Paper Title: Investigating a Student's Reasoning with Ratios: The Case of Gabriel

Author(s): Fetiye Aydeniz

Session Title: Investigating a Student's Reasoning with Ratios: The Case of Gabriel

Session Type: Brief Research Report

Presentation Date: April 13, 2016, 12:30 PM to 1:00 PM

Presentation Location: San Francisco, California

INVESTIGATING A STUDENT'S REASONING WITH RATIOS: THE CASE OF GABRIEL

To investigate how to differentiate instruction and to explore relationships between students' rational number knowledge and algebraic reasoning, an 18-episode design experiment was conducted with 9 cognitively diverse middle school students. As a part of the larger study, I conducted a case study of an eighth grade algebra student, Gabriel, who did not construct a distributive partitioning scheme (DPS) even though he had interiorized three levels of units and constructed advanced fraction schemes. To do so, I focused on five design experiments in which Gabriel worked on ratio problems. Findings indicate that Gabriel was mostly illustrating the results of his way of thinking as opposed to showing how he generated them. He also had some powerful ways of thinking and did not need to show a distributive partitioning operation (DPO) in reasoning with ratios.

Keywords: Distributive partitioning operation, distributive partitioning scheme, reasoning with ratios

Introduction

Proportional reasoning, described as a watershed concept and a cornerstone of higher mathematics, plays a crucial role in students' mathematical development (Lesh, Post, & Behr, 1988). One of the main problems for students in solving a task involving proportions is that they usually do not construct proportional reasoning but rather tend to use cross-multiplication to solve proportions of the A/B=x/D or to use additive strategies requiring a lower level of sophistication in cognitive schemes of operation (Steffe, 1992). However, the students need to show a second-order relationship, the relationship between two relationships, which is the salient characteristic of proportional reasoning (Inhelder & Piaget, 1958). Because proportional reasoning is a broad area, I will narrow the focus in this paper to reasoning with ratios.

Distributive reasoning plays an important role in the development of ratio reasoning (Steffe, Liss, & Lee, 2014). An illustration of this statement would be that one way of thinking about ratios is measuring one quantity in terms of the other. If there are two tbsp. of powder and five oz. of water, the two tbsp. must be distributed across those five oz. to determine the quantity of powder that goes with one ounce. Alternately, the five oz. must be distributed across those two tbsp. to know how much water goes with one tbsp. of powder. Establishing a unit ratio, say 2/5 tbsp. of powder for each ounce of water or 5/2 oz. of water for each tbsp. of powder, involves distributive reasoning (Figure 1). Making these unit ratios and being able to distribute them is a powerful way of establishing equivalent ratios. So, construing the distribution as measuring one quantity in terms of another quantity demonstrates that distributive reasoning is involved in proportional reasoning and reasoning with ratios.

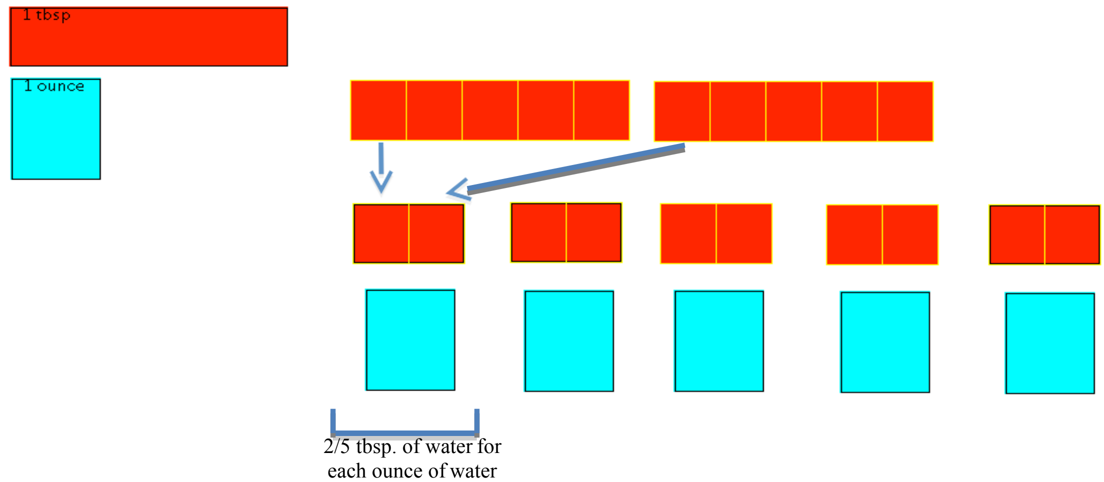

Figure 1: Distributing two tbsp. of powder across five oz. of water to establish a unit ratio

It has been found in prior research that diversity in students' abilities to reason with multiple levels of unit structures and their fractional knowledge influences their ways of distributive reasoning (Hackenberg, 2010; Steffe & Olive, 2010). For example, students operating with the third multiplicative concept (MC3 students, explained subsequently), have the potential, but are not guaranteed, to construct an advanced form of distributive reasoning called a distributive partitioning scheme (DPS, explained subsequently) (Liss, 2014; Steffe & Olive, 2010). The purpose of this paper is to examine the relationships between distributive reasoning and reasoning with ratios. For this reason, I will present a cognitive model (based on schemes of operation) of an MC3 student's mathematical reasoning as he attempted to solve problems involving ratios. Specifically, the research questions for this study are: (1) How does this MC3 student, Gabriel, solve problems that involve ratios? (2) How does the fact that Gabriel does not construct a DPS impact his problem solving with ratios?

Theoretical Framework

A Quantitative Approach

Thompson (2010) defines a quantity as a scheme consisting of an object, a quality of the object, an attribute of this object that has a unit of measure, and a process that the attribute's measure entails a proportional relationship with its unit. An extensive quantitative unknown refers to "the potential result of measuring a fixed but unknown extensive quantity (e.g. height, length, and steepness) before actually measuring it" (Steffe, Liss & Lee, 2014, p.4). Throughout this paper, following Thompson (2010), I interpret quantitative operations as mental operations and regard these operations as essential aspects of quantitative schemes because this approach allows me to account for Gabriel's schemes and operations when he reasoned quantitatively with ratios.

From a quantitative perspective, I view how students reason with ratios in two ways. One way to reason with ratios is by joining two quantities into a *composed unit*, which appears in a students' iterating (repeating) or partitioning (breaking apart into equal-sized sections) a composed unit (e.g., iterating 16 oz. of water and 3 tbsp. of powder two times to conclude that a 32:6 ratio has the same lemony taste as a 16:3 ratio—performing the same operation on both units and making the lemonade mixture twice) (Lobato & Ellis, 2010). Another way to reason with ratios is to create a *multiplicative comparison* of two quantities (e.g., considering that the number of tablespoons of powder is always 3/16 the number of ounces of water, and the number of ounces of water is always 16/3 the number of tablespoons of powder in a mixture comprising 16 oz. of water and 3 tbsp. of powder, which is different from creating two unit ratios) (Lobato & Ellis, 2010).

Schemes and Operations

The concept of schemes, or goal-directed ways of operating that involve an assimilatory mechanism, activity, and result, is a substantial part of Piaget's theory of knowledge (von Glasersfeld, 1995). My study is based on the mathematical thinking in students' operations, which are components of schemes. Based on Steffe and Olive (2010), I define a distributive partitioning operation (DPO) as partitioning *n* items among *m* shares by partitioning each of the *n* items into *m* parts and distributing one part from each of the *n* items to the *m* shares. For example, when asked to equally share three candy bars with five people, a student with a DPO would partition each of the three candy bars into five equal parts and take one part from each bar to determine a share. Steffe and colleagues (2014) also originated the ideas of a *distributive partitioning scheme* (DPS) and a *reversible distributive partitioning scheme* (RPDS). A child who has constructed a DPS could interpret one share as *n/m* of one unit by taking *1/m* of each *n* item, while a child with an RDPS could interpret one share as *1/m* of all the units and see that *n/m* of one unit is equal to *1/m* of all the units, and could also justify that iterating *n/m m* times does indeed produce *n* items (Liss, 2014).

Multiplicative Concepts

Students' multiplicative concepts are based on the interiorized results of their units-coordinating schemes (Steffe, 1992). Students operating with the third multiplicative concept, MC3 students, have interiorized three levels of units (Hackenberg, 2010; Steffe & Olive, 2010). MC3 students can take three levels of units as given and flexibly switch between three-levels-of-units structures. Thus, MC3 students can take as given both the distribution of six 5s and the structure of the result, 30, as a unit of six units, each consisting of five units. In the insertion of five parts into each of the six parts of the 30/30, MC3 students can retain the views of the 30/30 as a unit of six units, each containing five units, and they can switch to viewing the segment as a unit of five units, each containing six units (Figure 2). Additionally, it has been found in prior research that students who have interiorized three levels of units may construct a DPS (Liss, 2014; Steffe & Olive, 2010).

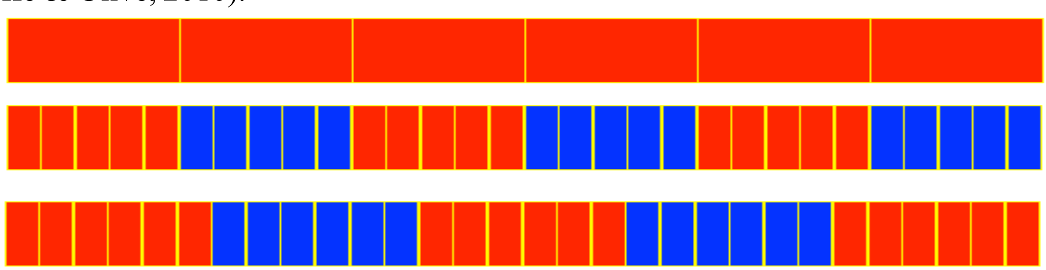

Figure 2: A length structured as a unit of units (top), a unit of six units, each containing five units (middle), or a unit of five units, each containing six units (bottom).

Methods

The larger project, of which this study is a part, has two overarching goals: to investigate ways to differentiate mathematics instruction for cognitively diverse middle school students, and to explore how students' rational number knowledge and algebraic reasoning are related. The data for this paper are drawn from an 18-episode-design experiment. To begin this experiment, 21 students engaged in a selection process (interview and worksheet) to assess students' multiplicative concepts and features of their fractional knowledge. Nine students operating with different multiplicative concepts were invited to participate in the design experiment. Each episode, in which students were engaged in solving particular math problems and participated in small or whole group discussions, lasted one hour and was video-recorded with three cameras. The three videos were subsequently mixed into a single video file. During the episodes, students often worked in groups of two or three at tables. When working in groups, they frequently used a software program called JavaBars (Biddlecomb & Olive, 2000). ScreenFlow software was also used to capture students' computer work and interactions. Each student participated in a follow-up interview at the conclusion of the design experiment.

The findings in this paper are based on data from five episodes in which MC3 student Gabriel worked on problems involving ratios in the contexts of lemonade mixtures and speed, episodes 13-17. Gabriel was an eighth grade student enrolled in an algebra class.

The data were analyzed retrospectively (Steffe & Thompson, 2000). Through retrospective analysis, I formulated a second-order model of Gabriel's reasoning with ratios. A second order model is a researcher's constellation of constructs to describe and account for another person's ways and means of operating (Steffe, von Glasersfeld, Richards, & Cobb, 1983). For this purpose, I repeatedly viewed the relevant video files, transcribed major portions, and took detailed analytic notes (Cobb & Gravemeijer, 2008). I also wrote memos and conjectures about changes and persistent constraints in Gabriel's ways of thinking and operating, and about interactions that may have supported these changes and constraints. Finally I wrote a document comparing these models and synthesizing the interactions that contributed to changes.

Findings

I present sample evidence of how Gabriel viewed constancy of taste in the experiment. First, Gabriel showed evidence that he had created a ratio as a composed unit. In episode 13, students were evaluating a list of mixtures to predict whether they had the same lemony taste as a mixture of 3 tbsp. of lemonade powder with 16 ounces of water, the ingredients of the "best" lemonade. Gabriel justified most of his responses in terms of indicating whether the mixture could be created by iterating or partitioning the 3 tbsp:16 oz. ratio. For example, he said 9 tbsp. of powder with 48 oz. of water has the same lemony flavor as the best lemonade, because 3 x 3 = 9 and 16 x 3 = 48. So, he either iterated the 3 tbsp. to 16 oz. ratio three times to justify the taste of the mixture with 9 tbsp. and 48 oz., or he scaled the quantities in the ratio by 3. In addition, he wrote that 1 ½ tbsp. of powder with 8 oz. of water has the same lemony flavor as 3 tbsp. of powder with 16 oz. of water by writing, "It fits the ratio; 3/2=1 ½ and 16/2=8." Here he operated on the 3 tbsp. to 16 oz. ratio by dividing each quantity by two. His work indicates that he had constructed a ratio as a composed unit because he operated multiplicatively on both quantities, 3 tbsp. of powder and 16 oz. of water, to assess other mixtures.

Gabriel also showed evidence that he had created a ratio as a multiplicative comparison. In episode 14, when Gabriel and his groupmates were trying to test whether ¼ tbsp. of powder with 1 ounce of water has the same lemony flavor as the "best" lemonade in JavaBars, Gabriel said "…3 is not one quarter of 16, but 4 is [4 is one quarter of 16] and ¼ of ¼ is 1 ounce. So that is a different ratio." There is evidence that Gabriel saw this ratio as a multiplicative comparison because he was aware that no matter how much water he put in, ¼ of that amount would always be the amount of powder that he needed to maintain the ratio. So for him, a ratio of ¼ to 1 meant that the amount of powder needed to be ¼ of the amount of water. Gabriel knew which equality had to remain constant for proportionality to hold. He also viewed the relationship as being constant regardless of the variation of the amount of powder and water.

In episode 15, the teacher had already created a sketch showing 3 tbsp. and 16 oz. bars in JavaBars (Figure 3, upper left). The main question students were asked was "How much water goes with exactly 1 tbsp. of powder?" When discussing this question with his group, Gabriel copied the 1 tbsp. bar one time and the 1 oz. bar six times. He joined five 1 oz. bars together. Then, he partitioned the remaining 1 oz. bar into three parts, broke them apart, pulled out one part, and joined that part to the five 1 oz. bars. Finally, he labeled that last bar as 5 1/3 oz. (Figure 3). Thus, while his way does not show evidence of a DPO, it does not necessarily contraindicate it, because he demonstrated a natural way to solve this problem. If someone was asked how much water goes with 1tbsp. of powder, it is quite unlikely that he would partition each of the ounces into thirds and share each of them among the three tbsp. bars. Gabriel shared 15 ounces equally among the three 1 tbsp. bars without having to do any partitioning. But there was a sense of distributing when he was partitioning the leftover ounce into thirds and give a part to each 1 tbsp. bar. Gabriel's way of thinking implies that he was distributing at least mentally but he did not show that process in the microworld. He might have engaged in a DPO, but we do not have any evidence of it here.

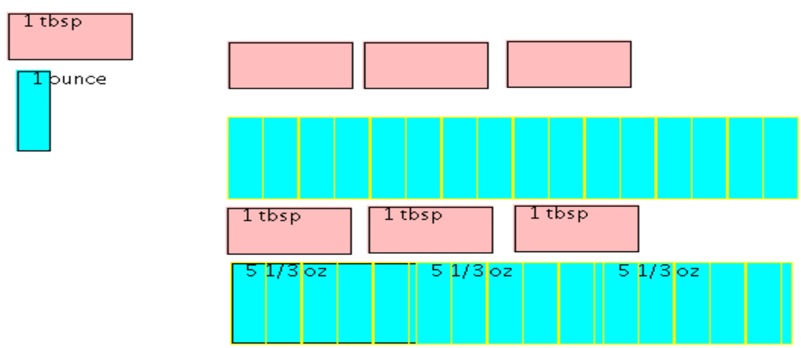

Figure 3: Gabriel's work on the problem "How much water goes with exactly 1 tbsp. of powder?"

I also conjecture that when he made his JavaBars picture, Gabriel already knew (e.g. He said "I literally know already") 16 divided by 3 is 5 1/3 because he had mentally calculated it. Therefore, he did not need to use a DPO here. When his groupmates started to move on to a new problem without agreeing on how much water goes with 1 tbsp. of powder, Gabriel said, "I am going to prove to them." He reopened the file that he had saved. He copied 5 1/3 oz. labeled bars two times and joined them (Figure 3, bottom bar). He also copied one 1 tbsp. bar two times without joining them. He moved those bars right under the 3 tbsp. and 16 oz. bars. He showed that those two pairs of bars were equal by iterating the quantities in the new ratio, which resulted in the same quantities as in the original ratio. Although I did not see Gabriel showing all the operations of a DPS, this justification could be evidence of an RDPS because he showed an equivalent ratio by iterating 5 1/3 three times to get the original bar. This way of iterating one quantity to get the whole is also definitely evidence that Gabriel was using a composed unit.

To test whether Gabriel might show evidence of implementing a DPO, I looked at what he did with the question "How much powder goes with 4 oz. of water?" because one would be more likely to use a DPO in this problem. In episode 15, Gabriel pulled out a 1 oz. part from the 16 oz. bar, repeated the part he pulled out three times, joined the four parts, and labeled the resulting bar as 4 oz. When the teacher asked him how many 4 oz. were in the 16 oz., he replied "four." Then, he led one groupmate to copy the 1 tbsp. bar, to break it into four parts, and then to pull out three parts from it (Figure 4, bottom bar). Gabriel said ".75, three quarters. I am talking about three quarters of one of these [pointing the 1 tbsp. bar]."

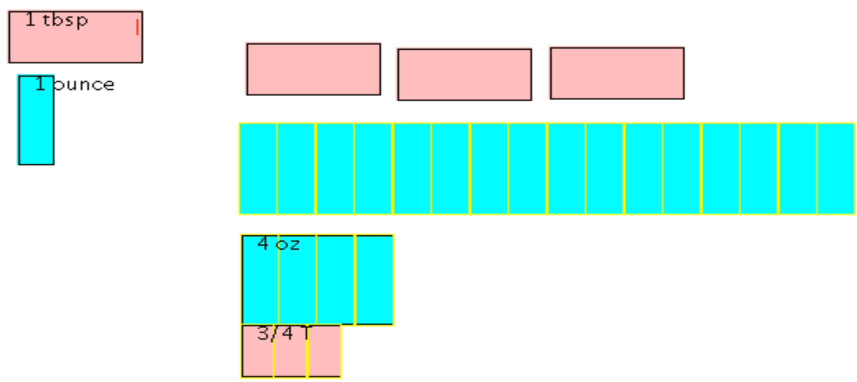

Figure 4: Gabriel and his group's work on the problem "How much powder goes with 4 oz. of water?"

I conjecture that because he already knew that 3 divided by 4 is .75, he did not need to show a DPO by partitioning each 1 tbsp. bar into four parts and pulling out one part from each. So, he was mostly illustrating the results that he had calculated or already knew. In this situation, he could at least figure out what ¼ of 3 is even if he was not thinking about how to get ¼ of the 3 tbsp. because the numbers were easy enough for him to calculate. The process might also have been more or less intuitive for Gabriel at that point. Therefore, he did not necessarily have a reason to use a DPO because he could figure out the answer mentally. That interpretation is strengthened by observation of how Gabriel worked in another situation where he could not compute easily. In an analogous problem given in episode 15, students were thinking about the coordination of distance and time—Rabbit travels 20 cm in 56 seconds. Turtle travels 5 cm at the same speed as Rabbit. How many seconds will he travel? Gabriel and his groupmates agreed on 14 seconds by dividing 56 by 4. Then they were asked to determine the number of cm Turtle will travel at the same speed in 1 second. In the latter part of this problem, there was a potential reason for Gabriel to use a DPO in a setting with 5 and 14 because he could not calculate the answer mentally. When the teacher asked how much distance goes with 1 second, Gabriel's response was "I am going to take a guess and say 1/3 of the cm", which was not the exact answer but was a good estimate. Although he did not know what exactly 1/14 of 5 is, he did not use a DPO but he made an estimate without operating to make the quantities for his ratio precise. Therefore, these two similar situations provide evidence that he was intuitively finding the answer or calculating without developing a warrant for why ¼ of 3 is ¾ and 1/14 of 5 is 5/14. In other words, he did not show that he had a way of thinking about how he could figure out how much powder goes with 4 oz. or how much cm goes with 1 sec.

In episode 16, students discussed how much powder goes with 4 oz. of water as a whole class. Although several students had ideas of how to draw it in JavaBars, Martin, an MC3 student, was the only one who justified the picture by suggesting the idea of taking a fourth of each tbsp. bar and iterating three ¼-bars four times to show equal ratios. Although Gabriel had the same justification method in his response to the question "How much water goes with 1 tbsp. of powder" by iterating the 5 1/3-bars three times, he did not contribute to the whole class discussion at that point. However, when the class was discussing how much powder goes with 1 oz. of water, Gabriel expressed some insightful ideas. Some students suggested dividing the 3 tbsp. bars into 16. When the teacher asked how they could divide 3 tbsp. bars into 16 or take 1/16 of it, Gabriel said, "You divide all them up [3 tbsp. bars] into 16ths because it is the same logic that Martin used with 3/4 tbsp. for 4 oz. It is the same logic, just different amounts." His response was evidence of the use of a DPO, even though he did not demonstrate a DPO in his small group work. So, Martin's ideas based on his DPO were sensible to Gabriel, although Gabriel did not necessarily produce the ideas independently. It appears that Gabriel found Martin's way of thinking valid and useful, but he himself did not need to use it in operating in general.

Conclusion and Discussion

The design experiment data presented above provides several important results. In the first place, Gabriel was mostly illustrating the results that he had calculated or already knew, so he did not use operating on the quantities to generate the results. Rather, he preferred doing calculations first and then making a picture rather than showing his reasoning with a picture first and then figuring out the answer. However, generating the result is more powerful than calculating it because it can help students work with the representations of quantities and develop a way of thinking that they can use in other situations. This may be a common issue with students like Gabriel, who are very good at quickly making mental calculations when the numbers are small or easy enough to calculate. This facility would be the main reason for Gabriel not to show a DPO.

Secondly, Gabriel had powerful ways of thinking and a desire to understand other people's reasoning. He could interpret others' ideas although he usually did not make them his own. He did not spontaneously produce these ideas when he was working on problems even though he could understand and justify them in interactions with other people in the class. Thus, while it was possible for Gabriel to take on others' ways of thinking that he recognized as useful and powerful, he did not necessarily do so, perhaps because he did not see them as more powerful than his own. Therefore, having his own powerful ways and not adopting others' ideas as his own could also help us understand why Gabriel did not need to show use of a DPO in reasoning with ratios. This conjecture suggests a further question to investigate: "Why didn't Gabriel take on others' ways of thinking even though he could understand and interpret them?"

In sum, the above descriptions and analysis are intended as a step in describing the specific roles of particular schemes and operations in MC3 students' reasoning with ratios. However, the case of Gabriel tells us that students could have powerful ways of thinking and might not need to use these important schemes and operations for reasoning with ratios. This observation might motivate us to further investigate what prompts MC3 students similar to Gabriel to use their own ways of thinking instead of using particular schemes and operations. Such exploration would help us discover new insights into the thinking of students who have interiorized three levels of units but do not demonstrate evidence of distributive reasoning.

References

Biddlecomb, B., & Olive, J. (2000). JavaBars [Computer software]. Retrieved June 4, 2002 from http://jwilson.coe.uga.edu/olive/welcome.html

Cobb, P., & Gravemeijer, K. (2008). Experimenting to support and understand learning processes. *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching*, 68- 95.

Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction, 28*(4), 383–432.

Inhelder, B., & Piaget, J. (1958). In: A. Parsons, & S. Milgram (Trans.), *The growth of logical thinking* (5th ed.). USA: Basic Books, Inc.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In: M. Behr, & J. Hiebert (Eds.), *Number concepts & operations for the middle grades* (pp. 93–118). Hillsdale, NJ: Lawrence Erlbaum Associates.

Liss, D. R. (2014). The development of distributive partitioning operations. Paper presented at the symposium of National Council of Teachers Mathematics.

Lobato, J., & Ellis, A. B. (2010). Developing the essential understandings of ratios, proportions, and proportional reasoning: Grade 6- 8. In R. M. Zbiek (Series Ed.), *Essential understandings*. Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259- 309.

Steffe, L. P., Liss II, D. R., & Lee, H. Y. (2014). On the operations that generate intensive quantity. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebraic students* (Vol. 4). Laramie, WY: University of Wyoming Press.

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267– 307). Hillsdale, NJ: Erlbaum.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application.* New York: Praeger.

Thompson, P. (1994). The development of the concept of speed and its relationship to concepts of rate. In: G. Harel, & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179–234). Albany, NY: State University of New York Press.

Thompson, P. (2010). Quantitative reasoning and mathematical modeling. In: Chamberlin S. A. & Hatfield L. L. (eds.) New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOMe. WISDOMe Monograph Volume 1. College of Education, University of Wyoming, Laramie WY: 33- 57.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning (Vol. 6).* London: Falmer.