

REFINEMENT OF THE CONCEPTUAL MODELS FOR INTEGER ADDITION AND SUBTRACTION

Nicole M. Wessman-Enzinger
George Fox University
nicole.enzinger@georgefox.edu

Investigations highlight that children are capable of sophisticated thinking about integers (e.g., Bofferding, 2014; Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014). There increased interest in the realm of understanding children's thinking about integers (e.g., Lamb et al., 2013). As we learn more about student thinking about integer addition and subtraction, there is an evident dichotomy in what we know about student thinking: research on student thinking about integers in contexts (e.g., Wessman-Enzinger & Mooney, 2014; Whitacre et al., 2012a, 2012b, 2015) and research on student thinking about symbolic problems, like integer addition and subtraction open number sentences (e.g., Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014). This research brief describes a study that aimed to address this dichotomy by investigating student thinking in both contextual and symbolic settings. Through this investigation, descriptions of student thinking initially generated from how students thought about integers in contexts (Wessman-Enzinger & Mooney, 2014 needed expansion and refinement to include how students thinking about solving integer addition and subtraction open number sentences.

Theoretical Perspective

Secondary Intuitions

As Fischbein (1987) showed, some concepts do not seem to emerge naturally out of intuition. Fischbein distinguished between primary and secondary intuitions. Primary

intuitions are those that emerge naturally and from personal experiences. Secondary intuitions are those that emerge from instructional influences. The evidence that negative numbers are not a primary intuition naturally emerging without the assistance of instructional experiences is found both from historical perspectives (e.g., Gallardo, 2002; Henley, 1999; Heeffner, 2011) and literature on student thinking (e.g., Mukhopadhyay, Resnick, & Schauble, 1990; Whitacre et al., 2012, 2015).

Conceptual Models for Integer Addition and Subtraction

The Conceptual Models for Integer Addition and Subtraction (CMIAS) were developed to describe student thinking about integers when using addition and subtraction. The five CMIAS were (i.e., Bookkeeping, Counterbalance, Translation, Relativity, Rule) were developed from a study where students posed stories for open number sentences (Wessman-Enzinger & Mooney, 2014). *Bookkeeping* describes a utilization of integers as gains and losses. *Counterbalance* describes a neutralization use of integers. *Translation* describes the use of integers as a shift or movement. *Relativity* describes the use of the integers in relative positions or orderings with an unknown referent. *Rule* describes the use of algorithms used with the integers.

Research Question

The research question for the larger study was focused on describing students' use and learning about of various conceptual models for integer addition and subtraction:

In what ways do fifth-grade students use conceptual models of the integers (e.g., Bookkeeping, Counterbalance, Translation, Relativity, Rule) as they (a) attempt to make sense of the integers and (b) learn about integer addition and subtraction?

In order to answer this research question, the current CMIAS needed to be refined, accounting for the ways that students reason in both context and symbolic settings. Thus, a subsequent research question, and focus of this research brief, became:

How did examining the use of students' responses for solving integer addition and subtraction open number sentences influence the refinement of the CMIAS?

Thus, the aim of this research brief is to describe the decisions and changes in the CMIAS descriptions.

Methods

Three Grade 5 students (i.e., Alice, Jace, Kim) from a rural Midwest school participated in a 12-week teaching experiment (Steffe & Thompson, 2000). Grade 5 participants were selected to allow for the instructional experiences within this study to be these students' first instructional experiences with negative integers while also being as close to the NGA and CCSSO (2010) recommended instructional age for integer instruction. The students were selected using an assessment tool that was developed after a pilot study with 131 Grade 5 students (Wessman-Enzinger, 2015). A key element of teaching experiment methodology is that generalizability occurs in the time spent with the children. This was an important decision made early in the study to account for investigating the learning of the students.

Across the 12-weeks the students participated in four Individual Open Number Sentence Sessions where they solved open number sentences (see Figure 1). The students' responses (i.e., verbal responses and all drawings) when they solved each open number sentence (see Figure 2) was considered the unit of data. In total, there were 279 units of data from the Individual Open Number Sentences.

Jace Individual Context Session 1	Alice Individual Context Session 1	Kim Individual Context Session 1
Jace Individual Open Number Sentence Session 1	Alice Individual Open Number Sentence Session 1	Kim Individual Open Number Sentence Session 1
Group Session 1		
Group Session 2		
Group Session 3		
Jace Individual Context Session 2	Alice Individual Context Session 2	Kim Individual Context Session 2
Jace Individual Open Number Sentence Session 2	Alice Individual Open Number Sentence Session 2	Kim Individual Open Number Sentence Session 2
Group Session 4		
Group Session 5		
Group Session 6		
Jace Individual Context Session 3	Alice Individual Context Session 3	Kim Individual Context Session 3
Jace Individual Open Number Sentence Session 3	Alice Individual Open Number Sentence Session 3	Kim Individual Open Number Sentence Session 3
Group Session 7		
Group Session 8		
Group Session 9		
Jace Individual Context Session 4	Alice Individual Context Session 4	Kim Individual Context Session 4
Jace Individual Open Number Sentence Session 4	Alice Individual Open Number Sentence Session 4	Kim Individual Open Number Sentence Session 4

Figure 1. Structure of teaching experiment.

Individual Open Number Sentence Session 1	Individual Open Number Sentence Session 2	Individual Open Number Sentence Session 3	Individual Open Number Sentence Session 4
$-20 + 15 = \square$	$-16 + 4 = \square$	$-18 + 12 = \square$	$-20 + 15 = \square$
$12 + -16 = \square$	$20 + -33 = \square$	$15 + -24 = \square$	$12 + -16 = \square$
$-4 + \square = 10$	$-6 + \square = 15$	$-3 + \square = 14$	$-4 + \square = 10$
$-7 + \square = -2$	$-6 + \square = -1$	$-9 + \square = -3$	$-7 + \square = -2$
$\square + -3 = 7$	$\square + -2 = 17$	$\square + -4 = 13$	$\square + -3 = 7$
$\square + 13 = -5$	$\square + 19 = -4$	$\square + 25 = -2$	$\square + 13 = -5$
$-8 + -7 = \square$	$-12 + -5 = \square$	$-17 + -6 = \square$	$-8 + -7 = \square$
$-2 + \square = -10$	$-4 + \square = -19$	$-5 + \square = -21$	$-2 + \square = -10$
$\square + -9 = -16$	$\square + -9 = -21$	$\square + -9 = -17$	$\square + -9 = -16$
$10 - 12 = \square$	$5 - 9 = \square$	$12 - 18 = \square$	$10 - 12 = \square$
$1 - \square = 3$	$4 - \square = 6$	$3 - \square = 4$	$1 - \square = 3$
$-5 - 4 = \square$	$-9 - 8 = \square$	$-5 - 3 = \square$	$-5 - 4 = \square$
$2 - -3 = \square$	$3 - -4 = \square$	$1 - -3 = \square$	$2 - -3 = \square$
$-1 - \square = 8$	$-2 - \square = 9$	$-2 - \square = 10$	$-1 - \square = 8$
$2 - \square = -10$	$6 - \square = -10$	$4 - \square = -12$	$2 - \square = -10$
$\square - -1 = 6$	$\square - -1 = 4$	$\square - -2 = 5$	$\square - -1 = 6$
$\square - 8 = -5$	$\square - 9 = -3$	$\square - 6 = -2$	$\square - 8 = -5$
$-15 - -4 = \square$	$-11 - -2 = \square$	$-12 - -4 = \square$	$-15 - -4 = \square$
$-12 - \square = -13$	$-15 - \square = -16$	$-10 - \square = -11$	$-12 - \square = -13$
	$\square - -3 = 2$	$\square - -3 = 1$	$\square - -2 = 1$
	$\square - -4 = 0$	$\square - -5 = 0$	$\square - -3 = 0$
	$12 + \square = 8$	$15 + \square = 9$	$17 + \square = 8$
		$8 + \square = -5$	$6 + \square = -2$
		$\square + 2 = 0$	$\square + 4 = 0$
		$-4 - 10 = \square$	$-2 - 8 = \square$

Figure 2. Open number sentences students solved during Individual Open Number Sentence Sessions.

Constant comparative methods (Merriam, 1998) were selected as an analytic tool to refine the CMIAS because the CMIAS already had initial descriptions and refinement or new possible CMIAS were being explored. Each of these units of data was first coded individually by two researchers with the initial CMIAS descriptions (Wessman-Enzinger & Mooney, 2014) and the option to code Other. Then, each of these units, in their respective groups (e.g., Bookkeeping, Rule), was examined and compared to the original descriptions. For example, each of the units that were coded as Translation was compared to the initial definition of Translation. If there were multiple units of data supported an attribute not present in the initial CMIAS definition, the definition was modified.

Overview of Results and Conclusions

The original CMIAS definitions (Bookkeeping, Counterbalance, Translation, Relativity) were refined to better accommodate how students' solved integer addition and subtraction open number sentences. Rule was expanded to Proceduralization, Analogy, and Algebraic Reasoning. This research brief will report on the decisions and the data that supported these refinements.

Example: The Refinement of Translation

Approximately 30% of the students' responses to the Individual Open Number Sentences (each coded as a unit of data) exhibited used of the Translation model of thinking. However, the original description of this model did not fully capture the themes present in the students' responses. Based on my analysis of the data, four modifications were made to the original description (see Table 1).

Table 1

Original and Refined Translation CMIAS

Original CMIAS Description from Wessman-Enzinger & Mooney (2014)	Refined CMIAS Description
<p>A conceptual model of <i>Translation</i> may be used if integers are treated as vectors, or directed numbers. With this model, integers are used to shift any kind of mathematical object. The zero may represent a zero vector, or no movement. Similar to relativity, the zero can also represent a relative position, with positive and negative numbers representing a movement in one direction or another from the relative zero. (p. 204)</p>	<p><i>Translation</i> is utilized when the integers are treated as vectors or translations. With Translation, the integers are used in ways that shift any kind of mathematical object (e.g., a number, a point, a curve). With Translation, the integers are often treated as vectors moving right or left or up and down a linear model, coordinate plane, or three-dimensional space. The zero in Translation is a zero vector or a translation of no movement. Similar to Relativity, the zero can also represent any arbitrary point with the addition and subtraction of positive and negative numbers representing the Translation in one direction or another from the relative zero.</p> <p>Also, movement and directed distances, or distances with direction, are considered to be Translation. However, sometimes distance may be used without direction. Although it is possible to conceptualize distance without direction, it is still considered to be drawing upon Translation because all distance may be considered as directed. Translation may also be employed with the use of counting strategies because counting fundamentally utilizes movement and order. The distinguishing features of Translation when compared to other models are the idea of order and directed movement.</p>

Clarification of wording. In 17 of the 85 student responses coded as Translation, student used Translation and empty number lines where the negatives were either left or right on a horizontal number line, or up or down on a vertical number line. Students made flexible translations along their number lines (adding sometimes being left, right, up, or

down). The definition was refined to incorporate the students' flexible use of movement left, right, up, or down.

Movement as a directed vector. In 67 of the 85 units that were coded Translation, there was evidence of “movement,” which was not previously clear in the definition. For example, Kim referenced this movement as she used Translation, “And, it was so strong it blew past negative nine (waves hand to the right), it blew past zero, and it stopped at 8.”

Distance without a clear direction. In 7 units of the 85 units coded as Translation, there was use of distance without a clear direction. For example, when Jace solved $-6 + \square = 15$, he drew a number line with two distances (see Figure 3). He first drew a distance from -6 to 0 and then a distance from 15 to 0. In this sense, Jace did not have a clear, singular “directed” vector. Instead, he had a directed distance going left to right and a second directed distance going right to left towards zero. He did not have a clear singular directed vector from -6 to 15. However, this type of response was related to Translation since Jace added the distances of 6 and 15 to determine the solution of 21, and he created two distances that were directed towards zero.

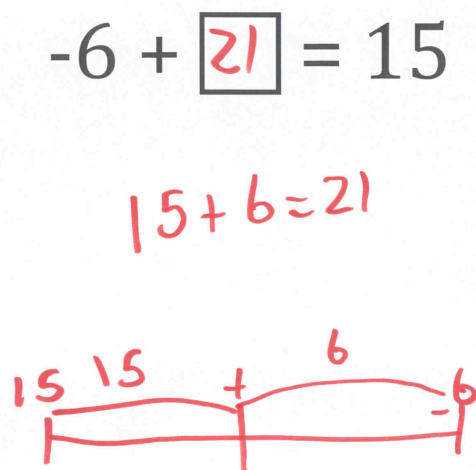


Figure 3. Jace's drawing for solving $-6 + \square = 15$.

Counting strategies. In 18 units out of 85 units of data coded as Translation, counting was used to enact a translation. For example, Alice used counting as she also solved $-6 + \square = 15$ (see Figure 4). She counted:

Well, I did six lines at first representing negative six. Then I did six, five, four, three, two, one (points at each tally mark), zero, one two, three four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen (continues pointing at each tally). And, then I had sixteen lines.

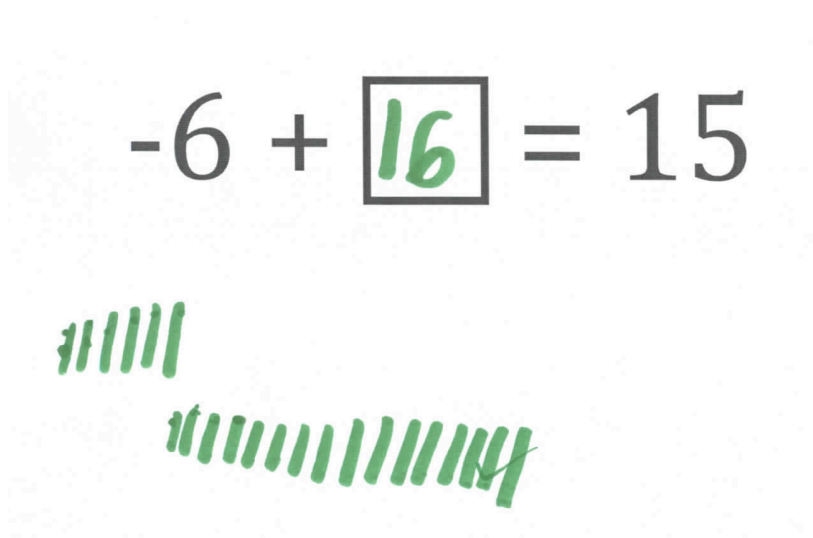


Figure 4. Alice's drawing for solving $-6 + \square = 15$.

Significance of the Research

The CMIAS provides a theory for describing student thinking derived from from both posing stories and solving open number sentences. The refinement of these models with descriptors for thinking extends the previous literature on student thinking about integers because it provides a framework on student thinking about integers connected to contextual and symbolic reasoning. Furthermore, this research brief provides a perspective into the refinement process of developing models for student thinking. This has implications for design of future tasks and research.

Acknowledgements

The Illinois State University Dissertation Completion Grant supported this research.

References

- Bofferding, L. (2014). Negative integer understanding: Characterizing first graders' mental models. *Journal for Research in Mathematics Education*, 45(2), 194–245.
- Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., Schappelle, B. P., & Lewis, M. L. (2014). Obstacles and affordances for integer reasoning: An analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education*, 45(1), 19–61.
- Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., & Schappelle, B. P. (2014). Using order to reason about negative numbers: the case of Violet. *Educational Studies in Mathematics*, 86(1), 39–59.
- Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht, The Netherlands: D. Reidel Publishing Company.
- Gallardo, A. (2002). The extension of the natural-number domain to the integers in the transitions from arithmetic to algebra. *Educational Studies in Mathematics*, 49, 171–192.
- Heeffer, A. (2011). Historical objections against the number line. *Science & Education*, 20(9), 863–880.
- Henley, A. T. (1999). *The history of negative numbers*. Unpublished doctoral dissertation. South Bank University.
- Lamb, L., Bishop, J., Philipp, R., Whitacre, I., Stephan, M., Bofferding, L., Lewis, J., Brickwedde, J., Bagley, S., & Schappelle, B. (2013). Building on the emerging knowledge base for teaching and learning in relation to integers. In M. Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1362–1366). Chicago, IL: University of Illinois at Chicago.
- Mukhopadhyay, S., Resnick, L. B., & Schauble, L. (1990). Social sense-making in mathematics; Children's ideas of negative numbers. In G. Booker & P. Cobb (Eds.), *International Group for the Psychology of Mathematics Education Proceedings of the Fourteenth PME Conference* (Vol. 3, pp. 281–288). Oaxtepec, Mexico: Program Committee of the 14th PME Conference.

National Governors Association Center for Best Practices. Council of Chief State School

- Officers. (2010). *Common core state standards for mathematics*. Washington DC: Author. Retrieved from <http://www.corestandards.org/the-standards>.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education*, (pp. 267–306). Mahwah, NJ: Lawrence Erlbaum Associates.
- Wessman-Enzinger, N. M. (2015). *Developing and describing the use and learning of Conceptual Models for Integer Addition and Subtraction of grade 5 students*. Normal, IL: Proquest.
- Wessman-Enzinger, N. M., & Mooney, E. S. (2014). Informing Practice: Making sense of integers through story-telling. *Mathematics Teaching in the Middle School*, 20(4), 202–205.
- Whitacre, I., Bishop, J. P., Lamb, L. L. C., Philipp, R. A., Schappelle, B. P., & Lewis, M. (2012a). Happy and sad thoughts: An exploration of children's integer reasoning. *The Journal of Mathematical Behavior*, 31, 356–365.
- Whitacre, I., Bishop, J. P., Lamb, L. L. C., Philipp, R. A., Schappelle, B. P., & Lewis, M. (2012b). What sense do children make of negative dollars? In L. Van Zoest, J. Lo., & J. Kratky (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 958–964). Kalamazoo, MI: Western Michigan University.
- Whitacre, I., Pierson Bishop, J., Lamb, L. L., Philipp, R. A., Bagley, S., & Schappelle, B. P. (2015). Negative of my money, positive of her money: Secondary students' ways of relating equations to a debt context. *International Journal of Mathematical Education in Science and Technology*, 1–16.