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Student Internalization of Representations and Its Role in Generalization

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In this paper we present the results of a study that investigated how students made use of their external and internal representations while generalizing algebraic tasks. A framework of internalization will be introduced and common internalization pathways used by students to generalize each task will be discussed.

Introduction

Representation is a broad term used to describe an important aspect of the learning and generalization processes. Leaders in mathematics education (RAND Mathematics Study Panel, 2003) have stressed the importance of representation as an area of research leading to improved teaching and learning and as a way to explain student thinking (Radford, 1999). For the purpose of this study, representation "applies to processes and products that are both observable externally as well as those that occur 'internally,' in the minds of people doing mathematics" (NCTM, 2000, p. 14).

During the process of generalizing, students often internalize their representations. The interaction of internal and external representations form a representational system that helps students build understanding (Goldin, 2003). The question then becomes, how are these internal representations developed? Internal representation systems are not simply carbon copies of external representation systems (Dreyfus, 1991), but rather the two-way nature of the representing system means that sometimes the internal represents the external and sometimes the external represents the internal (Goldin, 2003). Understanding more about how students internalize their representations when solving algebraic tasks may yield a new lens to viewing the generalization process. With this in mind, this study seeks to answer the following research question: How do sixth- and tenth-grade students make use of their internalization of representations when developing generalizations of algebraic tasks?

Theoretical Perspective

During the 1990s, researchers began investigating how external and internal representations interact, rather than simply studying the external objects that were produced (Lesh & Doerr, 2000; Yackel, 2000). Understanding the interaction between external and internal representations allowed for more informed decisions regarding how students were able to learn and use representations. However, the complex nature of these interactions requires the development of new frameworks and research methodologies.

Given the important role that internal representations play in the learning and generalizing processes (Goldin, 2003), a framework devoted solely to the process of internalization is needed. In 2013, the authors developed an internalization framework for K-12 students. This internalization framework was developed and modified over several years and consists of five representation categories: External, Near External, Near Internal, Internal, and Procedure. This framework was used to investigate the common pathways that students take when developing generalizations. A pathway is defined as a route through and among these five representation categories when attempting a generalization. The formal definitions for these representation categories are given in Figure 1.

External Representation (E) includes physically using representations (i.e. manipulatives, drawings) to solve problems in a way that uses only external representations (solved from beginning to end completely externally).
Near External Representation (NE) is when an external representation is the primary representation used to solve a problem, with some reliance on an internal representation. The external representation is necessary for the student to arrive at a solution, but external representations are not exclusively used.
Near Internal Representation (NI) is when an internal representation is the primary representation used to solve a problem, with some reliance on an external representation. The student uses the external representation as a stabilizing component of his or her thinking, meaning that he or she still could have arrived at a solution without using the external component.
Internal Representation (I) is when an internal representation is used to solve a task. Any use of, or reference to, a physical representation is superficial to solving the task.
Procedure (P) is when the student uses an articulated process either externally or internally without any reference to the context or physical representation.

Figure 1. Representation categories from the K-12 internalization framework.

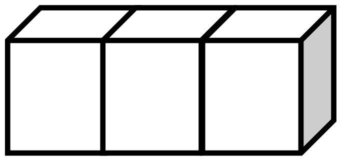
Methodology

Six sixth-grade and six tenth-grade students from a rural Midwestern K-12 school district were chosen to participate in semi-structured interviews. Each participant completed a series of algebraic tasks (linear and quadratic) designed to promote internal representations. Each participant was interviewed for approximately 60 minutes and completed between three and eight tasks. The researcher's role was to prompt the participant to verbalize his or her thinking, ask for clarification to ensure that accurate thoughts were being captured, and provide the appropriate sequence of tasks. All of the interviews were video and audio recorded to capture verbal and non-verbal cues. All written work produced by the students and field notes taken by the researcher were collected. An example task, labeled the Cube Sticker Task, was given to all students and is included in Figure 2.

The first step in the data analysis process was to separate the data into blocks, or portions of the interview containing a distinct approach to a case. Each case within a task was a single block with the exceptions being if a student did not produce a meaningful attempt for a case (these were ignored), or if the student completely changed approaches within a case without making a connection to their original approach (these were separated into different

blocks). To ensure consistency, three coders were asked to place the blocks into one of the five representation categories of the K-12 Internalization Framework. The level of coding agreement among all the blocks was 79.1% (219/277), and all but eleven of the discrepancies in coding were resolved through discussion among the coders. Those that were not resolved were removed from the study.

A company makes colored rods by joining cubes in a row and then uses a machine to place stickers on the rods. The machine places exactly one sticker on each exposed face of each cube. Thus, a rod of length three (as shown below) would require 14 stickers to cover all of the exposed faces.



(a) How many stickers are needed to cover a rod of length 4?
(b) How many stickers are needed to cover a rod of length 10?
(c) How many stickers are needed to cover a rod of length 20?
(d) How many stickers are needed to cover a rod of length 47?
(e) You have 282 stickers, how long of a rod can you cover?
(f) How many stickers are needed to cover a rod of length 48?
(g) How would you determine the length of the rod given any number of stickers?
(h) How would you determine the number of stickers used if you knew the length of the rod?

Figure 2. Cube Sticker Task. From *Developing Middle School Students' Understanding of Recursive and Explicit Reasoning* (p. 153), J. K. Lannin, 2001, Doctoral Dissertation, Illinois State University. Copyright 2001 by John K. Lannin. Adapted with permission.

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Once all the blocks of data were coded using the K-12 Internalization Framework, these codes were used to produce schematics for each task attempted by each student. The schematics provide the sequence of representation categories used by the student while working a task and whether their solutions were correct. An example of a partial schematic is provided in Figure 3.

			Representation Categories				
Block # (Case)	**Line Numbers from Transcript**	**Answer** (✓ = correct, x = incorrect)	**E**	**NE**	**NI**	**I**	**P**
1 (N = 4)	12-19				x		
2 (N = 4)	28-37	18 ✓		x			
3 (N = 10)	39-43	41 x	x				

Figure 3. Example schematic.

Findings

Using the K-12 Internalization Framework, we identified four common pathways that students used as they generalized individual algebraic tasks. The four pathways were defined based upon when (or if) the student used an external representation, and when (or if) the student used an internal representation. Additionally, the *External After Internal* pathway was divided into two sub-paths based on what type of representation the student used at the beginning of the task. A description of the pathways is provided in Figure 4.

External Only – Path A: Students relied on an external representation, either by choice or by necessity, for the duration of the task.

External After Internal – Path B: Students either began the task by using an internal representation or procedure (**Sub-Path B1**), or they used an external representation to produce an internal representation or procedure (**Sub-Path B2**). In either case, after they had used an internal representation they returned to use some form of external representation later in the task. Many of these instances involved either the verification of a rule and/or further investigation of a rule that became troublesome for later cases.

External to Internal – Path C: Students progressed from some form of external representation to an internal representation. Once they produced an internal representation (or procedure), they did not use any more subsequent external representations during that task.

Internal Only – Path D: Students used an internal representation for the duration of the task.

Figure 4. Internalization pathways.

These pathways represented the ways in which the students were making use of their internalization process and present a possible progression that students go through as they develop their skills in creating and using internal representations. When all of these pathways were viewed simultaneously, it provided clarity to the differences and similarities among the pathways. A model for the pathways is given in Figure 5.

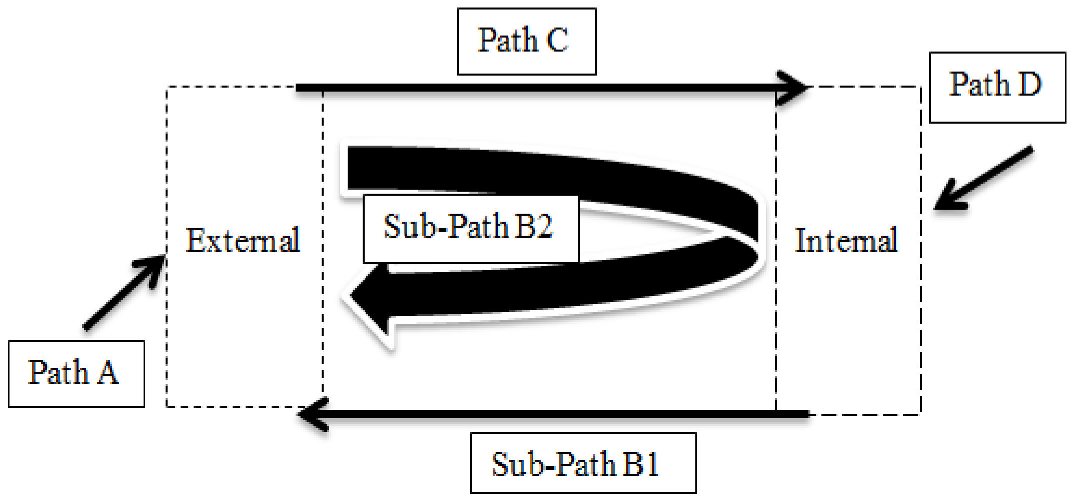

Figure 5. Pathways model.

In order to investigate student use of these internalization pathways we must understand the factors that influence pathway use. One of the major influences on pathway use was the level of challenge. During our analysis, each block of data was labeled as being too easy, too difficult, or appropriately challenged. Table 1 provides the overall distribution of pathway use relative to the level of challenge.

Level of Challenge	External Only (A)	External After Internal (B)		External to Internal (C)	Internal Only (D)
		Internal to External (B1)	External to Internal to External (B2)		
Too Easy	0	0	2	7	10
Appropriate	4	3	5	21	0
Too Difficult	5	1	0	1	0

Table 1. Comparing pathway use with level of challenge provided by the task.

From the table we can observe that the students in our study more readily internalized their representations when the task was too easy. Conversely, our subjects had difficulty moving beyond the external when the task was too difficult. When the students were not appropriately challenged, only 42% of the blocks of data (11 out of 26) contained elements of both external and internal representations. In contrast, when the students were appropriately challenged, 88% of the blocks of data (29 out of 33) contained elements of both external and internal representations. This illustrates the importance of the interplay between internal and external representations when student are appropriately challenged. In addition, the data provides some clues to teachers as to when students are not appropriately challenged. Of the 12 students in our study, all but one used Path C (external to internal) when appropriately challenged. In fact, 64% of the appropriately challenged blocks were classified as Path C.

When we look at the data at the student level, all of our students used at least some elements of both internal and external representations. However, there were differences in the distribution of their use of external and internal representations. There were students who never used Path D (internal only) and others who never used Path A (external only). Four students never used Path B, which is interesting since students generally used this pathway to verify and check their rules. Table 2 provides the distribution of pathway use for the 12 students in our study.

Grade Level	Student	Pathway				
		External Only (A)	External After Internal (B)		External to Internal (C)	Internal Only (D)
			Internal to External (B1)	External to Internal to External (B2)		
10th	**Nicki**	1	1	0	2	2
	Adam	1	0	1	3	1
	Linda	1	0	1	4	0
	Travis	1	0	0	5	0
	Michaela	3	0	1	1	1
	Tommy	0	0	2	2	0
6th	**Brad**	0	0	1	2	1
	Jason	0	1	1	2	1
	Tonya	0	2	0	1	1
	Abigail	1	0	0	2	1
	Eli	0	0	0	3	1
	Henry	1	0	0	2	1
TOTAL		**9**	**4**	**7**	**29**	**10**

Table 2. Distribution of pathway use by each student considering all tasks.

In our study we have identified some common pathways that students use as they develop generalizations for algebraic tasks. In addition, we have highlighted the influence of level of challenge on the use of these different pathways. The next step in our work is to investigate the connection these pathways have to successful generalizations and to uncover what instructional moves help students in the generalization process. For instance, would specific instruction on using external representations as a check improve student generalization? Would instruction on using the external to generate the internal be productive? If teachers understood these pathways, and the connection to level of challenge, would it help them identify where their students are at and what they need?

Significance

The importance of being able to interpret and produce accurate generalizations in mathematics has been well documented (e.g., Radford, 2006). Internalization plays a key role in this process and the findings from this study shed light on how these generalizations are developed. The pathways identified in this study could inform teachers about how students think as they attempt to generalize. Being able to identify these pathways within a classroom of students would be valuable as teachers make curricular and pedagogical decisions and as they react to student thinking.

References

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 24-41). Dordrecht, The Netherlands: Kluwer.

Goldin, G. A. (2003). Representation in school mathematics: A unifying research perspective. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 275-285). Reston, VA: National Council of Teachers of Mathematics.

Lesh, R., & Doerr, H. M. (2000). Symbolizing, communicating, and mathematizing: Key components of models and modeling. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 361-383). Mahwah, NJ: Lawrence Erlbaum Associates.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Radford, L. (1999). *Rethinking representations.* Paper presented at the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Cuernavaca, Morelos, Mexico.

Radford, L. (2006). *Algebraic thinking and the generalization of patterns: A semiotic perspective*. Paper presented at the Twenty-Eighth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mérida, México.

RAND Mathematics Study Panel (2003). *Mathematical proficiency for all students: Towards a strategic research and development program in mathematics education.* Santa Monica, CA: Author.

Yackel, E. (2000). Introduction: Perspectives on semiotics and instructional design. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 1-13). Mahwah, NJ: Lawrence Erlbaum Associates.