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Reasoning Paths From Exploration to Argument

Abstract: We present a framework for types of reasoning that can be used when generating and arguing for a general mathematical claim. The framework was used to analyze reasoning elicited in task-based interviews with 8th grade students. We illustrate with examples of student reasoning and arguments for the audience to analyze.

Audience Objectives:

1. Learn about a framework for types of reasoning people can use when generating and arguing for a general mathematical claim.
2. Learn about reasoning paths students used when generating and arguing for a general mathematical claim. These will be illustrated with examples from student interviews.
3. Understand the potential of this framework for teachers' use in understanding and supporting student reasoning and creation of viable arguments (CCSSM MP3).

Key Words
Reasoning and proof; Student Learning

As teachers pursue Common Core Practice 3, to have students construct viable arguments and critique others' reasoning (CCSSI, 2010), there is an increased need for theoretically grounded tools for helping students working to craft a viable argument without compromising their reasoning opportunities. To support student reasoning, teachers need to be able to identify the type of reasoning a student is employing, be aware of and recognize multiple different reasoning paths, and make judicious moves to either support the student in continuing on a particular reasoning path or redirect the student to a different and more productive path. In this brief research report, we present two components of a larger framework we have developed to identify the types of reasoning a student may use while striving to create a viable argument. This framework can be used to analyze a student's work toward an argument to detect conceptions of components for viable argument. We illustrate our framework with reasoning examples from two middle grades students.

Theoretical Background

We look for three types of signposts that indicate a person is using a particular kind of reasoning: an *inference* of the appropriate kind, or the *mental activity* a person is engaged in (regardless of whether an inference is produced), or a *goal* the person may be pursuing. We have a more general framework that includes various types of reasoning which can occur in various combinations and orders while a person develops and/or creates an argument for a general mathematical claim; here we focus just on two of those types: abductive and empirical reasoning. Figure 1 illustrates features of empirical and abductive acts and goals.

Figure 1. Conception of Viable Argument for Generalization

Activity*		**Goal***	
Empirical Acts	Abductive Acts	Empirical Goal	Abductive Goal
• Testing examples • Failing to look for a way to connect conditions and conclusion	• Looking for structure** in the conditions [what do we know about what we were given] • Can work from conditions toward conclusion or from conclusion toward conditions	• Looking for confirming examples • Testing examples is **not** exhaustive	• Looking for pertinent math structure** that connects the conditions to the conclusion

*Our larger framework includes additional types of activities and goals
** Structure includes results, objects, definitions, conceptions, algebra

Empirical Reasoning

A person with an *empirical goal* is trying to gather more evidence to gain certainty for or test a claim that has been conjectured. We consider an *empirical inference* to be a meta-level statement about a general mathematical claim, of the form, “This claim is true because I checked it with several examples.” Certainty about the claim is based on checking a subset of the cases in the claim’s domain, not on logical necessity. *Empirical acts* consist of testing one or more examples to confirm or refute the truth of the claim. Empirical activity is also indicated by failure to seek a structural way to connect the conditions and the conclusion of the claim.

Abductive Reasoning

In the context we focus on, a person with an *abductive goal* seeks to develop an analytic argument for a general claim (Pedemonte, 2007). An analytic argument for a general claim must attend to the claim’s generality and entail how the claim’s conclusion follows from the claim’s conditions as a logical necessity (Harel & Sowder, 1998). An analytical argument can attend to the claim’s generality by appealing to a feature of the objects in the claim’s condition that is structural and hence shared by all such objects. An *abductive inference* is made when a person has a general claim and proposes a general rule that could plausibly form a logical link somewhere between the conditions and conclusion of the claim in an analytic argument for the claim. The general rule being sought can be a known or discovered (a) transformation of a representation involved that can connect the conditions to the conclusion or (b) structural property that connects objects in the conditions to objects in the conclusion, or some intermediate object. This link could be used at any point in the argument. We consider two types of *abductive acts*: (i) searching for a general rule that will form a plausible link somewhere between the conditions and conclusion in an analytic argument for a general claim, and (ii) specifying the “case” of this general rule for the claim in question. These acts involve looking for structural links between the properties that define the objects in the conditions and conclusion, or some intermediate object, which make a rule applicable.

Research Question: For tasks that entail generalizing and justifying, what general kinds of reasoning paths do students pursue?

Methodology and Data Analysis

Task-based interviews requiring justifications were conducted with 8th grade students whose teachers were participating in the NSF-funded Learning Algebra and Methods of Proof (LAMP) project to develop students' methods of proof. Students were given the task "A student in your class claims that the sum of any three consecutive counting numbers is divisible by 3. Develop a viable argument for or against the student's claim. Explain why your argument is viable." We were interested to see if students would approach the creation of an argument for the claim based on empirical examples or via a search for a pattern or structure they could use in linking the condition and the conclusion of the claim (Harel, 2008; Harel & Sowder, 1998). Interviews were video-recorded, transcribed, and analyzed. Transcripts were read in conjunction with students' written work. Akin to Miles and Huberman (1994), we started with reasoning frameworks from existing literature, then we modified these as we developed models of the student reasoning to fit our data.

Findings

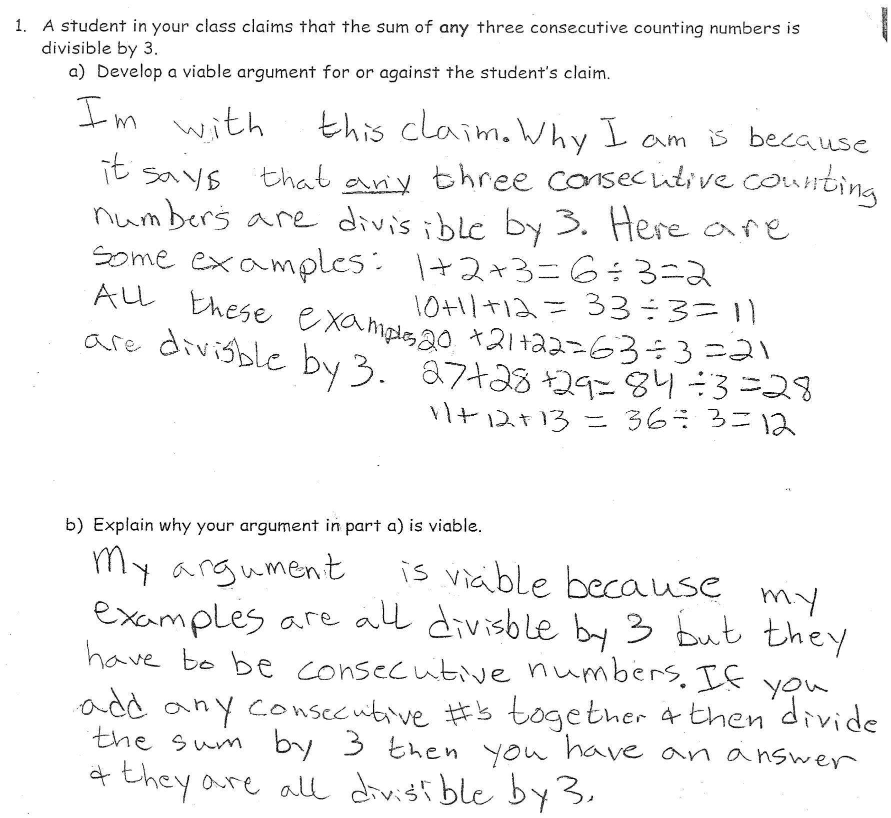
Students took multiple routes when exploring and trying to develop an argument for or against the following general claim: "the sum of three consecutive numbers will always be divisible by three". After generating and examining examples, students' reasoning tended to take one of two paths:

1) They displayed empirical reasoning, often pursuing an empirical goal and gathered examples in order to gain certainty for the claim. For example, one student said, "say you were adding two, three, and four. So, two plus three is five. Five plus four is nine. So…that'd be twelve. So, yeah it would be a correct claim." Another student validated her empirical conviction saying: "It's a viable argument because I tested five cases…Three cases…would not be enough." As we will demonstrate, this student was not pursuing any other type of reasoning beyond empirical.

2) They pursued an analytic argument and went back and forth between abductive reasoning (searching for structural features that connect a claim's conditions to its conclusion) and deductive reasoning (checking to see if parts of their argument entailed logical necessity).

The Case of Ray

Ray worked on the written task as a pre-assessment on one day and was interviewed about her responses on another day. Her written response expresses an understanding of the conditions and conclusion of the claim, and it uses words like "any" and "all," which suggests that she conceives of the domain of the claim as infinite. A subsequent interview confirmed this assumption. Hence, Ray's argument, illustrated in Figure 2, is empirical.



1. A student in your class claims that the sum of **any** three consecutive counting numbers is divisible by 3.
   a) Develop a viable argument for or against the student's claim.

Im with this claim. Why I am is because it says that any three consecutive counting numbers are divisible by 3. Here are some examples: 1+2+3=6÷3=2
All these examples are divisible by 3.
10+11+12 = 33÷3=11
20 +21+22=63÷3 =21
27+28 +29= 84 ÷3 =28
11+12+13 = 36÷ 3=12

   b) Explain why your argument in part a) is viable.

My argument is viable because my examples are all divisible by 3 but they have to be consecutive numbers. If you add any consecutive #'s together & then divide the sum by 3 then you have an answer & they are all divisible by 3.

Figure 2: Ray's response to a pre-assessment item.

A follow-up, activity-based interview was performed with Ray to assess her understanding of viable argument and proof. Ray was presented with her work on the pre-assessment and was asked a sequence of planned questions, in a semi-structured interview format.

> Interviewer: What would you say is the domain of your claim… the numbers to which your claim applies?
> Ray: Any three consecutive counting numbers 15, 16, 17, so on so forth. [There are] a lot. I don't know the exact number, but is a lot.
> Interviewer : More than a million.
> Ray: Uh-um. More than a trillion. I don't know how many more after a trillion.
> Interviewer : What would you change [about your argument]?
> Ray: I would make my claim to make it… [clearer]… maybe a few more examples to prove my claim more… I might use diagrams or pictures just to get someone else to understand the way I think… to explain it. I know a lot of people like to pictures or diagrams…

When Ray was asked to show how she would use a diagram for this purpose, she created the diagram in Figure 3. Her diagram did not contain any structural elements she could use in crafting a viable argument for the claim, nor did she indicate that she was aware that identifying such a structure would be a useful goal.

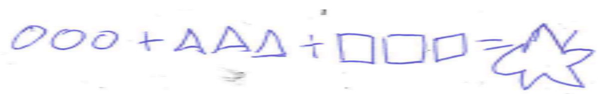


Figure 3: Ray's diagram for her "thinking" about summing three consecutive numbers.

Ray was then asked to develop a viable argument for or against the claim *the sum of any four consecutive numbers is divisible by 2*. She produced the argument illustrated in Figure 4, showing four examples and a drawing. When asked why she chose the particular 4 examples, Ray explained that they were the first numbers that popped into her head. She felt that testing four examples was enough to support her claim, saying: "Yeah. Between four and six, so you get lots of information, so people don't think there's only one and we're working with lots of different examples with lots of different numbers."

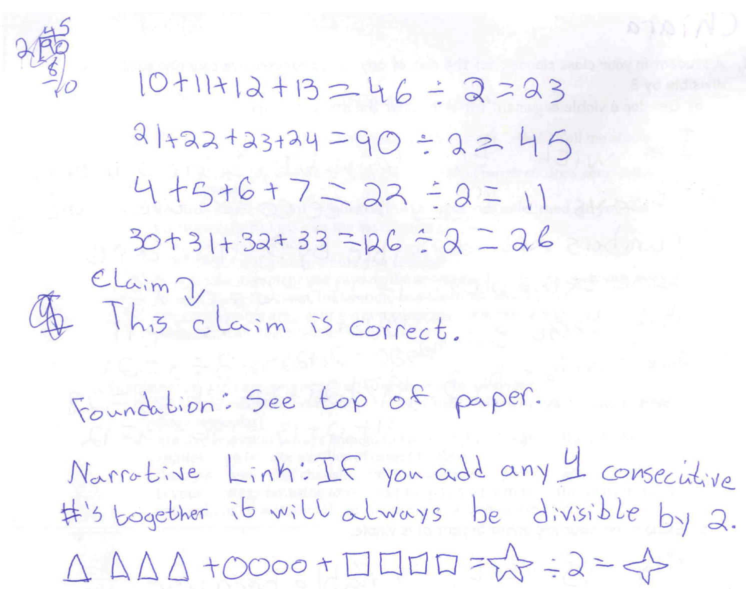


Figure 4: Ray's argument response to the four consecutive number claim.

Analysis of Ray’s Responses

Ray’s argument activity does not include a goal of developing an argument that demonstrates the pertinent mathematical structure that links the conditions to the claim and is shared by all cases. Her goal in this activity is to provide evidence that would convince someone skeptical of a single example and help others understand that her thinking applies to “lots of different examples.” While Ray does not present evidence that she understands “infinite domain,” she does see the domain of the generalizations as large and going on and on. Ray’s desire to include pictures in her argument is particularly telling of her argument goals. Her pictures express no mathematical structure that can be used to craft a structural argument. Our evidence suggests that Ray does not invoke an abductive goal, in general, when presented with a generalization to support. She does not appear to be looking for a structural argument in either of the argument contexts. Both her argument activity and her goal are empirical.

The Case of Emma

Emma’s approach was very different. When given the sum of three consecutive numbers task, she initially said the claim was not true, and attempted to provide an argument using her knowledge of numerical relationships. She said, “Well, any three consecutive numbers – counting numbers – divisible by three isn't a correct argument because to be divisible by three it has to be a multiple of three. The consecutive number can't be multiples of three because it must be counting up by three, and consecutive is counting up by one. … So, one number in the three consecutive numbers could be divisible by three.” However, when prompted by the interviewer to try an example, she quickly decided the claim was true.

Work on a few more examples increased Emma's certainty. When asked to create an argument for the claim, she tried more examples, looking across the examples for a pattern that she could connect to divisibility by three. She said, "So, it would be a correct claim because one, plus two, plus three equals six and that's divisible by three. And then every single number you add up – so, like one, two, three, two, three, four – that's adding one for every number in this. Those three ones would add up to three. So, then it would be that sum plus three, so that would be divisible by three." At this point, Emma was thinking about structural components that connect the structure of the summands to the structure of the sum, and she was applying her thinking about a specific example to subsequent cases.

Further investigation led Emma to identify a process she could use on any three consecutive numbers to decompose them into three copies of a single number plus one, two, and three, which she knew summed to six. For example, she took 22, 23, 24 and subtracted: 22 - 1 = 21; 23 - 2 = 21; and 24 – 3 = 21. Then she added the three 21's and the 1, 2, and 3 (totaling 6) she had subtracted to reach a sum of 69. She said, "sixty-nine is divisible by three 'cause sixty and nine are both divisible by three. Add those together. Both are divisible by three, so then the twenty-two, twenty-three, twenty-four – subtract one from twenty-two, subtract two from twenty-three, subtract three from twenty-four. You get twenty-one, twenty-one, twenty-one. Add those together, you get sixty-three. Plus the six from right here would equal sixty-nine, and that's divisible by three."

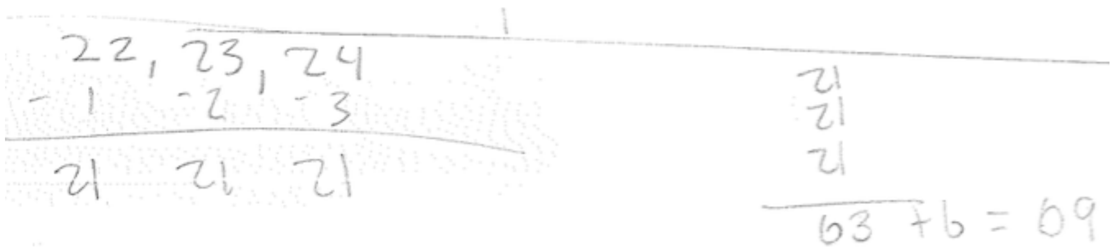


Figure 5: Emma applies her process for creating multiplies of three

Emma eventually realized that her strategy can be used for any sum of three consecutive numbers, "Because...you can always get – from any three consecutive numbers you can always get one number from subtracting one, two, and three. And then, you can always add something from that one, two, and three to get a number divisible by three. And then, that will be divisible by three. Her argument appears in Figure 6.

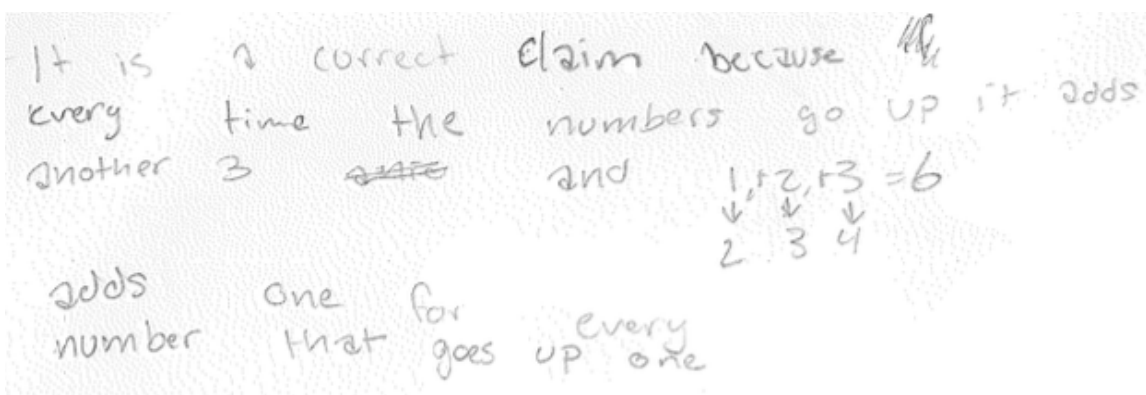


Figure 6: Emma's argument for the claim

Analysis of Emma's Responses

Throughout Emma's interview, she provided evidence suggesting she had an abductive goal and she pursued it for an extended period of time. While like Ray, Emma used examples to confirm the claim, she repeatedly went beyond such confirmation, seeking to explain why the claim was true by searching for a general rule that connects the conditions of the claim to the its conclusion—in this case, a structural property. She engaged in multiple abductive acts as she searched for rules, perhaps invented, to apply to an argument. She used abductive reasoning to identify a general rule she knew but wasn't sure how to use, that if two numbers are each divisible by three then their sum is also. She decomposed 16 + 17 + 18 into 10 + 10 + 10 and 6 + 7 + 8, then reasoned, "since thirty is divisible by three and twenty-one is divisible by three, I'm pretty sure fifty-one would be divisible by three because [using her rule:] two numbers divisible by three added together would be divisible by three." However, she could not yet apply the rule generally, so returned to an abductive goal. She noticed a pattern in all the results she had checked: all were divisible by three, confirming the claim for her. She then performed numerous abductive acts, seeking, but not finding, a rule that would account for this result. She changed her focus and began searching for a process that would transform consecutive numbers into a set of numbers that she knew was divisible by three. Through Process Pattern Generalization (PPG) (Harel, 2008), she found a general process and illustrated it with 15, 16, 17. "You would subtract one from fifteen, subtract two from sixteen, subtract three from seventeen.....We've got three of the same number [three 14's]. Add those together. It gets divisible by three. Then you add that six, because it's one, two, three….You add that six there and you get what this answer would be, and that answer is divisible by three." Her argument here is general; without calculating the sum, she knows it must be divisible by three because she earlier stated, "a number divisible by three plus a number divisible by three will be divisible by three." The process from her PPG provided the rule that could be used in her analytic argument. Emma eventually identified a rule for decomposing the numbers in the sum that would transform any sum of three consecutive numbers into 3n + 6, which, in combination with her earlier rule, allowed her to create an analytic argument.

Implications

With our framework, we can make sense of and provide in-depth analysis of a student's reasoning and goals, even when it appears the student is flailing. If an instructor can identify the type of reasoning and goal a student is employing, he/she can then interact purposefully with the student to promote analytical justifying. In particular, by also focusing on students' goals as well as their acts, an instructor can distinguish between a student who is presenting empirical support for a generalization because she is not able to or has yet to find structural support for the claim and a student who is pursuing empirical support for a generalization without any analytic goal at all. For instance, if the instructor discerns that a student is using abduction, he/she could direct the student's attention to features or representations that might allow the student to identify the missing rule. If the student is pursuing an empirical goal instead, such attention-directing may not help the student, since they are not pursuing an analytical argument.

References

Common Core State Standards Initiative (2010). *Common Core State Standards for Mathematics.* National Governors Association Center for Best Practices and the Council of Chief State School Officers. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Harel, G. (2008). DNR perspective on mathematics curriculum and instruction, Part I: Focus on proving. *ZDM*, *40*(3), 487–500. http://doi.org/10.1007/s11858-008-0104-1

Harel, G., & Sowder, L. (1998). Students' proof schemes. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research on Collegiate Mathematics Education* (Vol. III, pp. 234–283). AMS.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative Data Analysis: An Expanded Sourcebook*. Thousand Oaks, CA: SAGE.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, *66*(1), 23–41.