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Author(s): Elise Lockwood
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Eliciting Student Understanding of Mathematical Aspects of the Multiplication Principle

Elise Lockwood
Oregon State University

Introduction

Counting problems foster rich mathematical thinking, (e.g., Kapur, 1970; Martin, 2001; Tucker, 2002) and have important applications in fields like probability and computer science. However, correctly solving counting problems is challenging, and many studies report on students' difficulties with counting (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Eizenberg & Zaslavsky, 2004). Given these difficulties, there is a need for studies that investigate particular aspects of students' counting, including how they reason about central combinatorial ideas. The multiplication principle (MP) is a fundamental aspect of combinatorial enumeration. It is generally considered to be foundational to many of the counting formulas students learn and is called by some "The Fundamental Principle of Counting" (e.g., Richmond & Richmond, 2009). The MP can also be a much-needed source of justification for why many common counting formulas (such as the formulas for factorials, permutations, and combinations) work as they do. In spite of its importance, little has been studied about the MP in and of itself. As part of a larger goal of understanding student thinking about the MP, we had two undergraduate students reinvent a statement of the MP. In this paper, we describe a couple of subtle mathematical issues that is entailed in the MP (the independence of stages in a counting process and the need for distinct composite outcomes), and we report on a particular task that helped students address these issues as they reinvented a statement of the MP. We seek to address the following research question: *What tasks facilitated student reasoning about key mathematical issues in the MP, and how do students reason about these issues?*

Literature Review and Theoretical Perspectives

Previous work (Lockwood, Reed, & Caughman, 2015) has demonstrated the importance of the MP in counting, and the lack of a well-developed understanding of the MP appears to be a significant problem and hurdle for the students, particularly in terms of their ability to justify or explain formulas. We have found that students can easily assume that they completely understand the MP in counting because multiplication is a familiar operation for them. As a result, they use the operation frequently but without careful analysis, and they tend not to realize when simple applications of the operation are problematic. While some researchers have discussed multiplication within combinatorial contexts (Lockwood, et al., 2015; Tillema, 2013), there have not yet been studies that explicitly target student understanding of the MP.

We recently conducted a textbook analysis that explored statements of the MP in university combinatorics and discrete mathematics textbooks (see Lockwood, et al., 2015 for details). This study revealed a handful of mathematical issues that are important for statements of the MP to include. We highlight two of these key issues in the MP – we briefly mention them here and provide more details in the Results section.

First, there is the notion of *independence* of stages in the counting process, which captures the idea that a choice of options at a given stage does not affect the number of outcomes in any subsequent stage. Tucker (2002) provides a thorough statement of the MP (Figure 1) and addresses the issue of independence when he says "the number of outcomes at each stages is independent of the choices in the previous stages."

The Multiplication Principle: *Suppose a procedure can be broken down into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, ..., and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and if the composite outcomes are all distinct, then the total procedure has $r_1 \times r_2 \times \ldots \times r_m$ different composite outcomes.*

Figure 1: Tucker's (2002) statement of the MP

Independence is a necessary condition in order to apply the MP, or else overcounting may occur. To see this, we can consider to the Cards problem: *How many ways are there to pick an ordered pair of cards from a standard deck (without replacing the first card) so that the first card is a face card and the second card is a heart?* The following explanation highlights how non-independent stages might result in overcounting. One way to solve the problem is to specify two stages: first pick a first card, and then pick a second card. Here, the number of ways to pick the second card is not independent of the first choice, because if a face card that is also a heart is drawn first, then the number of hearts decreases by 1 in the second stage of the process. If, however, a face card that is not a heart is drawn first, then the number of hearts does not decrease by 1 in the second stage. A student who applies the MP in such a case, without making sure the number of ways to complete the steps are independent, could get that there are 12×13 = 156 outcomes, when in fact there are only 153 outcomes. Thus, an important mathematical issue that a statement of the MP should address is that stages in the counting process must be independent. Multiplying when the stages are not independent may result in an incorrect answer to a counting problem.

Another mathematical issue that arose from the textbook study is that the MP must yield *distinct composite outcomes*, which means that when applying the MP we want to ensure that there are no duplicate outcomes. This qualification, too, prevents instances of overcounting. To see this, we can consider a problem like the 3-Letter Sequences problem (from Tucker, 2002). This problem states: *How many 3–letter sequences made from the letters a, b, c, d, e, f must* *contain the letter e, where repetition of letters is allowed.* A tempting approach to this problem is to argue that there are 3 options for where to place an *e* in the sequence, and then once that happens there is no more restriction. Because an *e* is guaranteed to be in the password, and because repetition is allowed, there are then 6 options for the next available position, and 6 more options for the last available position. This yields an answer of 3*6*6. However, note that this actually overcounts, as some sequences are generated more than once by this counting process. For example, the sequence *eae* is counted once when *e* is placed in the first position, then the sequence is filled out with *ae* in the 6*6 stages, but it is counted again when *e* is placed in the third position, and *ea* is the result of the 6*6. Thus, this problem highlights that if we are not careful about ensuring that our multiplication does not generate *distinct* outcomes, we may overcount. In the Results section we highlight how students interacted with counting problems that demonstrate the need for each of these mathematical issues in statements of the MP.

Prior research (e.g., Larsen, 2013; Lockwood, Swinyard, & Caughman, 2015; Oehrtman, Swinyard, & Martin, 2014; Swinyard, 2011) suggests that having students *reinvent* (i.e., construct for themselves) mathematical definitions or formulas facilitates coherent reasoning for students. In the same vein, we chose to have students reinvent a statement of the MP through generalizing their work on a number of counting problems, with the goal of gaining insight into how students might reason meaningfully about the MP.

Methods

Data collection. We conducted an eight-session teaching experiment (e.g., Steffe & Thompson, 2000) in which a pair of undergraduate students solved a number of counting problems over eight hour-long sessions. The students were enrolled in vector calculus in a large university in the western United States, and they were chosen because they had not been explicitly taught about the MP in their university coursework (and thus would not simply try to recall it). The interviews took place outside of class time over a period of four weeks. Broadly, in Sessions 1-2 the students solved a series of counting problems, and they were asked to write down and characterize when they were using multiplication as they solved these problems. Then, in Sessions 3-7 they wrote down several iterations of statements of the MP as they solved more counting problems. (In Session 8, which we do not discuss here, they evaluated existing statements of the MP). Throughout the study the interviewer selected tasks to highlight various aspects of the MP and regularly asked clarifying questions.

Data analysis. The interviews were videotaped and transcribed, and overall the videos and transcripts were analyzed so as to construct a narrative about the teaching experiment (Auerbach & Silverstein, 2003). We used prior understanding of the MP that had emerged from the textbook analysis to guide our focus of particular mathematical issues. For this paper, key episodes involving independence and distinct composite outcomes were flagged and reviewed, and we scrutinized the students' statements of the MP and their explanations for insights about their reasoning.

Results

In this section we detail the students progress on both of the subtle mathematical issues introduced above – independence and distinct composite outcomes. In doing so, we highlight particular tasks that were useful in eliciting discussion about these mathematical issues for the students. The point is to highlight student reasoning about these ideas and the instructional moves that brought them to light.

Independence. As noted above in the mathematical discussion section, independence of stages in a counting process is a key aspect of a statement of the MP (especially an operational or a bridge statement). This came up for students in Problem 1.4, which states, *You have 4 different Russian books, 5 different French books, and 6 different Spanish books on your desk. In how many ways can you take two of those books with you, if the two books are not in the same language?*. Note that in this problem, one cannot simply consider that there are just 15 options for the first book in the pair (from 4+5+6 = 15 total books), because subsequent books might depend on what language the first book was. To handle this, a correct solution may break down the problem into cases according to which pair of languages are being chosen. This is exactly what happened for the students. They started to reason about whether they could just consider 15 options for which book is first in the pair, but they soon realized that they could not simply multiply 15 by anything to get the answer. As the exchange below shows, they became aware of the issue of independence. In order to fix this, they broke the problem into cases according to which language book was first, arriving at the correct answer or 4*5 + 4*6 + 5*6 = 74).

Int.: Um so why... so I think I don't, I mean I think your idea of like the fifteen options for the first one, why did that break, which I think is a good idea, but what, what about the problem made it so you couldn't just do like fifteen times something?
Pat: Um because the, whatever you select for the first one, then determines what kind of book you can select for the next one.
Int.: Okay.
Pat: So there are, technically there are fifteen options for the first book, but you have no way of knowing if they selected a Russian, French, or Spanish. And so then you don't know if you have four times five, or five times six, or six times four as your next option.
Caleb: Yeah.

In a subsequent session, the students worked on the Cards problem. First, the students articulated an issue with dependence, and they realized that the answer could not just be 12 (for the number of face cards) times 13 (for the number of hearts).

Pat: How many ways are there to pick two different cards from a standard 52 deck such that the second card is heart. Ok well in that case.
Caleb: Oh yeah then 12.
Pat: Yeah there are 12 options for your face card.

Caleb: Yeah and then you'd multiply that by how many hearts there are which is 12, saying that your first one was a face card.
Pat: So there's 13 or 12 options depending on if the first one's a heart or not.
Caleb: 13, oh yeah.
Pat: So.
Int.: Say that again, what are you, you said there's 13 or 12 depending on what?
Pat: Depending on if the face card you pick is a heart face card.
Int.: Okay.

The students thought for a bit, with Caleb articulating that this was a difficult task. They then realized that they can separate out the two possible situations for the first card, and Pat noted that they could add them.

Caleb: Well I think we can just say that, oh gosh, this is hard. Oh we could do if there is no heart and then if we could do if there is a heart, so we could just have two different ones and then multiply them.
Pat: Or add them.
Caleb: So saying that this is, this is non hearts right? So we have 8 times 13, plus 4 times 12.
Pat: Okay.
Caleb: Right?

We had some clarification about whether or not an Ace is a face card, and they decided that if it was not a face card, they needed to change their answer slightly. They arrived at the answer in Figure 2.

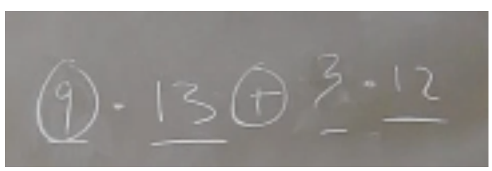

Figure 2 – The students' answer to the Cards problem

We asked them to follow up and explain their reasoning, and they were able to articulate the need for independence if they are to multiply. We then asked them to explain why they are using multiplication and addition in their expression. The following exchange sheds further light on how they are thinking about multiplication.

Int.: Cool. Can you talk about why you're multiplying and adding when you are in that expression?

Pat: Ok. So these ones are, these individual components here (points to both expressions on either side of the plus sign, Figure 2), are these ones are groups. so in this case you would have selections that could have, for each of these selections (points to the 9) you'll have 13 corresponding selections (points to the 13) that can go with that, which makes groups, which is multiplication, groups of things added together. Cause multiplication really at it's heart is kind of a fancy way of doing addition. So that's this is the way to add nine groups of 13 options for every one of those nine. And this is the same thing, 12 groups for each one of these three. So that's why you have multiplication. And you add these together because once you have all of those possible selections and all of these possible selections, there's counting together as components of the total. You wouldn't multiply them together because you're not gonna say for every one of these selections there's an, this number of selections for each of those. Cause that would, we're not selecting four cards, we're selecting two cards.

Int.: Ok. Ok nice. Yeah. Good, anything to add to that?

Caleb: No, not really. I just think cause they are two separate components of the same problem, and they're generally independent of each other, so then that they come to the same conclusion you add them, but other than that.

By Session 6, then, the students had spent time refining statements, and they had come up with the following statement for the MP (See Figure 3): "For every selection towards a specific outcome, if one selection does not affect any subsequent selection then you multiply the number of all the options in each selection together to get the total number of possible outcomes." Notice that they are attempting to address independence with the phrase "if one selection does not affect any subsequent selection."

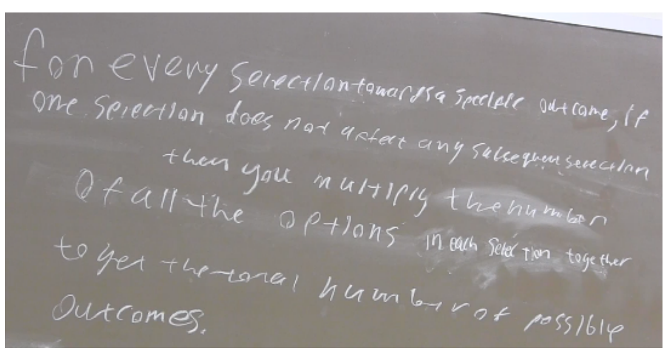

Figure 3 – The Students' statement of the MP

However, there is an even more subtle issue that is related to independence – namely, it is not that the stages themselves must be independent, but rather that the number of options at each stage must be independent. As stated, the statement in Figure 3 is not quite accurate, because multiplication still applies if the actual options change from stage to stage. Instead, it is the *number* of options that cannot change (as the following episode demonstrates).

To draw the students' attention toward this issue, we presented them with the following problem: *How many 6-character license plates consisting of letters or numbers have no repeated character?* They immediately recognized that they could use multiplication on this problem – they knew the answer would be 36*35*34*33*32*31, and they had the following exchange as they tried to justify the answer.

Pat: It's still multiplication, but it's not the same as multiplication that we were thinking of. So as we, it has to change things now doesn't it?
Caleb: That one definitely put a damper on our –

They then thought about the problem a bit more, and Pat had the following realization:

Pat: I'm just concerned about the idea that we're saying, that the selection is affecting the next selection, because technically in this case, the selections…affect subsequent selections, but it still is multiplication…

Pat saw that their wording would not allow for the multiplication in a problem like the License Plates problem, because their statement in Figure 2 was too restrictive about how subsequent selections might be affected. The two students then had the following exchange (the A and 9 refer to choices for a license plate character):

Caleb: Maybe you pick like A and then 9, and you can't do either one of those again.
Pat: Yeah but it's like A or 9 won't affect the number of next selections. 'Cause no matter what it's gonna be 36, 35, 34, 33…See what I'm saying? So like how do we incorporate that? Because like, so as long as it… – as long as the next, the subsequent selections still have the same number of selections, it's okay. But if like the first selection changes the number of options…in the second or in the subsequent selection.

The students then proceeded to write down language that might help address this issue (Figure 4): "…does not cause a difference in the number of options in any subsequent selection." Note they are now emphasizing the *number* of options to which they had not previously attended.

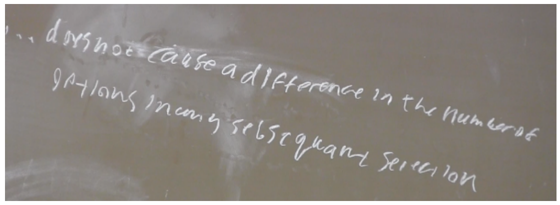

Figure 4: The students' edited statement about independence

This exchange provides evidence of how students reason about a subtle and important mathematical issue in the MP. By having students reinvent a statement of the MP, and by closely analyzing aspects of multiplication to which they attend, we gain insight both into how students can and do reason about the MP, but also how productive reasoning about the MP might be developed. The Cards task was chosen to emphasize the need for independent stages, and the License Plate task was carefully chosen to refine an already-existing idea – that the *number* of options (not the options themselves) must be independent.

Distinct composite outcomes. As noted above, another key mathematical issue is the need to count distinct composite outcomes, which helps to avoid overcounting. The issue of overcounting first arose for students as they solved some initial problems in Session 1. They tried to solve the Rooks problem, which states *How many ways are there to place two different-colored rooks in a common row or column of an 8x8 chessboard?* In solving this problem, the students figured out that there were 64 positions for the first rook, and then 14 positions for the second rook to be in the same row or column. They also multiplied by 2 to account for the different colored rooks, depending on which rook was placed first. Unfortunately, this additional multiplication by 2 is not necessary and leads to an overcount.

The students did not recognize the overcounting issue, and I eventually drew their attention to it by highlighting that the particular outcome in Figure 5 would get overcounted. We had the following exchange, and they ultimately came to reason why overcounting would occur.

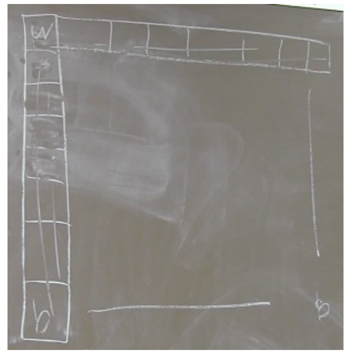

Figure 5 – The students' work on the Rooks problem

Int.: Ok. So this, you, you're counting I place the white rook first, and then I place the black rook. What if you, could you, would you also count where you placed the black rook there and then placed the white rook up there?

Caleb: You'd do it, you'd place it...[*trails off*]

Pat: Oh hey. That's fair. That's a good point.

Int.: Does that make sense what I?

Pat: That's a good point. Cause if you're saying you put white rook here, and then selecting to put the black rook here, but that would be the same as if saying I selected to put the black rook first here and put the white rook there. So it is, everything is being counted twice this way.

Caleb: Yeah, with the color.

Pat: Yeah, because you could put it somewhere starting, and put another one somewhere else, but you could also say put the other one starting where it ended up and putting the white, the first one there. Ok so everything is being counted twice. Hmm.

Int.: Does that make sense?

Caleb: Yeah.

As the students started to articulate and refine their statement of the MP, we see that this experience with overcounting stuck with them. For example, after some work, they arrived at an intermediary statement "For each possible pathway to an outcome there is an equal number of options leading to that path" (Figure 6).

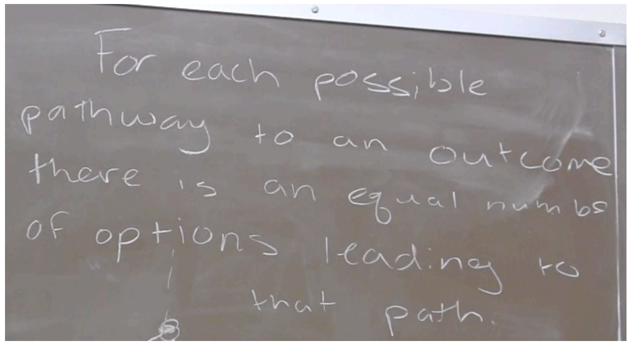

Figure 6 – The students articulate a statement involving pathways

I then asked them to revisit the Language Books problem, which involved independence and also overcounting. After thinking about why they needed a case breakdown (as discussed in the mathematical discussion above), the students added the clause "but w/o repeating the same pathway more than once" (Figure 7) and then had the following exchange.

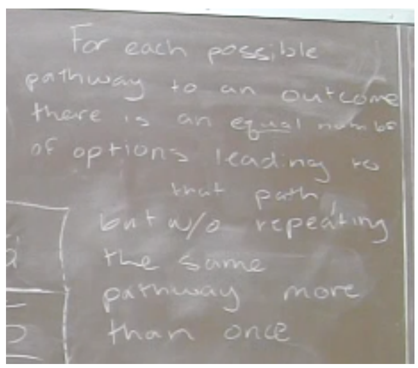

Figure 7 – Attending to overcounting in their statement

Int.: Do you see what he's saying there?
Pat: Yeah I see what's going on.
Int.: And why is that important to include a statement like that?
Pat: Well because, like let's just put it out there again [*he writes SF = FS in large letters, which refers to the fact that a pair of Spanish and French books is the same as a pair of French and Spanish books*].
Int.: Alright sure. Hahaha.
Pat: You don't want to count twice, you don't want to count selections twice, or multiple times, any number of multiple times. I'm assuming it could get larger if you have more options.

We see from this discussion that the students were aware that they wanted to avoid overcounting, and they realized that they wanted to ensure that their statement of the MP would not allow for repeated outcomes. The issue of overcounting came up again as they wrote their final statement of the MP (Figure 8): “If for every selection towards a specific outcome, if there is no difference in the number of options, regardless of the previous selections, then you multiply the number of all the options in each selection together to get the total number of possible unique outcomes.” Note that the addition of the word “unique” was inserted after some discussion about wanting to avoid overcounting.

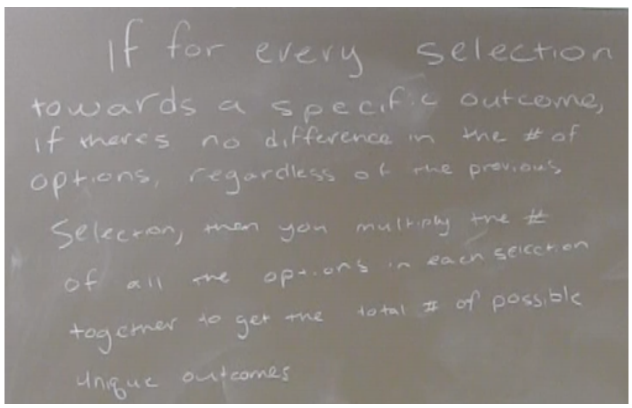

Figure 8 – The students’ final statement of the MP

Ultimately, this final statement emphasizes the mathematical issues in which Caleb and Pat engaged as they thought about and reinvented a statement of the MP.

Discussion and Conclusion

We have highlighted two issues in the MP and have demonstrated some tasks that were effective in prompting students to think about and address these issues. As a summary we now list some of problems that we used in the TE that get at both of these issues (Table 1). We also include We recognize that these two issues are certainly related, and problems like the Language Books problem get at independence and also issues of overcounting. Nonetheless, these are two important issues that students can and should attend to when thinking about when to multiply in counting problems.

Mathematical Issue	Sample Problems[1]	Criteria
Independence	***Cards***: *In a standard 52-card deck there are 4 suits (hearts, diamonds, spades, and clubs), with 13 cards per suit. There are 3 face cards in each suit (Jack, Queen, and King). How many ways are there to pick two different cards from a standard 52-card deck such that the first card is a face card and the second card is a heart?* ***Even numbers***: *How many even, five digit numbers are there with no leading zeros and no repeated digits?* ***Dominos:*** *A domino is a small, thin rectangular tile that has dots on one of its broad faces. That face is split into two halves, and there can be 0 through 6 dots on each of those halves. Suppose you want to make a set of dominos (i.e., include every possible domino). How many distinguishable dominos would you make for a complete set?*	Problems should have stages in counting process that are not independent, in which the choices in one stage affects the choices in a subsequent stage.
Distinct Composite	***Language Books:*** *You have 4 different Russian books, 5 different French books, and 6 different Spanish books on your desk. In how many ways can you take two of those books with you, if the two books are not in the same language?* ***Rooks:*** *How many ways are there to place two different-colored rooks in a common row or column*	Problems should be susceptible to overcounting, in which multiple ways of completing the same counting process yield the same outcome (thus resulting in outcomes getting counted more than once).

[1] The Even Numbers, Language Books, Rooks, and 3-letter Sequences problems are adapted from Tucker (2002).

Outcomes	*of an 8x8 chessboard?* ***3-letter Sequences:*** *How many ways are there to form a three-letter sequence using the letters a, b, c, d, e, f: (a) with repetition of letters allowed? (b) without repetition of any letter? (c) without repetition and containing the letter e? (d) with repetition and containing e?*	

This study speaks specifically to key mathematical ideas underlying the MP. Combinatorially, an implication of this work is that students may benefit from time and explicit energy devoted to learning even subtle aspects of the MP, because doing so may strengthen their understanding of a fundamental combinatorial topic that is traditionally not well understood. More broadly, this study highlights the ways in which particular tasks can elicit student thinking about key mathematical issues, and this provides evidence of the fact that students are capable of developing for themselves robust understandings of even subtle and complex mathematical ideas. Furthermore, it seems to be beneficial for students to have meaningful, salient experiences with key mathematical issues, and these issues can be illuminated through engagement with particular, carefully-chosen problems.

References

Auerbach, C. & Silverstein, L. B. (2003). *Qualitative data: An introduction to coding and analysis.* New York: New York University Press.

Batanero, C., Navarro-Pelayo, V., & Godino, J. (1997). Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics, 32*, 181-199.

Eizenberg, M. M., & Zaslavsky, O. (2004). Students' verification strategies for combinatorial problems. *Mathematical Thinking and Learning, 6*(1), 15-36.

Kapur, J. N. (1970). Combinatorial analysis and school mathematics. *Educational Studies in Mathematics, 3*(1), 111-127.

Larsen, S. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *Journal of Mathematical Behavior*. Doi: 10.1016/j.jmathb.2013.04.006.

Lockwood, E., Swinyard, C. A., & Caughman, J. S. (2015). Patterns, sets of outcomes, and combinatorial justification: Two students' reinvention of counting formulas. *International Journal of Research in Undergraduate Mathematics Education, 1*(1), 27-62. Doi: 10.1007/s40753-015-0001-2.

Lockwood, E., Reed, Z., & Caughman, J. S. (2015). Categorizing statements of the multiplication principle. In Bartel, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.), *Proceedings of the 37th Annual Meeting of the North American Chapter of the Psychology of Mathematics Education,* East Lansing, MI: Michigan State University.

Martin, G. E. (2001). The Art of Enumerative Combinatorics. New York: Springer.

Oehrtman, M., Swinyard, C., & Martin, J. (2014). Problems and solutions in students' reinvention of a definition for sequence convergence. *Journal of Mathematical Behavior, 33,* 131-148.

Richmond, B. & Richmond, T. (2009). *A Discrete Transition to Advanced Mathematics.* Providence, RI: American Mathematical Society.

Swinyard, C. (2011). Reinventing the formal definition of limit: The case of Amy and Mike. *Journal of Mathematical Behavior, 30,* 93-114.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education*. Mahwah, NJ: Lawrence Erlbaum Associates.

Tillema, E. S. (2013). A power meaning of multiplication: Three eighth graders' solutions of Cartesian product problems. *Journal of Mathematical Behavior, 32*(3), 331-352. Doi: 10.1016/j.jmathb.2013.03.006.

Tucker, A. (2002). *Applied Combinatorics* (4th ed.). New York: John Wiley & Sons.