverse relations to do computation or checking, initial unknown problem, a groups of inverse word problems, or two-step word problems where inverse relations were indicated by the two steps (Ding, 2016). Notably, both part-part-whole and comparison word problems were involved because these are major problem structures (Ng & Lee, 2009) that can be used to facilitate inverse relations (Carpenter et al., 2003). As shown in Table 1, teachers in G1 and G2 taught additive inverses lessons while teachers in G3 and G4 taught multiplicative inverse lessons.

Interestingly, the selected teachers used different mathematics textbooks. It happened to be the case that one used *Investigations* while the other *Go Math* for each grade. Both textbooks claimed to be aligned with the common core based standards. *Investigations* is an NSF-supported curriculum focusing more on student explorations. Go Math is a new textbook series that absorbed recent research assertions (e.g., the use of schematic diagrams) and was newly adopted by the school district. Regardless of curriculum differences, the project researchers were able to identify corresponding lessons that aligned with the proposed structure (see Table 1).

Coding Framework

To code teachers' use of worked examples, representations, and deep questions in the enacted lessons, we used a rubric adapted from Ding and Carlson (2013). The original rubric was developed for coding lesson plans based on the IES recommendations. Because a lesson plan is essentially an image of a lesson, it is appropriate to use this rubric as a basis for coding. This coding framework contains three large categories (worked examples, representations, and deep questions) each containing two subcategories (worked examples and practice problems, concrete and abstract representations, deep questions and deep explanations). We expect that a teacher could sufficiently discuss a worked example with practice problems clearly connected to the worked example. We also expect that discussions especially that of worked examples can be situated in rich contexts such as pictures and story problems so as to help students make sense of a concept. Meanwhile, we expect concrete representations to be well connected to the abstract representations of the underlying concept. Finally, we expect a teacher can ask deep questions to elicit students' explanations of the underlying concept. A 0-2 scale is used to score each subcategory with "0" denoting not addressing well and "2" being fully addressed (see Table 2).

Table 2. Coding Framework for Videotaped Lessons

Category	Subcategory	0	1	2
Worked Examples	Example	Examples and guided practice cannot be differentiated.	Worked examples are discussed in a brief manner	Worked example is sufficiently discussed
	Practice	Practice problems have no connection to the worked examples.	Practice problems have some connections to the worked example.	Practice problems have clear and explicit connection to the worked example.
Representations	Concrete	Discussions, especially of worked examples, are completely limited to the abstract. No manipulatives, pictures, or story situations are used.	- Concrete contexts (e.g., story problems) are involved but not utilized sufficiently for teaching the worked example; - Semi-abstract representations such as dots or cubes are used as a context for teaching the worked example	Discussions, especially of worked examples, are well situated in rich concrete contexts (e.g., pictures and story problems). Concrete materials are used to make sense of the target concepts.
	Abstract	Discussions are limited to the concrete and are not at all linked to the abstract representations of the target concept.	- Both concrete and abstract representations are involved but the link between both is lacked; - Since all discussions remain abstract, the link between the concrete and abstract is invisible; - Opposite: from abstract to concrete.	Concrete representations are used to purposefully link the abstract representations to the target concept.
Deep questions	Question	No deep questions are asked when discussing a worked example or guided practices.	Some deep questions are posed to elicit deep explanations/	Deep questions are sufficiently posed to elicit student explanation of the targeted concepts.
	Explanation	- No deep student explanations are elicited. - Teacher provides little or surface explanations.	- A few deep student responses are elicited. However, most of the student explanations still remain at a surface level. - Teacher rephrases students' explanations without promoting to a higher level. - Teacher directly provides deep explanations.	- Deep student explanations are elicited. In particular, these explanations are related to the target concepts. - Teacher rephrases student explanations to make them deep.

This coding framework has been validated through four rounds of revisions. Initially, given that a lesson plan is indeed different from an actual lesson, two authors who were experienced video analysis researchers discussed the needed changes of the original framework. For instance, the original descriptions for the 0-category for worked examples were “no examples is visible” and “no practice problems are provided for students” (Ding & Carlson, 2013). We thought that even though there could be no examples or practice problems in a lesson plan, there will always be some kind of problems to be solved in the actual lessons. As such, we changed these descriptions to “Examples and guided practice cannot be differentiated” and “Practice problems have no connection to the worked examples.” The revised coding framework was further used to code two US videotaped lessons and each round of codes were compared to inform the revision of the framework. For instance, the 2-category for worked examples in the original framework used the words “well-discussed.” This did not allow the two coders to have a consistent interpretation. After discussion, we elaborated it as “worked examples are sufficiently discussed with the underlying idea made explicit.” However, after another round of coding, one coder pointed out that when the underlying idea was stressed, some participants would lose credits twice for the same reason because the underlying idea was also emphasized in deep questions. Due to this consideration, we took off the requirement of making the underlying idea explicit for worked examples. In other words, as long as a teacher made sufficient discussion (e.g., 10 minutes) of a worked example, we would consider that the teacher grasped the worked example effect. In addition, the pilot coding suggested adding a new bullet to 1-category of student explanation, which initially referred to student explanations. However, we found that some teachers directly provided deep explanations by themselves. This is a better situation than “no deep explanations or teachers provide little or surface explanation” (0-category) but worse than “students provided deep explanations or a teacher rephrased student explanation to make it deep” (2-category). As such, we agreed to give a 1 point credit to this case. After finalizing the coding framework, we put it into use for coding the larger amount of videotaped lessons.

Procedures and Data Analysis

All 32 videos were transcribed. The first author read all of the transcripts and commented on teachers’ use of worked examples, representations, and deep questions for a general sense. Typical screen shots were saved for each teacher’s classroom instruction. To obtain a more accurate measure, the finalized coding framework was used to score each subcategory for each lesson, resulting in 32 scored sheets. Reliability was ensured. The second and third authors who were familiar with the codes selected about 20% of the lessons (n=6) and independently conducted coding and compared their results with the first author. Among the 84 codes, 8 were different, resulting in a reliability of 90.5%.

The resulting 32 coding sheets that were further compiled using an excel spreadsheet. Pattern of teacher’s use of worked examples, representations, and deep questions across eight teachers were illustrated. This revealed both successes and challenges in classroom instruction. Most importantly, this round of coding also revealed a few key aspects that matter in teachers’ use of worked examples, representations, and deep questions.

To obtain an enriched understanding of the identified key aspects, the authors went back to all transcripts (and videos when needed) to systematically inspect teachers’ classroom instruction with a focus on quantifying certain features and identifying typical teaching episodes. Using an excel spreadsheet, the authors recorded the total length of each lesson, the number of worked examples, the length of each worked example, and the sequence between worked example and practice problems. Using a separate excel spreadsheet, the authors documented the types of concrete and semi-concrete representations used during instruction. Those that were suggested by students were highlighted as were the episodes of when connections were and were not made between concrete and abstract representations. Finally, using another separate excel spreadsheet, the authors documented teachers' typical "deep questions," as well as instances where deep questions were obviously missed. After the excel spreadsheets were completed for all 32 lessons, the authors computed the average proportion of worked examples and analyzed typical sequences between worked examples and practice problems. Features and purposes of representation uses and connection making between concrete and abstract were also identified. Finally, typical ways to ask deep questions to elicit students' deep understanding were noticed. These findings have enabled an identification of teachers' AKT based on the existing classroom data.

Results

An Overview of Expert Teachers' AKT

Figure 1 illustrates each teacher's average AKT scores for each category. Among the eight teachers, T1 (G1 teacher) and T7 (G4 teacher) performed the lowest (M_{T1} = 6.5, M_{T7} = 6.75). T2's score (G1 teacher) was also relatively low. The other five teachers performed much better with T5 (G3 teacher) receiving the highest score (M_{T5} = 11.5).

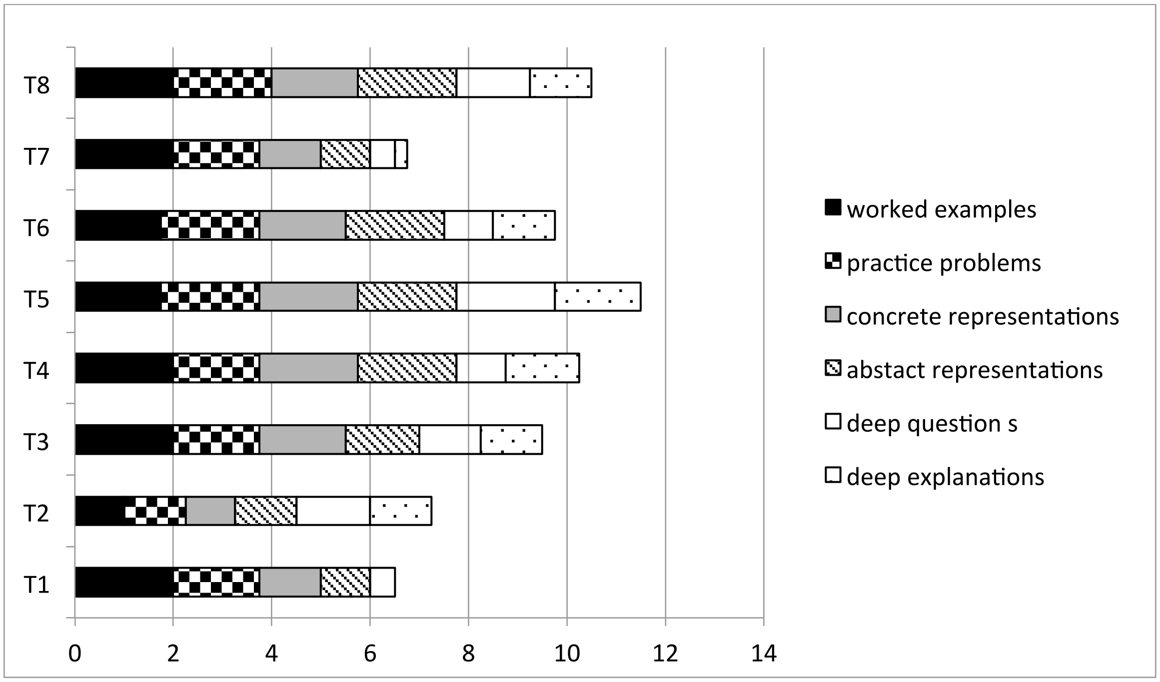

Figure 1. Teacher's average AKT scores for each category.

A closer inspection of the Figure 1 shows that T1 and T7 indeed performed well in using worked examples and practice problems. However, their teaching demonstrated challenges in representation uses and in asking deep questions, which seemed to separate them from the other teachers. Indeed, this finding seemed to be consistent with the overall analysis. When comparing teachers' use of worked examples, representations, and deep questions across participants (see Figure 2), it was found that teachers performed best in using worked examples and arranging relevant practice problems; yet they performed worst in asking deep questions to elicit deep explanations.

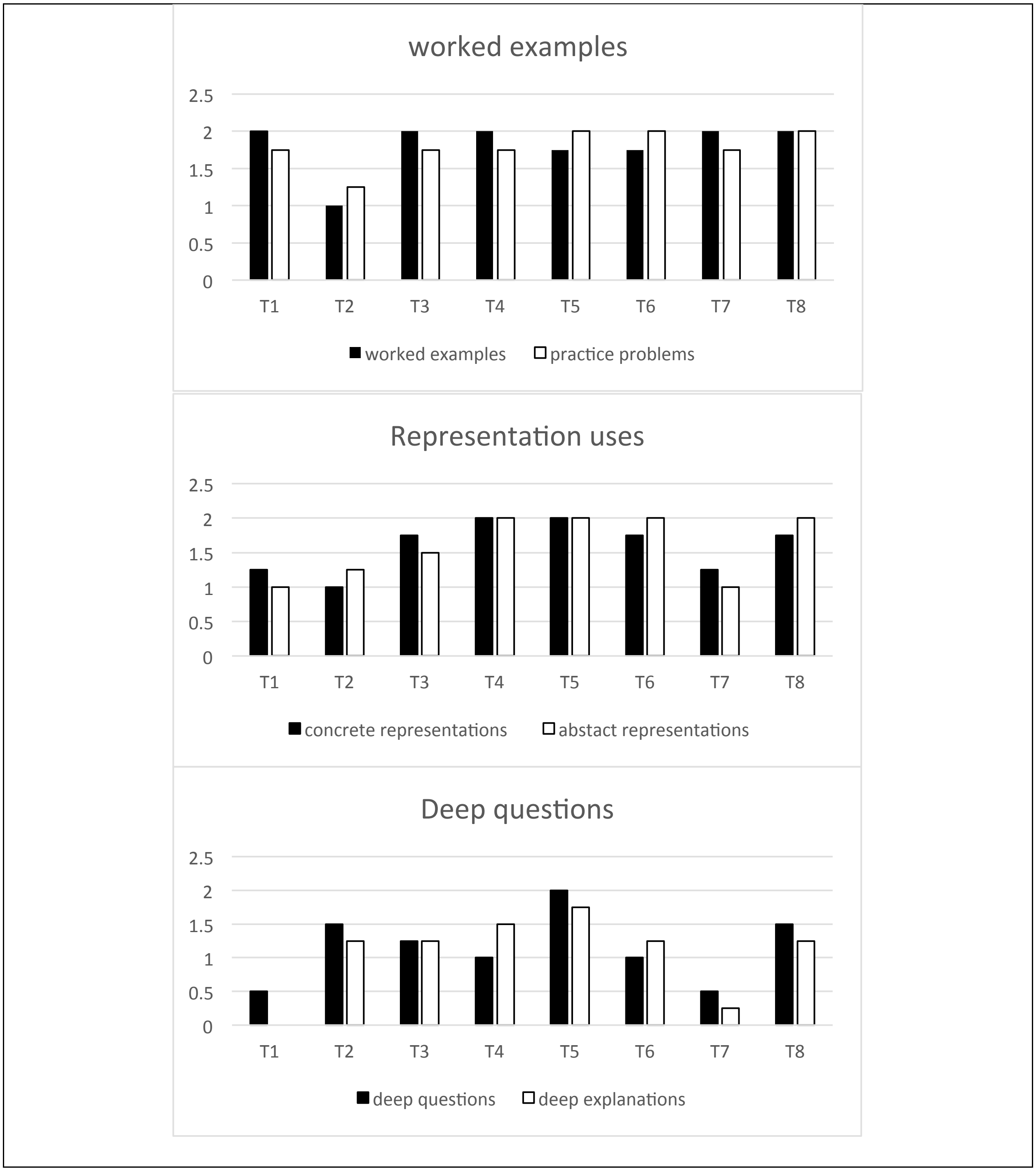

Figure 2. Teachers' AKT scores for worked examples, representations, and deep questions.

The Use of Worked Examples

Seven out of eight teachers (87.5%) spent more than 10 minutes on worked examples before students' own practices. Even for those teachers who used inquiry-based textbook (e.g., *Investigations*), teachers in this study were still able to set up plenty of time for worked example before students' own exploration. This finding shows that expert teachers generally embraced the use of worked examples, which is different from other classrooms where minimum guidance is provided (Kirschner, Sweller, & Clark, 2006).

However, a close analysis of teachers' worked examples indicates differences. For instance, while T5 used 14 minutes to discuss one worked example, T1 used the same amount of time covering three examples, resulting in an average of only 4-5 minutes per example. As such, T1's worked examples were presented in a rather quick manner. This may result from a common misconception that the more examples, the better (Ding & Carlson, 2013). In fact, all lessons involved multiple worked examples ranging from 2-6 with 3 being most common. In most of the classrooms, the nature of the worked examples was often reparative.

With regard to the sequence from worked examples and practice problems, we found that except for T1 who interweaved practice problems during her second lessons, the rest of the teachers presented multiple examples (e.g., three examples together) prior to practice problems. This is different from the literature assertion about interweaving worked examples and practice problems. This may also be attributed to the common belief that if students don't get the idea, they should see another example until they eventually get the idea (Ding & Carlson, 2013). Given that all US teachers discussed multiple examples, the difference between higher and lower scores in worked example seems to be associated with whether at least one of the worked examples has been sufficiently unpacked. This indeed relates to teachers' knowledge and skills needed for representations and deep question as elaborated in the next sections.

The Use of Representations

In all of the eight classrooms, multiple representations were used including both concrete and abstract. These representations were either suggested by students or provided by the teacher. These concrete representations included using fingers, cubes, dominos, base-ten blocks, fact triangles, ten-frame sheets, number grid, number lines, tape diagrams, pictures, or story problems. Typically, there were about 3-4 different types of representations involved in each lesson. Students in all classrooms were allowed to choose own manipulatives (e.g., cubes, blocks, fingers) when solving inverse relations problems and T3 even asked students to use their favorite animals to create story problems based on a given model. With these concrete aids, some students came up with creative strategies. Figure 3(a)(c)(d) shows a variety of representations suggested by the teachers for solving word problems. Figure 3(b) shows students' generated representations in T8's class. Regardless of the multiple solutions, it seems that there were two aspects of representation use that separated high from low quality instruction: (a) purpose of representation uses and (b) connection-making between representations.

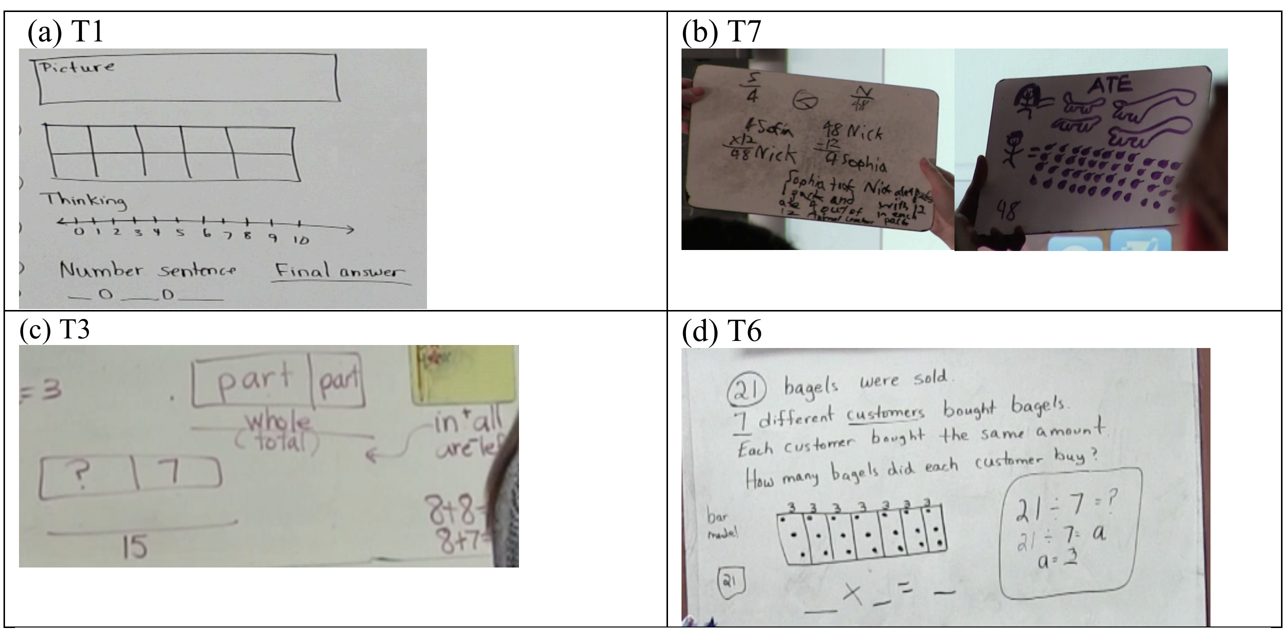

Figure 3. Multiple representations in the US classrooms

Purpose of representation uses. Teachers in this study seemed to use representations for finding the answers rather than for modeling the quantitative relationships. In Figures 3(a), 3(b), the purposes of T1 and T7's representations served as a means to find the answer. For instance, T1 taught first graders to solve the following problem, "Jim held 7 balloons in his hand. Some of them flew away. Now he has only 2 balloons. How many balloons flew away?" In particular, she taught students to use a picture, ten-frame, number line, and number sentence to solve this problem. After the answer of "5" was obtained, the teacher then guided students to check for their answers using each representation, "Count with me, 1, 2, 3, 4, 5. This gives me 5… This gives me 5. This also gives me 5. So, the answer is 5."

In contrast, the purpose of T3 and T6's representations in Figure 3 appeared to be different. Both teachers used the "bar model" which clearly illustrated structural relationships. For instance, T3 drew the bar diagram to represent the part-part-whole relationships embedded in two word problems that were inverses to each other. Excerpt 1 below shows T3's instructional focus.

Excerpt 1:

T: Our bar model is a way for us to think of parts and another part and how those equal one whole (draws a bar and labeled it with "part"). Sometimes we call the whole "the total." Or we say "in all." And we are going to be using that today to understand how addition and subtraction are related. …So I'm going to tell you a story about the bar model that's at the top of page 137... *A soccer team has 8 red balls and 7 yellow balls. How many soccer balls does the team have?* Well in our bar model we see 8 and 7. I want to know how many balls the soccer team has all together.

In the above episode, T3 clearly stated that her purpose of using the diagrams was to help students learn inverse relationships. In discussing this addition story problem, the teachers' attention was placed on the part-part-whole relationships rather than on finding the answer. This was confirmed by the teachers' action of reminding the class to use connected cubes to find the answer for 8+7 if needed. After the discussion of this addition story problem, T3 presented a related subtraction story problem (taking away 7 yellow balls from 15 soccer balls), which was modeled by another bar model (see Figure 3c). After this pair of word problems was solved, T3 furthered inverse reasoning by asking students to compare the similarities and differences between the two diagrams. Likewise, T6 used bar diagrams to help students understand the process of distributing 21 bagels evenly within 7 customers (see Figure 3d), which led to the corresponding numerical solutions.

Note that the "bar model" is a new type of schematic diagram recently emphasized by the CCSS and adopted by the *Go Math* textbook. As such, this model was quite new for US teachers (and students) who may be uncomfortable with it. As indicated by our videotaped lessons, among the four teachers (T1, T3, T6, T7) who used Go Math, only half (T3 and T6) used this model in the classroom and both teachers' representations show room for improvement. As shown by Excerpt 1, T3 directly drew this diagram for students and matched the quantities and the diagram for students. Students in this class seemed to passively receive the instruction from T3. Given that schematic diagrams are less transparent for students, it was unclear to what extent T3's students have understood the diagrams. With regard to T6's use of bar diagrams, in addition to illustrating the fair sharing process, she also used it as a way to find the answer of 3 (see Figure 3d). In addition, even though division sentences were generated from this bar diagram, she asked students to directly comp up with the multiplication sentences based on the division sentences. T6 could have sufficiently used the bar diagram to generate the multiplication sentences so as to enable them to visualize the inverse relation.

Connection-making between representations. Making connections between concrete and abstract is critical (Pashler et al., 2007). In this study, connection-making in different classrooms reached different degrees. Figure 4 shows examples at different levels ranging from low to high.

(a)T1 (b) T2 (c) T5

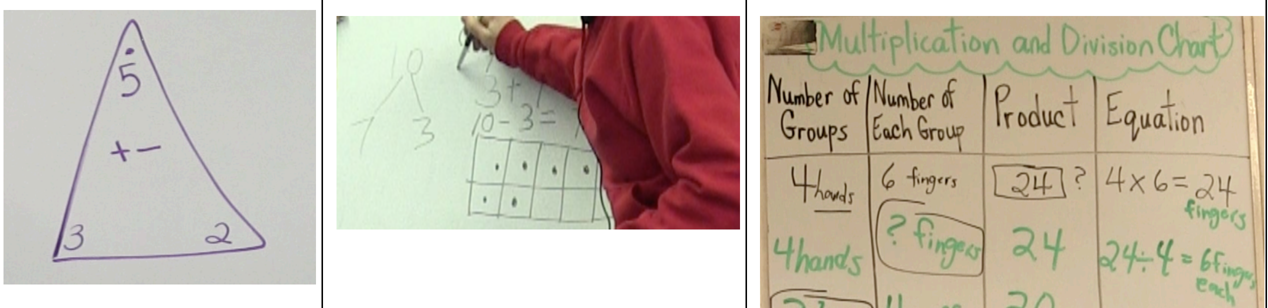

Figure 4. Connection-making between representations in different classrooms

In Figure 4a, T1 introduced a fact triangle for students to learn fact family (3+2=5, 2+3=5, 5-3=2, and 5-2=3). Similar to teachers' over-reliance on concrete representations reported in last section (e.g., encouraging finger and cube using to find answers), there was over-reliance on abstract representations. Excerpt 2 below is from T1's classroom, which shows typical class discussions on number manipulation.

Excerpt 2:

T: (Make a large fact family triangle as shown in Figure 4a).

T: What goes to the top?

S: The biggest number

T: The biggest number. If I have these numbers, 3, 2, and 5. Which would go to the top? Which one goes to the top? Which is the biggest numbers? I know a lot of you already did that? What goes to the top?

(Students were writing a fact family on the slate. Teacher walked around and corrected student mistakes. Next, T1 checked the answers by asking four individual who suggested the fact family. T reminded that all they did was to use the same numbers)

T: you've already know your fact families. Now let's play a game

In the above excerpt, T1 emphasized the location of the numbers on the fact triangle. The generated fact family was based on abstract number manipulation without any connection-making to concrete contexts.

A higher level of connect-making was indicated by Figure 4b. T2 in this lesson used a ten-frame. Based on this, T2 guided students to understand number composition leading toward a fact family (3+7=10, 7+3=10, 10-3=7, and 10-7=3). Clearly, connections were made between semi-concrete and abstract representations. However, if one checks the corresponding textbook, the lesson clearly suggested two inverse word problems that may be used to generate the set of number sentences, which can then be linked to the fact triangle.

> Lyle and Isabel were in charge of collecting pencils during clean up time. Lyle found 7 pencils and Isabel found 3. How many pencils did they collect?
>
> Isabel and Lyle put the 10 pencils in a pencil basket. Then Diego came by and took 3 of them for the students at his table. How many pencils were left in the basket?
>
> After posing the second problem, ask students to consider how the problem they have already solved might help them solve the new problem. Keep in mind that for many first graders, the second problem will seem like a new, unrelated problem. However, some students will connect 10 − 3 = ? (problem 2) to 7 + 3 = 10 (problem 1).[1]

Figure 5. Textbook suggestions for T2's lesson

The highest level of connection-making illustrated the research assertion of concreteness fading (Goldstone & Son, 2005). The aforementioned bar diagrams in T3 and T6's lessons offered a sense of this feature (fading from story problem situation to bar diagrams to corresponding number sentences). In the same vein, T5 made these connections by using an organization chart to help students understand and compare the structure between multiplication and division story problems (Figure 4c).

The Use of Deep Questions

Figure 1 indicates that few teachers asked deep questions. For many teachers, instead of asking questions (regardless of deep or not), they directly provide students with deep explanations, which likely deprives students' reasoning opportunities. For those teachers who did ask questions, two aspects differentiated their quality of instruction: (a) whether a teacher stressed the meaning, and (b) whether a teacher stressed the quantitative relationships.

Stressing the meaning. In this study, teachers' deep questions first related to the meaning of operations. For instance, when teaching multiplication, T5 asked, "What makes you to think that it is a multiplication story problem?" This question requested students' explanations of the meaning of multiplication. T5 also reminded students, "Think about it. 4 hands, is it a group or a group of things?" This reminder oriented students' understanding toward the equal-group meaning. More interestingly, when discuss the following word problem - *DJ picks 7 apples. Teacher Kelly picked 4 times as many apples. How many apples did Teacher Kelly pick?* - T8 in Excerpt 4 focused the word of "times" and asked a series of questions to help students understand this concept:

Excerpt 4:

T7 – what does "times" mean?

S1: Uh, it means that when she has 7 more but 4 times

T: What do you mean, she has 7 more but 4 times?

S1: Like, she has 7 more and then she has another 7 more, so its like…..

T: Call on someone to help you out to clarify your thinking.

S2: Can I give an example?

T: Please.

S2: Do you see how you have 7 apples?

T: I do see I have 7 apples. That's my favorite number.

S2: You just add on 4 more like. They saying like you're adding on 4 more bags of 7 apples.

T: Well, what does it mean that I have to add on 4 more bags of 7 apples?

S2: Cause it says 4 times

T: Okay, because it says 4 times, but why do the bags have to have 7?

S2: Because of the number that you already have, that's like the …

T: Ah, because of the number I have already. I picked 7 and then we said that teacher Kelly picked 4 times as many apples I did. You can't just pick a number out of the sky and say that I'm going to do 4 times 27. Okay, because it's 4 times as many apples as I already picked. So she has to have 4 groups, with that same number inside of it. So now, who can tell us an equation that can represent how many apples teacher Kelly picked?

In the above conversation, T8 asked a deep question, "What does "times" mean?," which has elicited students' self-explanations. With continuous guidance, students were able to reach a deep understanding that "4 times" meant 4 equal groups of the reference number. T8'own rephrasing of students' explanations have made the meaning explicit to all students.

Unfortunately, there were teachers (T1, T7) who frequently asked students to look for key words when solving word problems. For instance, T7 reminded students, "Here, look at the wording, look at the wording. What operation might use? The wording should help you figure out." Even though looking for key words has been criticized for a long time (Stigler & Hiebert, 1999), it appears to still occur in current classrooms. Occasionally, T3 also guided students to look for the key words as evident by the statement "In all told us to add and "How many are left" told us to subtract."

Stressing the quantitative relationships. Deep questions in this study also targeted relationships, which were often through using comparison techniques. For example, T3 in her lesson 2 summarized the connections between addition and subtraction as the following:

> So let's think about the relationship between these two facts. They're both talking about parts and a total, but when we add, we are taking two parts, maybe 6 red cookies and 4 green cookies and putting them together, (puts counters together) to find our total number of cookies; when we subtract we are starting with all of our cookies and then maybe we are taking some away (takes away the 6 red counters).

In the above discussion, T3 compared the meaning of addition and subtraction based on the part-part-whole structure through the contexts of red and green cookies illustrated with counters.

T5 also frequently asked comparison questions to relate multiplication and division. When teaching division story problems, T5 constantly asked "what are the difference and similarities between the new example and what we learned last week?" In addition, T5 tended to ask students to create a "reverse problem" to a given word problem. Excerpt 5 shows how T5 asked students to change a multiplication story (Each robot has 4 hands. How many hands do 6 robots have?) into the corresponding division story problems.

Excerpt 5:

T: (After the multiplication problem was discussed) How may I change this question? Challenging you: See if you can work with your partner if you can change this problem to a division problem. Who can come up with the reverse?

(Students are working on this; Students have difficulties to change this problem by themselves).

T: What is the product? What does 24 refer to? 24 what? Robots? Hands?

S: There are 24 fingers from a robot. This robot has 4 hands.

T: Is 4 hands groups or number in each group?

In Excerpt 5, T5 not only requested an inverse word problem from students, but she also provided guidance based on a structural view. Later on, when this pair of story problems were created, T5 asked the class to compare the problem structures between multiplication and division and recorded the comparison on a chart (see Figure 2d).

In contrast with T3 and T5's deep comparison questions, opportunities to target quantitative relationships were missed in many classrooms. For example, even though many teachers encouraged multiple solutions, they often did not ask comparison questions to elicit students' understanding of the underlying structure inherent in these solutions.

Discussion

The issue of US students' algebraic readiness has received increasing attention, marked by its recent emphasis on fundamental mathematical ideas like inverse relations in elementary school (CCSSI, 2010). However, it is unclear how teachers may stress these ideas to facilitate students' algebraic readiness. Our findings based on US expert teachers' videotaped lessons shed light on the needed AKT components to support students' algebraic thinking.

In terms of worked examples, it is most important for teachers to unpack a worked example into its subcomponents. Simply showing one or even many worked examples may not be helpful for developing students' necessary schema for problem solving (Sweller & Cooper, 1985). As such, in a short class period (40 minutes – 1 hour), it might not be feasible to ask teachers to interweave worked examples and practice problems (Pashler et al., 2007). Otherwise, when examples and practice problems were interweaved in a quick pace, discussion time on each worked example will be sacrificed, possibly reducing the quality of classroom instruction.

With regard to how to unpack a worked example to support learning, we found that representation uses and deep questions play a key role. In this study, concrete representations were often used as a tool for finding answers, which is consistent with Cai's (2005) findings on teachers' views of representation uses. Without shifting teachers' views of the purpose of representations, discussions of worked examples cannot go in depth. In contrast, when concrete representations are used to model the quantities relationships and are timely linked to abstract representations, they can potentially maximize the worked example effect (Sweller & Cooper, 1985).

It should be noted that schematic diagrams such as bar diagrams are used by some teachers. This type of schematic diagram has been widely used in East Asian textbooks (Ding & Li, 2014; Murata, 2008) and found to be effective in supporting student learning (Ng & Lee, 2009). However, schematic diagrams were non-transparent for students, which brings challenges to teachers. In fact, this type of diagram is quite new for many US teachers. As seen from teachers' striving for success with the bar diagrams to teach inverse relations in this study, there is a need to help US teachers improve the AKT related to schematic diagram uses. For instance, instead of drawing and explaining the diagram for students, are there any ways to engage students to co-construct such diagrams (Ding & Li, 2014)?

To unpack a worked example, deep questions should be asked to stress the meanings and quantitative relationships. Comparison questions appear to be most powerful and have been constantly used by expert teachers. This echoes the literature where comparison techniques were found to be effective in promoting relational thinking (Rittle-Johnson & Star, 2007). Prior studies found that comparison technique is indeed used in U.S. classrooms but with rare success (Richlan, Zur, & Holyoak, 2007). Future studies need to further explore what, how, and when comparison questions should be used to better support student learning.

The present study is significant in that (a) it is among the first to investigate AKT focusing on inverse relations, (b) the theoretical framework is guided by research-based IES recommendations (Pashler et al., 2007), and (c) this study is based on expert teachers' actual classroom practice. It is expected that findings from this study will inform teacher educators, professional developers, and curriculum designers about existing opportunities to better support teachers with developing students' algebraic readiness. However, we acknowledge the limitations of this study. For instance, all findings were only based on videotapes without linking to actual students performance either within this class or after class. Future studies may design experimental studies to test out how our findings such as schematic diagrams and deep comparison questions can play a role in supporting the learning of worked examples, especially the targeted concepts.

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