Title: Eliminating Counterexamples: A conception of contrapositive proving for adolescents

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Session Title: Eliminating Counterexamples: A conception of contrapositive proving for adolescents

Session Type: Brief Research Report

Presentation Date: Tuesday, April 12, 2016: 11:30 AM-12:00 PM

Presentation Location: 3010 (San Francisco Convention Center) San Francisco, California

Abstract: This study is predicated on the hypothesis that adolescents perform poorly on contrapositive reasoning tasks because they lack a meta-theory for validating contrapositive argumentation. By studying an adolescent's actions and conceptions as they develops and validates not-the-conclusion-implies-the-conditions-impossible arguments for conditional claims, a promising conception of indirect argumentation is developed and used to improve her understand contrapositive proving. Data from a Grade 8 student's constructions and critiques of contrapositive arguments are used to illustrate the conception's usefulness.

*Key words***:** reasoning, proving, contrapositive proving, conceptions, teaching experiment, viable argument

Acknowledgements: This material is based upon work supported by the National Science Foundation under Grant No. DRL 1317034. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

Introduction

Indirect reasoning arises spontaneously in mathematics courses for students at all ages (Antonini, 2003; Antonini, 2004; Antonini & Mariotti, 2008; Freudenthal, 1972; O'Brien, 1972; Thompson, 1996; Reid & Dobbin, 1998). Common Core (NGACBP & CCSS, 2010) does not explicitly mention indirect reasoning; however, indirect reasoning arises naturally when addressing the Grade 8 content standards. Standard 8.G.5 states that students will "use informal arguments to establish facts about… the angles created when parallel lines are cut by a transversal" (p. 56). When students construct intersecting lines cut by a transversal, they create a triangle whose interior angles sum to 180 degrees. Students can then observe that congruent alternate interior angles are impossible and argue indirectly that congruent alternate interior angles correspond to parallel lines.

Students tend to do poorly on indirect reasoning tasks. They fail to recognize indirect arguments as proofs (Antonini, 2004), and they struggle negating conditions and conclusions (Leon, 1985). Students familiar with indirect proving methods may reject indirect proofs as operating in an "absurd" world (Antonini, 2008).

Authors have suggested possible sources of students' struggles with indirect argument that may be useful in crafting an intervention. In particular, Antonini (2004) asserts that there is a gap between students' intuition about indirect proving and the logical equivalence between a conditional statement and its contrapositive. In a later article, Antonini (2008) posits that a meta-theory might be needed to recognize the validity of contrapositive reasoning across contexts. Loosely defined, meta-level reasoning is reasoning about reasoning. In argumentation, a meta-theory is the theoretical framework that one acknowledges to validate a particular type of argument independent of context. I conjecture that students do poorly in contrapositive reasoning tasks because they lack a sufficient and appropriate meta-theory for understanding, constructing, and accepting contrapositive arguments.

A meta-theory useful to adolescents during indirect reasoning activities might be very different from that used by experts. The link between $p \rightarrow q$ and not $q \rightarrow$ not p can be so strong for experts (e.g., a university teacher) that it requires no explanation (Antonini, 2004, p. 47). When pressed, an expert mathematicians might validate and understand contrapositive arguments by appealing to a meta-theory grounded in axioms, definitions, truth tables, and theorems of logic. Unfortunately, adolescents may not have access to such conceptions. I posit that a meta-theory grounded in adolescents' spontaneous indirect reasoning may be useful in constructing a contrapositive metatheory distinct from that of experts', yet helpful to adolescents in constructing, understanding, and validating contrapositive arguments.

The goal of this paper is to explore an alternative meta-theory for understanding and validating contrapositive arguments and to demonstrate its usefulness. This works is intended to offer the bases for an intervention for improving adolescents' indirect reasoning. Interventions that improve students' performance on argumentation tasks (Stylianides and Stylianides, 2015) are sparse, and this is particularly true for indirect argumentation.

Theoretical Framework

Concepts and Conceptions

The mathematics education research community lacks consensus a definitions of *concepts* and *conceptions*. Simon, Placa, and Avitzur (2016) define concept "as made up of a goal and an action to achieve that goal (p.76)." Sfard (1991), on the other hand, does not define these terms and instead indicates how these terms are used in her work. Sfard refers to a mathematical idea in its theoretical construct within the formal universe of ideal knowledge as a concept and refers to the human internal and subjective knowing of the mathematical idea as a conception. A debate about what definition of concept and conception are most useful is beyond the scope of this article. I do, however, take the point of view these terms should be used and defined in ways that align with the researchers' context.

For this study, the term *concept* is used to describe an idea of something formed by mentally combining all the characteristics or particulars associate with that idea. A conception is individual-based and is determined by what every characteristics or particulars are called upon in that moment in pursuit of some goal. A conception is witnessed by the observing an individual's coordination of extant knowledge, understandings, actions, etc. as a "united" notion used to satisfy some goal. These definitions allow me to address my goals of describing an adolescent's conceptions of contrapositive proving as she engages in activities that involve direct or indirect reasoning.

Pivotal Intermediate Conceptions (PICs) and Conceptual Learning Goals (CLGs)

Lobato et al. (2012) assert the existence of *pivotal intermediate conceptions* (PICs) that may be important to learners, but are forgotten when learners become experts and no longer need them. PICs are "image[s], meaning[s], idea[s] or way[s] of comprehending a situation that can be leveraged in a students' mathematical development" (Lobato et al., 2012, p. 87). "Comprehending" is described as noticing and isolating some mathematic aspect, elaborating on characteristics of the aspect, and coordinating mental actions. A PIC is *pivotal* in that it can be leveraged to reason about related problems and more complex ideas, and it is *intermediate* in that it may no longer be needed by experts to think about and operate with an idea. PICs can be founded in students in the moment of understanding of objects and tasks. It is in these moments that students may express thinking that might be hidden from experts.

PICs can inform conceptual learning goals (CLGs) (Lobato et al., p. 88, 2012), which are statements of concepts that are the targets of instruction. Ideally, CLGs are empirically derived from student thinking. CLGs are "psychologized version of mathematics" (Lobato et al., p. 88, 2012) in that the concepts can be defined relative to students' personal meaning makings, intuitions, and constructions. This is in contrast to "Platonic" ideals of concepts (i.e., the notion that concepts are external of human construct). Like PICs, CLGs can be productive ways of "coming to know" (e.g., developing mathematical meanings, comprehending situations, and mentally organizing information about a situation) that may be distinct from experts' ways of knowing.

PICs and CLGs are important to this study because they underscore my hypothesis that a meta-theory for indirect argumentation that is distinct from that of experts might be useful in improving Grade 8 students' indirect reasoning practices. This hypothesis is predicated on the prediction that adolescents may come to comprehend and validate indirect argumentation differently than how experts do. Along these lines, I conjecture that is possible to develop a viable meta-theory for why contrapositive arguments can be taken as proof without necessarily building formal logical theories.

Mental Reasoning Schemes

The Mental Models reasoning scheme (Johnson-Laird, 1983) asserts that reasoners engage in a sequence of stages involving representations analogous to the structure of the situation being reasoned about. In the comprehension stage, the reasoner constructs a mental model of the situation. In the description stage, the reasoner tries to develop a concise description of the model that asserts things not explicated stated in the premises. In the validation stage, the reasoner searches for alternative models (e.g., counterexamples) that might refute assertions developed in a previous stage. If no alternative model is found, the assertions are taken as valid. Johnson-Laird (1983) summarizes this process as *eliminating the possibility of counterexamples*.

From the perspective of Mental Models, a reasoner concludes that *p implies q* is true if and only if no counterexamples (*p* and *not q*) exist. This biconditional inference is equivalent to any definition of proof that attends to validity and soundness. A proof of *not q* implies *not p* eliminates the possibility of counterexamples *not q* and *not (not p)*, which is of course *p* and *not q*. The point is that a proof of *not q implies not p* eliminates the possibility of counterexamples to a claim *p implies q* and hence is a proof of this claim as well. Thus, a conditional claim and its contrapositive can be viewed as logically equivalent by observing that the two claims share the same counterexamples, which is the same observation experts make with truth tables in propositional logic.

Proof as eliminating counterexamples

Validating arguments as proof requires a conception of "proof." From a teaching point of view, this assumes a definition of the term proof. The mathematics education community lacks consensus on the meaning of the term proof (A. J. Stylianides, 2014), and attempts to define proof as a list of characteristics are doomed to failure (K. Weber, 2014). Yet, a definition of proof as a cluster concept holds promise. Clusters allow for multiple models to come together and define a concept. Weber's models for proof include arguments that would convince mathematicians and arguments that are blueprints for writing "complete" proofs. Weber's cluster seems to include conceptions of "what proof is" and "what proof does."

Following the logic of Lobato et al. (2012), *coming to know* what proof is may be different from *knowing* what is proof. I argue, in the results section, that a conception of proof and proving as eliminating the possibility of counterexamples is useful to developing a meta-theory for contrapositive argumentation. In the results section, I demonstrate that such a conception is ground in adolescents' reasoning and that such a conception is useful to adolescents in developing more sophisticated reasoning skills. Expert mathematicians surely hold this conception of proof—it is a trivial observation from truth tables that define *p implies q* in propositional logic. However, experts may not communicate this conception to learners, and this conception is not explicitly presented as learning framework in the mathematics education literature. Thus, defining and thinking about proof and proving as eliminating the possibility of counterexamples is novel and new to the mathematics education literature.

Methods

Assumptions and research questions

As mentioned above, Lobato et al. (2012) assert that there exists *pivotal intermediate conceptions* (PICs) that occur and are important to learners, but are perhaps forgotten when they become experts because these conceptions may no longer be needed when reasoning. My inquiry was predicated on the assumptions that students posses ways of reasoning that are distinct from mathematicians' (Lobato et al, 2012) and that students distinct ways of reasoning are useful in developing sophisticated understandings. To explore this, I address the following research questions:

1. In what ways do Grade 8 students develop indirect arguments and how do students validate the arguments they produce.
2. In what ways can Grade 8 students' conceptions of indirect argumentation be used to improve their indirect argumentation skills and understanding?
3. In what ways can Grade 8 students' conceptions of indirect argumentation be used to develop a meta-theory for contrapositive argumentation that is useful and accessible to adolescents?

Data Sources

Data comes one student, Alison, who participated in teaching experiments designed to improve her viable argument knowledge and skill as they learned CCSS-M content. This "overarching" teaching experiment (OTE) was presented to Alison's entire Grade 8 class, taught by her usual teacher. Students engaged in activities in which they learned Grade 8 content as they made claims and arguments about relationships among content ideas. Students engaged in constructing and critiquing arguments using exhaustion, counterexamples, and deduction, including proof by contrapositive and contradiction. These lessons occurred between late October and late February.

Alison experienced the OTE twice: once when she was a Grade 7 student placed into Grade 8 mathematics because of her high scores on state mathematics assessments, and again when she officially entered Grade 8. In her second Grade 8 mathematics experience, she essential retook Grade 8 mathematics and experienced the teaching experiment again. The setting was a K-8 charter school, so no other mathematics course was available for Alison to take. The teacher accommodated her advanced skill with Grade 8 content in the second year by augmenting her assignments with additional practice with Algebra I concepts and skills.

Each year, in mid-April, after the OTE was complete, Alison participated in one-on-one cognitive, activity (task)-based interviews (Ginsburg, 1997), in which I was the interviewer. The purpose of these task-based interviews was to assess the efficacy of the overarching teaching experiment, and to guide the development of new activities to bring about intended practices and conceptions. In the second year, the interviews included learning activities though sought to improve Alison's conceptions and skills with contrapositive reasoning. These activities employed a methodology similar to the Learning Through Activity approach (Simon et al., 2010; Simon, Placa, and Avitzur, 2016) that builds on Piaget's (2001) theoretical construction of reflective abstraction. This processes (modified slightly from that presented in Simon et al., 2016) involves:

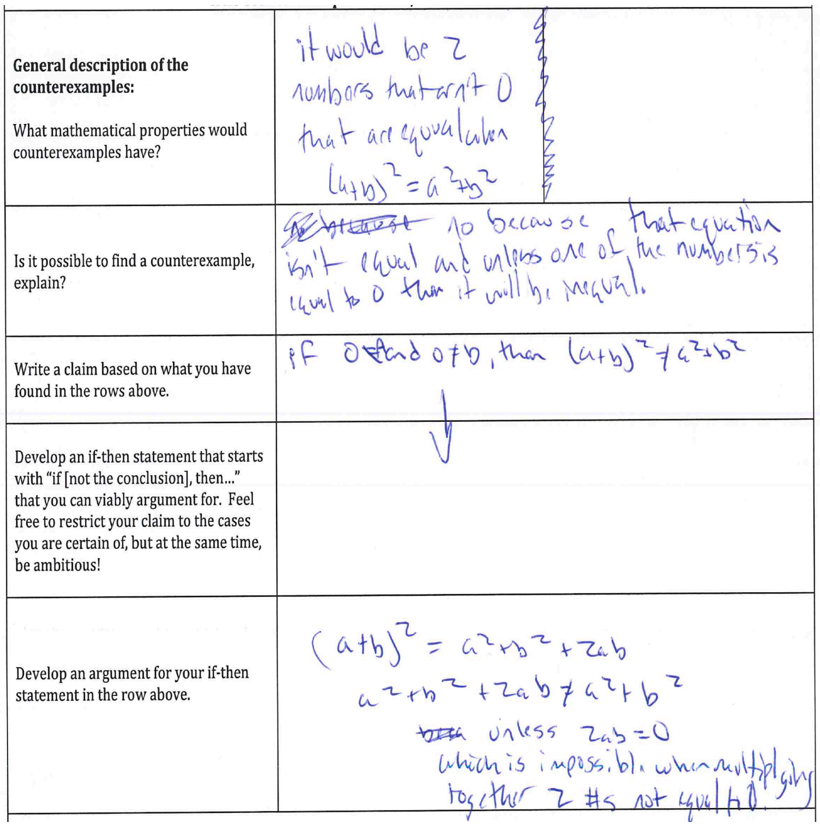
1) Assessing the learners' extant understanding.
2) Specifying a learning goal.
3) Engaging the learner in an activity designed as the basis for new abstraction yet involving processes and conceptions already available to the learner.
4) Engaging the learning in a sequence of tasks designed to bring about the intended abstraction.

My activity design was heavily influenced by notion of *pivotal intermediate conceptions* (PICs, Lobato et al., 2012), which suggests that students' less-than-formal, incomplete, and different-from-experts' conceptions can be leveraged by students' in developing sophisticated conceptions, and by the Mental Models (Johnson-Laird, 1983) reasoning theory, which offers ways of interpreting reasoning approaches, as well as interventions for improving reasoning.

Figure 1 illustrates one of these activities. The activity is grounded in the indirect reasoning expressed by Alison during the interviews that followed the first iteration of the OTE. Her reasoning at this time point can be described as eliminating "alternatives" to claims she perceives as true. The activity encourages her to leverage this reasoning approach, as well as her understanding of counterexamples and the contrapositive form, in developing more sophisticated conceptions of indirect argumentation.

A critical mechanism for growth is expressed in the activity's encouragement to employ Mental-Models style reasoning. The student is encouraged to develop descriptions of all possible counterexamples and then encouraged to use these descriptions to eliminate the possibility of counterexamples.

When a student engaged in the activity argues that objects satisfying "not-the-conclusion" of the claim cannot satisfy the conditions of the claim, that student expresses an understanding of the activity. When a student asserts this type of argument "eliminates the possibility of counterexamples," that student is expressing indirect reasoning. When a student justifies this last assertion with the clause "because objects satisfying the counterexample description above are impossible," that student is justifying her/his indirect reasoning. When a student coordinates the claim her/his indirect argument directly supports, with the original claim, and with the contrapositive form, either in her/his eliminate-counterexamples argument or in her/his justification for the validity of that argument, that student is expressing an understanding of the link between contrapositive argumentation and the eliminating counterexamples approach. Finally, when the student justifies the contrapositive proving approach by noting that the contrapositive and the original claim have the same class of counterexamples, thus an argument for one form is an argument for the other, the student expresses a meta-theory for contrapositive proving. Table 1 summarizes this progression to a meta-theory that justifies contrapositive argumentation.



General description of the counterexamples: What mathematical properties would counterexamples have?	it would be 2 numbers that aren't 0 that are equal when $(a+b)^2 = a^2+b^2$
Is it possible to find a counterexample, explain?	[illegible] no because that equation isn't equal and unless one of the numbers is equal to 0 than it will be inequal.
Write a claim based on what you have found in the rows above.	if $0 \neq a$ and $0 \neq b$, than $(a+b)^2 \neq a^2+b^2$
Develop an if-then statement that starts with "if [not the conclusion], then..." that you can viably argument for. Feel free to restrict your claim to the cases you are certain of, but at the same time, be ambitious!	
Develop an argument for your if-then statement in the row above.	$(a+b)^2 = a^2+b^2+2ab$; $a^2+b^2+2ab \neq a^2+b^2$ unless $2ab=0$ which is impossible when multiplying together 2 #s not equal to 0.

Figure 1: Alison's response to an activity designed to encourage students to develop a description of all possible counterexamples and leverage the description to craft an argument that shows counterexamples are impossible.

Student Activities	**Student Learning Expressed**
The student argues that objects satisfying "not-the-conclusion" of the claim cannot satisfy the conditions of the claim.	The student is expressing an understanding of the activity.
The student asserts a not-the-conclusion-makes-the-conditions-impossible argument "eliminates the possibility of counterexamples" to the original claim.	The student is expressing an understanding of the structure of indirect reasoning.
The student justifies the validity of a *not-the-conclusion-makes-the-conditions-impossible* argument with the clause "because objects satisfying the counterexample description above are impossible."	The student is justifying indirect reasoning as a viable mode of argumentation.
The student coordinates the claim her/his indirect argument directly supports, with the original claim, and with the contrapositive form, either in her/his eliminate-counterexamples argument or in her/his justification for the validity of that argument	The student is expressing an understanding of the link between contrapositive argumentation and the eliminating counterexamples approach.

The student justifies the contrapositive proving approach by noting that the contrapositive and the original claim have the same class of counterexamples, thus an argument for one form is an argument for the other.	The student expresses a meta-theory that justifies for contrapositive proving.

Table1: Summary of student activities and learning in a progression toward viable justification of contrapositive argumentation.

Analysis

Alison's data was analyzed using the Mental Models reasoning scheme theory (Johnson-Laird, 1983) and the approaches described in Lobato et al. (2012). In particular, PICs were identified using retrospective analysis of Alison's data immediately after she completed an activity. These PICs were used to develop CLGs, which were modified as PICs were identified. Models of Alison's reasoning were developed and compared to the data until a model that fit the data emerged.

Results

Findings from the Year 1

During the interview, after the first iteration of the overarching TE, it was found that Alison could construct the contrapositive of a claim and even construct a contrapositive argument when explicitly asked to do so. Alison reported believing that the contrapositive argument "proved" the claim but offered no meta-theory that explained why a contrapositive argument should be taken as proof of a claim. Alison failed to recognized contrapositive arguments during argument critiquing activities and did not construct any arguments that she labeled as indirect or contrapositive, except as previously mentioned, when she was explicitly prompted to construct a contrapositive argument. Alison did construct indirect arguments that eliminated alternatives to the conclusion of a claim, yet, she did not refer to these arguments as "indirect" or "contrapositive" arguments. Figure 2 illustrates one of these *eliminate-alternatives* arguments that Alison constructed.

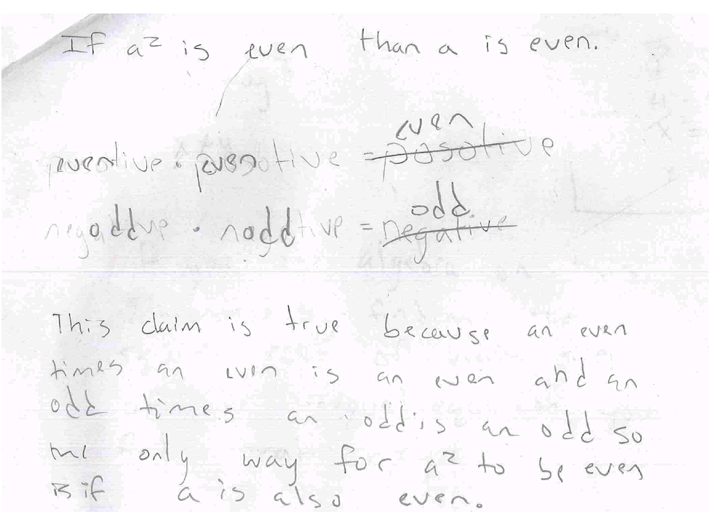


Figure 2: Alison's informal "contrapositive" argument.

In her argument, Alison eliminates alternatives to the conclusion of the claim without explicitly using a mode of argumentation such as contrapositive or contradiction proving approaches. Thus, her argument can be described as a *not-the-conclusion-makes-the-conditions-impossible* argument. To view the argument as a contrapositive argument, one would need to apply her or his own understanding of contrapositive argumentation to extract a *when-the-conclusion-is-not-met, the-conditions-cannot-be-met* argument format.

Alison's reasoning is more akin to a Mental Models reasoning scheme than a formal contrapositive-reasoning scheme. This is because Alison models alternatives to the claim's conclusion when she addresses cases of "*a is odd*" and eliminates the only possible alternative conclusion. Her reasoning is that an odd times an odd is odd, so this alternative conclusion cannot occur.

Her reasoning can also be viewed as "pragmatic" reasoning (Cheng & Holyoak, 1985) because it is based in the particular and does not explicitly appeal to any theoretical scheme such as equivalent logical forms. Moreover, she interprets her results as demonstrating that "the only way for a^2 to be even is if a is also even," and this statement can be rewritten as "a^2 is even only if a is even." This last statement is akin to a *permissive rule.* Essentially, Alison asserts that "a even" is a precondition for a^2 to be taken as even.

When asked "why her argument is viable" and "why her approach supports the claim," Alison offered no meta-level support. Her comments where restricted to "I don't know; it make sense to me" and "I can't think of anything that would disprove it." This raises concern about whether Alison can apply the approach in other contexts, and it suggests that she lacks a meta-theory for validating the approach as a mode of argumentation in general.

An assertion that Alison lacks a meta-theory for supporting this approach as a viable mode of argumentation in general is also supported by the fact that Alison explores the case "even times an even" and summaries her findings. A person who views "*eliminating-alternatives-to-the-conclusion-of-the-claim*" as a viable mode of argumentation (or "proof") should feel no need to explore and comment on cases that conform to the claim's assertions. A person with a complete conception of contrapositive proving should feel no need to summarize the findings from exploring cases that conform to the claim, as Alison does in her prose (see Alison's prose at the bottom of Figure 2).

Findings from the Year 2

After engaging in the overarching TE again and after engaging in activities that leveraged Alison's extant knowledge, conceptions, and practices to develop new conceptions of indirect argumentation (e.g., Figure 1), Alison demonstrated the ability to construct and critique contrapositive arguments. Moreover, she based these activities in a meta-theory that validated arguments that show not the conclusion makes the conditions impossible as a viable mode of argumentation because it eliminated counterexamples to either claim, the contrapositive or the original claim, and because the two forms have the same class of counterexamples. The remaining analysis presented in this paper documents that Alison expresses the *eliminating-counterexamples* conception of proving and that she uses this conception as she constructs, critiques, understands, and validates contrapositive arguments.

The Zero Product Activity

Alison was prompted with the activity: *develop a viable argument for or against the claim if $ab = 0$ then $a = 0$ or $b = 0$.* After asking some clarifying questions about the "or" in the conclusion and whether it included "and," Alison constructed the response illustrated in Figure 3.

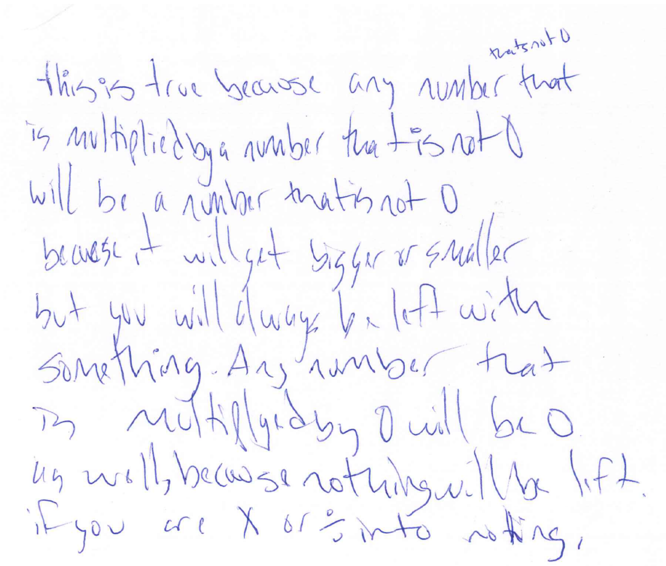


Figure 3: Alison's response to an open ended-argument activity.

When asked whether her argument is viable, Alison responsed: "Not really; it doesn't have enough evidence, and I don't have any mathematical writing, or whatever, to prove it as well." When asked about the mode of argumentation she has used, Alison replied:

> Is it, like, an indirect counterexample? Because I was sort of arguing for the contrapositive. Because I was saying that any number you're multiplying that's not zero is going to be a number that's not zero. Which is the contrapositive of that [points to the original claim].

Analysis of Alison's response to the Zero Product activity

Alison's argument expresses indirect reasoning because she is asserting that when the conclusion is not met, the conditions cannot occur. In the first sentence, Alison incorrectly describes indirect reasoning as "indirect counterexample," but this may be a vocabulary issue.

Alison expresses a meta-theory for validating her mode of argumentation when she states that she argues for the contrapositive. Her critique of her own argument is about the strength of the support she gave; it is not about her mode of argumentation. Alison's response coordinates elements of her extant conceptions (e.g., contrapositive form, indirect argument, and counterexample); however, she fails to articulate a coherent, unified (and correct) description of all of these conceptions. Thus, it is unclear whether or not Alison meta-theory is complete and adequate at this point.

Sum of Odds Activity

Alison was prompted with the following activity:

> Evaluate the argument: Claim: If a and b are odd counting numbers, then $a + b$ is even.
> Support: Let a and b be odd counting numbers. Then, a can be written as $2k+1$ and b can

be written as $2n+1$, where k and n are counting numbers. The sum $a + b = 2k+1 + 2n+1 = 2k+2n+2 = 2(k+n+1)$, which show that $a+b$ is even.

Alison gave the following critique:

> This argument is viable because they proved that a+b is equal to 2(k+n+1), which is an even number because it is being multiplied by 2… they proved without a doubt there aren't any counterexamples… if you prove something mathematically in a way that shows that all cases where this is true meets the conditions and the conclusion of the claim, then the argument is always true.

Analysis of Alison's response to the Sum of Odds activity

Alison asserts that the argument is viable because it eliminates counterexamples by showing that all cases meet the conditions and the conclusion of the claim. Here, she applies the conception of eliminate counterexamples to the a direct argument. This is the first instance in which Alison expressed the *eliminating-counterexample* conception of viable argumentation in the context of direct argumentation.

Squaring Negatives Activity

Alison was prompted with the following activity:

> Evaluate the argument: Claim: If $x < 0$, then $x^2 > 0$. Support: If $x < 0$, then $–x > 0$. In other words, $-x$ is a positive number. Because the product of two positive numbers is also positive, $(-x)(x) > 0$.

Alison gave the following critique:

> [The argument is] viable because they proved that there aren't any counterexamples by proving that it's true for all cases. I'm not sure what they meant by this part, which is, like, x is a positive number… . If x is less than zero, then… negative x is greater than zero… That's saying that if x is less than zero, then that means that it's a negative number. So, the negative of that number would make it a positive number… I'm not sure why this is necessary…

Analysis of Alison's Response to the Squaring Negatives activity

Alison asserts that the argument is viable because it eliminates counterexamples proving the claim is true for all cases. Alison does not appear to fully understand the argument, and she appears to view it as addressing cases, when x is positive and when x is negative. Alison's concerns are about the notation in the argument and why an intermediate step is needed. These are not meta-level concerns. Alison offers no information about whether or not she views the argument as direct or indirect.

Unequal Squares Activity

Alison was prompted with the following activity:

> Evaluate the argument: Claim: Let x be any real number. If x^2 is not equal to y^2, then x is not equal to y. Support: Let $x = y$. Then $x^2 = y^2$.

Alison gave the following critique:

> This argument is not viable because the support is not proving the original claim. They have just written the contrapositive and provided no explanation. … They haven't really explained why ... [The argument could be viable] since it's the contrapositive, if you've proven that then you've also proved the original claim… because they have the same kind of counterexamples. … If you prove then the contrapositive, then you've proved that … the original claim doesn't have counterexamples.

Analysis of Alison's Response to the Unequal Squares Activity

Alison's concerns are with the argument's details and the explanation given. These are not meta-level concerns. At a meta-level, Alison tells us that proving the contrapositive is a viable approach (mode of argumentation). She asserts that such a proof shows that "the original claim doesn't have counterexamples." She supports this mode of argumentation with an assertion that the contrapositive and original claim have the same class of counterexamples. These statements are taken as meta-level statements because they argue that the mode of argumentation is viable.

Even Squares Activity

Alison was prompted with the following activity.

> Evaluate the argument: Claim: Let n be a counting number. If n^2 is even, then n is even. Support: Let n be an odd counting number. Then n can be written as $2k+1$, where k is an integer. Squaring, $n^2 = (2k+1)^2=4k^2+4k+1=2(2k^2+k)+1$.

Alison gave the following critique, failing to recognize the contrapositive structure of the argument:

> I said that the argument wasn't viable… They were proving that the – what's it called again? The converse. Because, they were saying that if n was an odd number, then n squared is an odd number. Even if they proved the converse, that doesn't necessarily mean that they proved the claim because they don't always have the same set of counterexamples.

After Alison responded to all planned assessment activities, I asked her to revisit the claim in the Even Squares activity. The purpose of doing this was to better understand her incorrect response and what conceptions and misconceptions it expressed.

I asked Alison to construct a description of possible counterexamples to the claim. After Alison developed a correct description of the counterexamples, the following exchange occurred:

> Yopp: Could you… convince me that there are no objects like this [pointing to Alison's description of counterexamples]?
>
> Alison: … So, I'd have to prove that any odd number[s] multiplied together would have to be odd.

I nodded, and Alison constructed the expression (2k+1)(2b+1) and attempted to expand it. Alison needed assistance completing the multiplication correctly, but once the expression was expanded correctly, Alison collected like terms and constructed the equivalent expression 2(2bk+b+k)+1. I asked Alison to reflect on her earlier response to the Even Squares activity. Below is an excerpt from our exchange:

> Alison: The argument is viable… because they're proving that there can't be a counterexample by saying that any odd counting number multiplied together is going to

be odd. And, that would be… one of the counterexamples would be if an odd number was squared to be even.
Yopp: What changed about your assessment? …
Alison: Um. I don't know. I guess the first time I just didn't think about what the counterexample would look like. …
Yopp: So when you validate an argument, you usually look for what? Earlier… before…
Alison: Earlier I was looking if it was, like, directly affecting this.
Yopp: How do you think [describing the CEs] helped you look for these indirect arguments? …
Alison: I guess that I should figure out what the counterexample would look like and see if the support is proving that instead of the original claim.
Yopp: Proving? What do you mean by –
Alison: Proving whether or not there could be a counterexample.

Analysis of Alison's response to the Even Squares activity

It was disappointing that Alison failed to recognize the indirect argument as a viable mode of argumentation on her first assessment. However, this article does not assert that the TE is sufficient for students to develop proficiency in constructing and recognizing contrapositive arguments. This article is concerned with the conceptions of indirect reasoning Alison expressed at certain time points and whether the conceptions were useful to her in constructing, critiquing, understanding, and validating contrapositive arguments.

A positive aspect of Alison's first response to the task is that even though she mistaken labels the argument as a converse argument, she critiques that mode of argumentation as not viable because it does not eliminate counterexamples. She offers a meta-level argument for her critique: noting that in general, conditional claims and their converses do not share the same collection of counterexamples.

In Alison's second response to the task, Alison expressed a conception of eliminating counterexamples by showing not the conclusion makes the conditions impossible. However, Alison required considerable prompting and scaffolding to invoke the *eliminating-counterexample* practices. Yet, after being prompted to do so, Alison constructed a general description of the counterexamples and leveraged this description to construct a contrapositive argument. She then corrected her earlier, incorrect assessment of the contrapositive argument presented to her. Her final comment—"proving whether or not the could be a counterexample"—suggests that she reflected on her habit of look for direct arguments and laments not "figure out what the counterexample would look like and see if the support is proving that instead of the original claim."

Discussion

Indirect arguments occur naturally in eighth-grade curriculum. Indirect argumentation is within the conceptual reach of adolescents. Interventions for improving eighth-grade's indirect reasoning in mathematics content are sparse. This study provides evidence that a conception of proving as *eliminating counterexamples* and an associate practice of describing all possible counterexamples can be useful to adolescents' when constructing, critiquing, and validating contrapositive arguments. As Allison demonstrates, the conception of proving as eliminating counterexamples can be used as meta-level support for contrapositive argumentation. This meta-level support can be in the form of an acknowledgement that a contrapositive argumentation eliminates counterexamples by showing that when the conclusion does not occur the conditions are impossible. Or, it can be in the form of an acknowledgement that a conditional claim and its contrapositive have the same class of counterexamples.

Future studies might explore the utility of the eliminate-counterexamples conception and the practice of describing the collection of all possible counterexamples as tools for improving reasoning with a larger group of students. Future students might also explore ways to encourage students to look for indirect reasoning by determining if the argument eliminates counterexamples by showing not the conclusion implies not the conditions. While Alison reported that she should adopt the approach of checking whether or not the argument addresses the existence of counterexamples direction in her future critiques of arguments, no evidence was collected to determine if she did.

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