2016 NCTM RESEARCH CONFERENCE
April 11–13 • San Francisco

Paper Title: Constructions of Coordinate Systems: Four Ninth-Grade Students

Author: Hwa Young Lee

Session Title: Constructions of Coordinate Systems: Four Ninth-Grade Students

Session Type: Brief Research Report

Presentation Date: Tuesday, April 12, 2016 9:15 AM–9:45 AM

Presentation Location: San Francisco Convention Center, 3010

Constructions of Coordinate Systems: Four Ninth-Grade Students

Introduction

Mathematical representations play an important role in the learning and doing of mathematics. According to Kaput (1987), "mathematics learning and application are concerned with representation and symbolization because these are at the heart of the content of mathematics and are simultaneously at the heart of cognitions associated with mathematical activity" (p. 22). Like Moore (2014) and Thompson (2004), I posit that representations do not represent anything in themselves and consequently, I find it important to attend to *how* students construct mathematical representations and use them in their reasoning.

Coordinate systems are often considered as representational tools for reasoning with mathematical concepts. For example, in the Common Core State Standards for Mathematics, students are expected to use the conventional Cartesian plane to investigate mathematical ideas in various grade levels, such as number systems, geometric figures, ratios and proportional relationships, and equations and functions (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010). However, the Common Core State Standards do not address how students might construct coordinate systems or how students might use their coordinate systems in the understanding of mathematical concepts and ways of operating. Further, prior research focused more on students' understandings of graphs of functions, with the conventional Cartesian plane assumed as an already given structure (e.g., Herscovics, 1989; Leinhardt, Zaslavsky, & Stein, 1990; Schwarz & Hershkowitz, 1999; Oertman, Carlson, & Thompson, 2008; Lloyd, Beckmann, & Cooney, 2010). Some researchers have investigated how children and middle-grades students draw coordinate planes and graphical representations of given situations (e.g., DiSessa, Hammer, Sherin, & Kolpakowski, 1991; Moritz, 2003) or how college students understand covariation of two or more quantities with the use of coordinate systems (e.g., Moore, Paoletti, & Musgrave, 2013). Lacking is research investigating the mental operations and reasoning that is involved in the construction and use of coordinate systems, especially at the high school level. Such investigations are critical because high school students are expected to use coordinate systems when exploring various mathematical concepts as mentioned above.

The investigation that I report on in this paper involved four ninth-grade students' constructions of coordinate systems throughout a two-year teaching experiment. In particular, I investigated (1) how students construct frames of reference and coordinate measurements in locating points or defining the motion of one point to another in two- or three-dimensional spaces, and, (2) how students use their frames of reference in their reasoning about spatial objects. I present data to highlight the differences in the four students' constructions of frames of reference and their uses of frames of reference in coordinating measurements. Also, I present data to demonstrate how the four students used their frames of reference in inserting and coordinating units in three spatial dimensions. I argue that the levels of units that students can coordinate in reasoning are related to their constructions of frames of reference and coordination of measurements in three-dimensional space.

Theoretical Orientation and Constructs

I find it important to attend to how students construct their mathematical knowledge and organize their mathematical experiences into coherent ways of thinking (Steffe, 1991; von Glasersfeld, 1995). In line with this perspective, I base my work on modeling students' constructive activities with a focus on schemes and operations. A *scheme* is a goal-directed basic sequence of events consisting of three parts: the individual's recognition of an experiential situation, a specific activity or operation that the individual associates with the situation, and the result or sequel of the activity in the situation (von Glasersfeld, 1980). *Operations* are conceptual or internalized mental actions that are used in the activity of the scheme, "which can return to its starting point, and which can be integrated with other actions also possessing this feature of reversibility" (Piaget & Inhelder, 1967, p. 36). When referring to construction of coordinate systems, I refer to the *mental construction* of coordinate systems to fulfill either of two goals: 1) *spatial organization*, which is to re-present space by establishing a frame of reference and locating points within the space using coordinated measurements; or 2) *quantitative coordination*, which is to co-ordinate sets of quantities by establishing frames of reference and obtaining a representational space using geometrical objects. This study focused on constructive activities related to spatial organization. In discussing the differences in students' organization of space, I use Piaget and Inhelder's (1967) distinction between perceptual space and representational space. Perceptual space is the space constructed in a figurative sense through sensorimotor and perceptual activity on elements of raw material; representational space is a re-presentation of perceptual space constructed through the interiorization of perceptual space, which entails a symbolic function in which the individual could regulate spatial behavior in a systematic way.

I also use the construct of levels of units (Steffe 1991; 2012). That is, the levels of units that students are able to coordinate and hold mentally together in one structure. To illustrate, imagine a long strip of candy is shared among three people and then one of those shares is shared again by five people. When asked what fraction of the candy is one of the five person's share out of the entire candy, students who have interiorized three levels of units can partition the unit (entire candy) into three parts, disembed one of these parts from the three parts, and then make part-to-whole comparisons at three levels of units (Steffe & Olive, 2010). Further, they can recursively partition the given unit into a unit entailing the three units, and project five units into each of those three units without having to carry out the actual sharing activity. In other words, they can mentally construct a unit of unit of units structure as shown in *Figure 1*. Further, after producing such a unit structure, they can use that unit structure as material in further operating. The ability to hold this structure in mind allows an individual to understand the multiplicative relationship between the number of pieces and thus find that one share would be one-fifteenth of the entire candy. On the other hand, a child with two levels of units has difficulty in using the unit structure of three units of five units as material in operating. Two levels of units students typically lose track of either the three units within the original unit or the five units within each of the three units which limits them when solving the problem at hand.

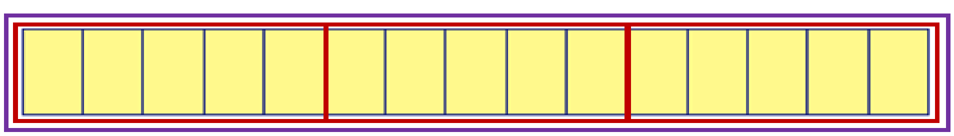


Figure 1. A unit of three units of five units structure.

Students' units coordinating activities have been linked with their understandings of various mathematical concepts and reasoning, such as fractional knowledge (Steffe & Olive, 2010), multiplicative relationships (Hackenberg, 2010), combinatorial reasoning (Tillema, 2012), and proportional reasoning (Steffe, Liss, & Lee, 2014). Students' levels of units are associated with different mental operations that a student can utilize in reasoning. I hypothesized that the disembedding operation—taking a part of the whole as an independent entity while being aware of the part in the whole as well—and the units coordinating operation—mentally inserting units into another level of units (Steffe & Olive, 2010)—are essential in the coordination of spatial dimensions. Therefore, the construct of levels of units was used to select participants at the beginning of the study and also in analyzing data.

Methodology

Participants

A two-year teaching experiment (Steffe & Thompson, 2000) was conducted with four ninth-grade students at a rural high school in the southeastern United States. I conducted initial interviews with the four students to infer the levels of units they could coordinate and utilize in reasoning. For example, consider the aforementioned candy-sharing problem. After explaining the sharing situation to the students and showing them a long strip of candy or a long strip of paper as a model of the candy, I covered the visual material with a cloth and asked what fraction of the candy is one person's share out of the entire candy. Dan initially had a difficult time reasoning about the fractional amount of the candy without manipulating the visual material. After cutting off one fifth of the original strip of candy and then one third of the one-fifth piece, Dan said that it would be one-eighth of the entire candy, since there are three people and five people sharing the candy. In Dan's case, he was able to mentally partition the long strip of candy into a unit of three units and he was able to partition the one-third unit into five units. However, he was not able to posit the five units within the one-third unit, within the original candy as a unit and showed a typical reasoning pattern of a two levels of units student. That is, Dan added the number of pieces required (three and five) and determined that the fractional amount of one share should be one-eighth of the entire candy, as illustrated in *Figure 2*.

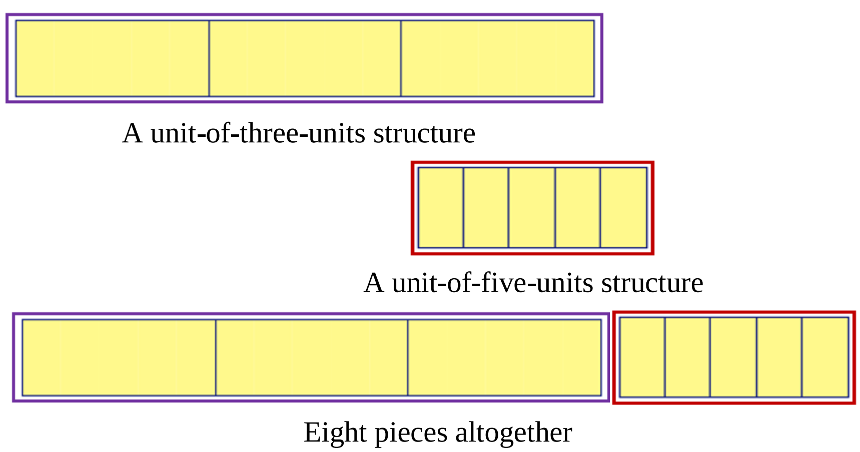


Figure 2. Dan's candy sharing activity.

When asked how many of the small pieces would fit within the original candy, instead of reasoning reversibly, using his reply of one-eighth, Dan moved the small strip of paper along the remaining paper to count how many times it would fit in the entire candy.

Similar to Dan, Morgan initially had a difficult time reasoning about the fractional amount of the candy without manipulating the visual material. However, once she made the cuts of the one-third piece and the one-fifth piece, she was able to coordinate the number of pieces together and said that the fractional amount should be one-fifteenth of the entire candy. When asked why it would be one-fifteenth, Morgan explained that since there were five in each three, there should be fifteen pieces in total. Although Morgan also had to carry out some activity in order to assist her enactment of coordinating three levels of units, she was able to recursively partition the given unit into a unit entailing the three units, and project five units into each of those three units, as shown in *Figure 3*.

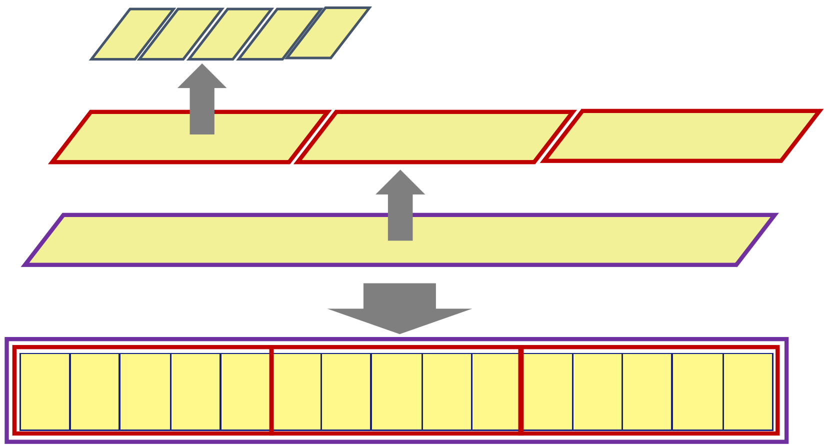


Figure 3. Morgan's candy sharing activity.

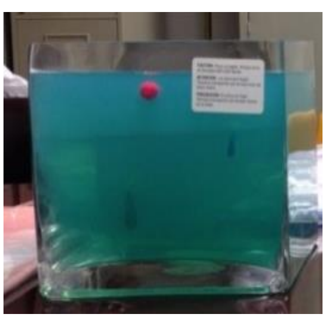
Similar to Morgan, Craig had to carry out some activity in order to assist his enactment of coordinating three levels of units; once he carried out that activity, he was able to recursively partition the given candy and hold a three levels of units structure within the original candy. As such, Morgan and Craig were identified as reasoning with three levels of units in activity. In this task and others in the initial interview, Kaylee demonstrated the ability to do so without having the carry out activities; she was able to reason with three levels of units as given. Therefore, I concluded that Kaylee reasoned with three levels of units as given; Morgan and Craig reasoned with three levels of units in activity; Dan reasoned with two levels of units.

Procedure, Data Sources, and Analysis

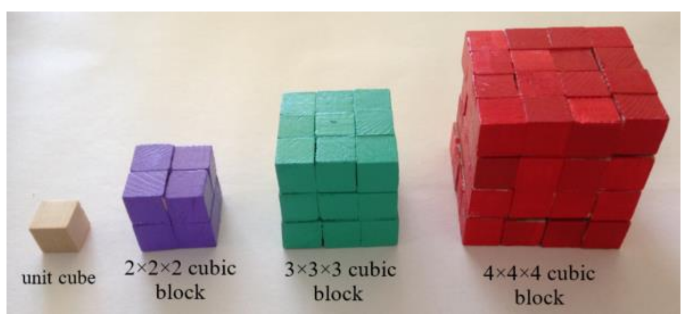
I acted as the teacher-researcher in the teaching episodes that comprised the teaching experiment. Kaylee and Morgan participated as a pair in teaching episodes the first year, each session lasting for approximately 20–25 minutes; Craig and Dan participated as a pair in teaching episodes the second year, each session lasting for approximately 15–20 minutes. The pairings were made given the pool of participants each year and were primarily based on similar levels of units they were able to operate with. The students participated in the teaching episodes twice a week mostly as a pair but sometimes individually due to absences. In most teaching episodes, I asked the students to work individually on the task until both students were finished organizing their thoughts and then asked each student to share their solutions. In many cases, I also asked each student to compare, contrast, or critique their partner's solutions. Video recordings (one wide-angle view, one close-up angle on the students' work, and occasionally screen recordings if computers were used) and student work were collected as data sources. In analyzing data, my main focus as a researcher was to build working models of ways students mentally construct and use frames of reference and coordinate measurements. The investigation of mental constructions, which I do not have direct access to, was only possible by me making inferences from observing the physical, observable activities the students carried out. Therefore, I concentrated on the students' visual illustrations, verbal descriptions, and physical gestures. These elements of our interactions were analyzed based on on-going and retrospective analyses (Steffe & Thompson, 2000). On-going analyses involved testing and formulating new hypotheses throughout the teaching episodes based on the ways students engaged in the tasks. Retrospective analyses of data involved identifying instances that would offer insights in building working models of students' spatial organization. For such instances, I transcribed the video and performed a conceptual analysis (Thompson, 2008) in order to refine my models of the students' constructions of coordinate systems. Weekly research meetings were used to come to a consensus on the interpretations and analyses of the students' engagement in tasks and in testing and reformulating hypotheses.

Tasks

In this paper, I present three tasks that were used in the teaching experiment. In Task A, I asked the students to describe the location of four fish contained in a cubic fish tank (*Figure 4 (a)*) and how one fish would get to another fish. This task was designed to explore how the students made sense of the location of a point or described the motion of a point in three-dimensional space. The goal was to investigate the frames of reference and measurements (if any) students make use of and how they coordinate these elements together. Tasks B and C were developed in order to investigate the students' coordination of three dimensions of spatial objects, when the interior of these spatial objects were not available in their direct perceptual fields. An additional goal was to explore how the students might partition, insert, and coordinate units in relation to the frames of reference I observed in Task A. In Task B, students were asked to reason with cubic blocks of various sizes, each consisting of 2×2×2, 3×3×3, and 4×4×4 unit cubes (*Figure 4 (b)*). In Morgan and Kaylee's case, the cubic blocks were pre-constructed and were painted on the exterior. After showing them the cubic blocks I covered the blocks and asked them to find the total number of unit cubes in the cubic block and the number of unit cubes that would be painted if the exterior of the block was painted for each of the cubic blocks, one at a time.



(a) Cubic Fish Tank (Task A)



(b) Cubic Blocks of Various Sizes (Task B)

Figure 4. Task Artifacts.

In Craig and Dan's case, based on their earlier activities in the teaching experiment, I modified the task to entail the activity of building the blocks. Also, when Craig and Dan investigated the cubic blocks, I asked them to focus on only the total number of unit cubes required to construct a given cubic block rather than the number of painted or unpainted unit cubes. To explain their reasoning, I asked all four students to illustrate how they were thinking about the situation. Task

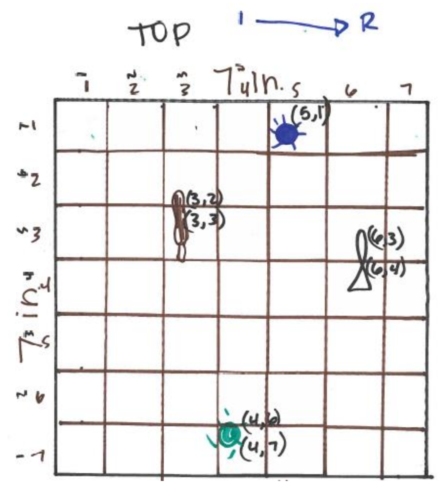
C was an extension of Task B in which Dan and Craig were asked to find the number of unit cubes needed to fill a shoebox.

Results

Task A: The construction of frames of reference and coordination of measurements

In locating the four fish in the cubic tank, Morgan located all four fish taking the perspective of looking down onto the tank. She made a diagram depicting the top view of the tank, looked at the fish in the tank from the top and plotted each fish in her diagram without any explicit coordination of measurements (see *Figure 5 (a)*). It is possible that Morgan was utilizing some frame of reference in making visual estimations of the location of each fish; however, because she did not carry out any observable measuring activities and did not verbalize her estimating activities, I was limited in making inferences of any use of frames of reference. After she located each fish based on visual estimations, Morgan added a grid of horizontal and vertical lines in 1 inch intervals on top of her diagram (as shown in *Figure 5 (b)*). However, these grids were used for a communicational purpose in that she wanted to use the grid and associated inscriptions based on which sections the fish were located in; the grids were added *after* she had located the fish and not used in coordinating measurements in locating the fish.

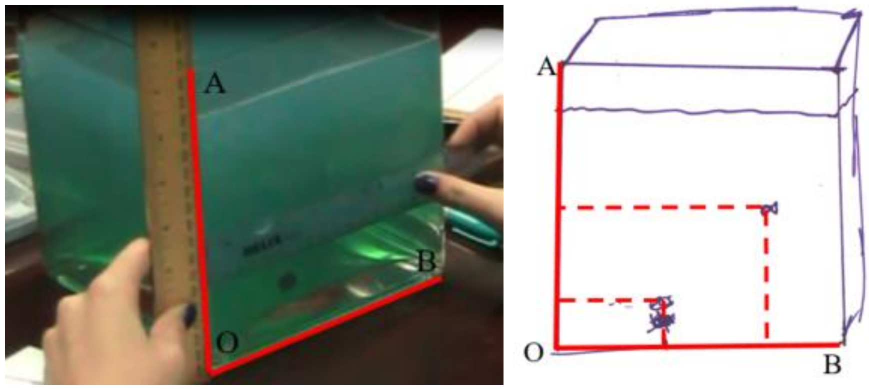
(a)



(b)

Figure 5. Morgan’s diagram to explain the location of the four fish in the cubic tank.

On the other hand, Kaylee coordinated three measurements, which she referred to as “length, width, height” for each fish in the tank taking the side view of the tank. Kaylee picked two faces of the tank where the fish were most visible and located two fish in each of the side views. I will illustrate her locating activities with two fish she located. In order to locate the two fish, Kaylee first measured horizontal and vertical distances along one face using two rulers as shown in *Figure 6 (a)*. Then, in her diagram of the tank, she coordinated those measurements to locate two of the fish as shown in *Figure 6 (b)*. From this, I infer that Kaylee had superimposed a set of horizontal and vertical lines onto the face of the tank such that a set of perpendicular lines were coincident with edges OA and OB and intersected at point O in the tank. Using those set of perpendicular lines and point of intersection {O, OA, OB} as her frame of reference, Kaylee constructed two sets of horizontal and vertical lines that would each pass the two sea creatures in the tank (*Figure 7*). As a result of utilizing this rectangular frame of reference, Kaylee was able to coordinate the “length” and the “height” of the position of each sea creature (*Figure 6 (b)*).



(a) (b)

Figure 6. Kaylee’s coordinated measurements in the first face of the tank.

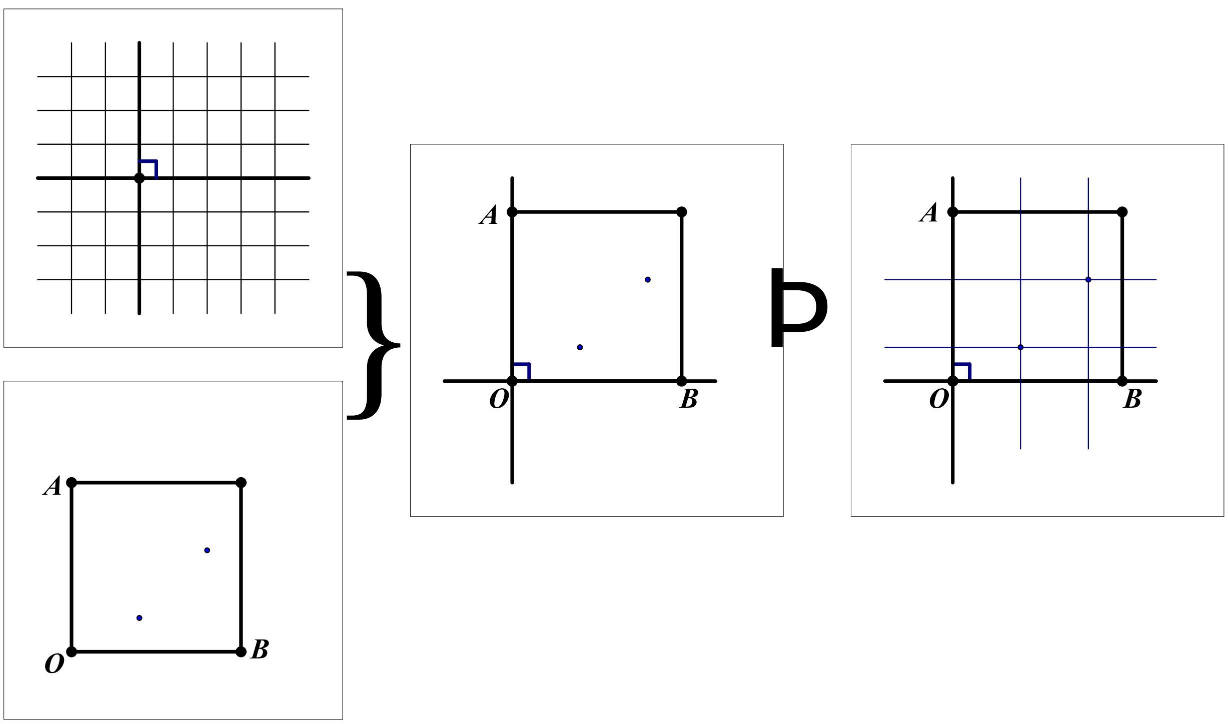


Figure 7. Kaylee’s rectangular frame of reference superimposed onto the face of the tank.

Once Kaylee coordinated the vertical and horizontal distances along one face of the tank in her sketch, Kaylee rotated the tank counterclockwise with respect to a vertical axis through the center of the tank such that vertex O was now on her bottom right. She then picked up her ruler and measured the distance from edge OA to the first fish with her ruler parallel to edge OC (see *Figure 8 (a)*). Then in her sketch, she drew an arrow sign and wrote the length she just measured (1.25 in) next to the first fish (see *Figure 8 (b)*). Kaylee repeated the same activity for the second fish: she measured the distance from edge OA to the second fish with her ruler parallel to edge OC and marked the measurement (3.75 in) next to the fish in her sketch (see *Figure 8 (b)*).

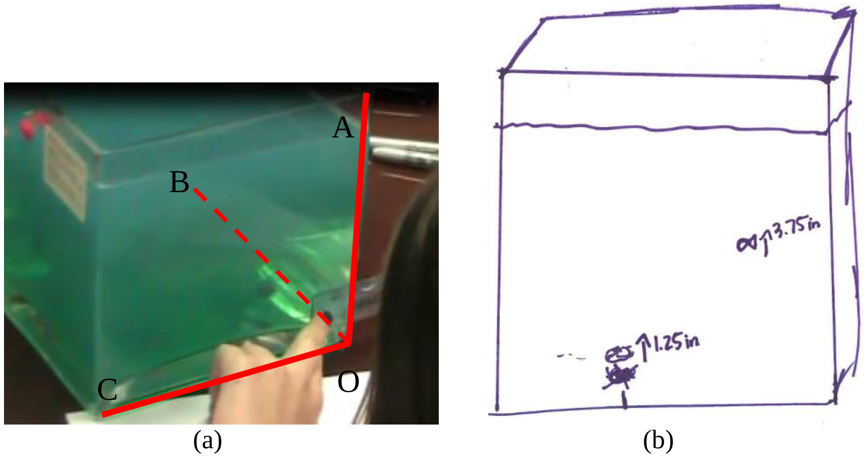


Figure 8. Kaylee's coordinated measurements in the second face of the tank.

Kaylee's goal in finding these measurements (1.25in, 3.75in) was to consider how far into the tank the fish were from the face including {O, OA, OB}. In order to do so, she superimposed a rectangular frame of reference onto the new face including {O, OA, OC}. Using this frame of reference, Kaylee coordinated the horizontal distance from edge OA parallel to OC to each fish. Here, there are two things to note. First, Kaylee only measured the horizontal distance and did not measure the vertical distance from edge OC parallel to OA to the sea creatures. Second, she wrote the measurements she obtained (1.25in, 3.75in) in her original sketch (as shown in *Figure 8 (b)*), instead of making a new sketch for this new face she used to locate the two sea creatures. For these two reasons, I claim that Kaylee was inserting her second frame of reference that she had imposed onto the second face (the face including {O, OA, OC}) into her frame of reference of the first face (the face including {O, OA, OB}). That is, through disembedding, she held her frames of reference of the first and second faces each as a unit structure, translated and inserted her second frame of reference into the first frame of reference resulting in a three-dimensional structure (See *Figure 9*). Taking each frame of reference as a given structure, Kaylee was combining and uniting these two structures, which allowed her to coordinate all three spatial dimensions that build up the three-dimensional space.

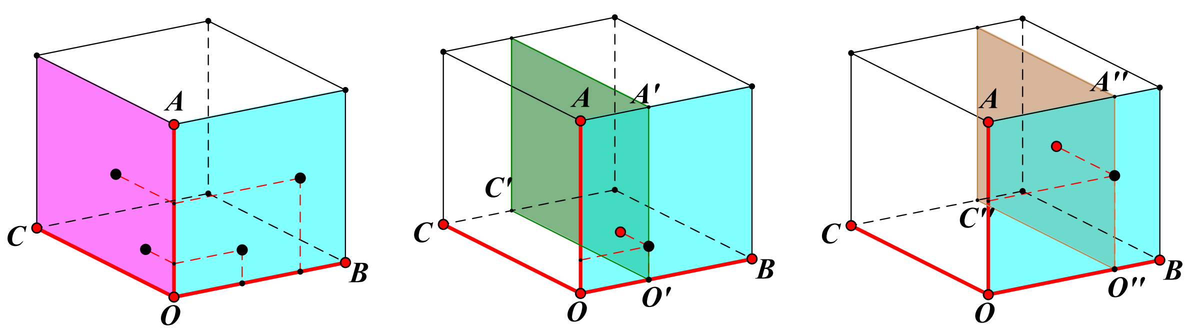


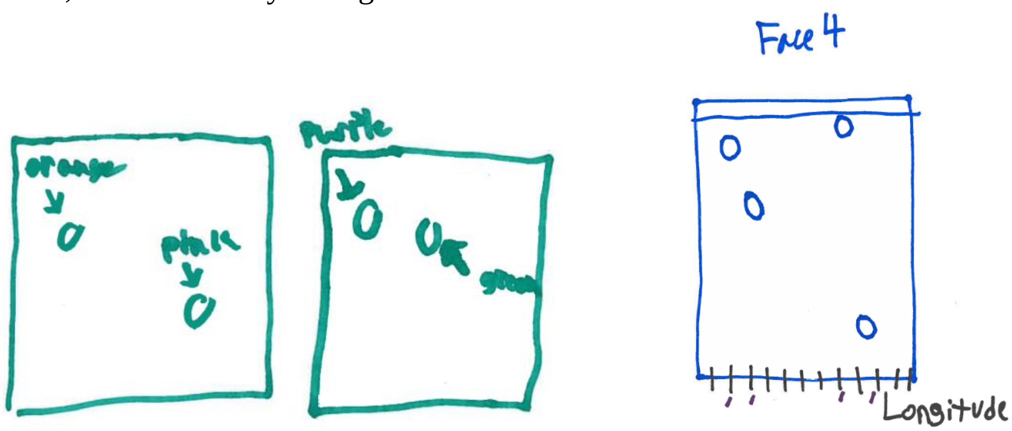
Figure 9. Kaylee's insertion of her {O, OA, OC} frame of reference into her {O, OA, OB} frame of reference.

Because Kaylee combined and united the two structures, she did not measure both the horizontal and vertical distances in her second frame of reference. She knew that she only needed to measure the horizontal distance in order to coordinate the measurements to locate the two sea creatures. Because Kaylee only measured the horizontal distances of the two fish from edge OA and added only those distances to her first sketch, I infer that she was aware of her coordinated measurements of the first face (the face including OA, OB, and O) in relation to the second face (the face containing OA, OC, and O). Kaylee's actions were consistent in locating the other fish and her locating activity was systematic and unhesitant as if she had an already made plan of action in her mind. Although she carried out the actions of rotating the tank, the way her actions were coordinated seemed as if she had already decentered her perspective and mentally rotated the tank in planning such activities. From this, I claim that Kaylee has constructed and enacted her rectangular frame of reference scheme. That is, she has constructed a recognition template for a situation in which she could associate the use of a rectangular frame of reference, activated the activity of superimposing a rectangular frame of reference onto the spatial object, and resulted in a coordinated system of measurements. Because her measurements were systematically coordinated using a frame of reference, the physical location of the sea creatures in the tank did not matter; Kaylee would have been able to locate fish in any location in the face she was considering.

When I asked Kaylee to comment on Morgan's method, Kaylee offered a critique explaining that Morgan's sketch only accounted for the top view of the tank. Kaylee argued that all the fish could be "floating on the top," and Morgan's sketch did not account for "how far down" the fish were in the tank. Kaylee was able to project the fish into a two-dimensional plane and take that projection as an object in further operating. Consistent with her earlier behaviors, Kaylee was able to disembed the projection of the fish into the top layer of the water in the tank as a unit and insert that into any point along the third dimension. She was also aware that Morgan's sketch could be of any instance of those insertions of the top layer of the water in the tank into any depth of the tank. Although the top layer of the water was a different face from what Kaylee had used earlier in her sketch, she was able to flexibly transfer her frames of reference and coordinate them to the face that Morgan was using in organizing the locations of the sea creatures. From Kaylee's flexibility in transferring her coordination of frames of reference to any face of the tank I claim that Kaylee had constructed a frames of reference coordinating scheme. That is, she recognized of a situation in which she could posit a frame of reference as a unit and insert it into another frame of reference resulting in a coordinated frames of reference. Moreover, the coordination of two-dimensional frames of reference allowed her to coordinate all three spatial dimensions that build up the three-dimensional space of the tank; this was something Morgan did not consider initially. I claim that Kaylee's use of mental actions of decentering, rotating, disembedding, inserting, uniting, and coordinating frames of reference resulted in locating the four sea creatures in a re-presentational sense (Piaget & Inhelder, 1967).

After listening to Kaylee's way of locating the four sea creatures in the tank and Kaylee's critique of her sketch, Morgan seemed to carry on some of Kaylee's mental operations that were activated in activity in front of her and made modifications in her spatial organizing activities. When I asked Morgan to consider how one fish would swim to another, she initiated activities that she has not done before that, to me, indicated her construction and superimposition of a rectangular frame of reference that allowed her to decompose the movement of one fish to another into three spatial dimensions. Her illustrations, actions, and explanations gradually shifted from making visual estimations of the location of the fish to using spatial references to coordinate measurements.

In the first teaching episode with Task A, Dan claimed that "it's impossible" and Craig claimed that "this blows my mind. I've never done this before." While Morgan and Kaylee worked on Task A in one teaching episode, it took a total of four teaching episodes for Task A to be concluded with Dan and Craig. In the first episode, after some encouragement, Both Dan and Craig started with diagrams of the fish tank (*Figures 10 (a)* and *(b)*, respectively) and located the four fish in the tank based on where they saw them, similar to Morgan's locating activities in making visual estimations. Dan was aware that there were different perspectives that you can take at looking at the tank and chose to make a diagram based on the two faces where he could best see the fish. Similar to Morgan's approach, Craig located all fish from one perspective of the tank, which he called Face 4. After locating the fish in the tank based on visual estimations, Craig wanted to use the idea of longitude and latitude to describe the locations, which was similar to the way Morgan superimposed a grid onto her diagram *after* locating the fish. However, he did not carry through with this idea.



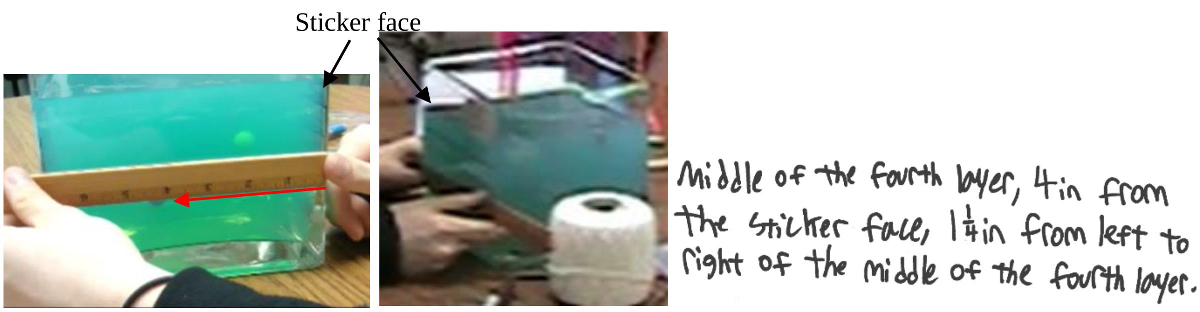
(a) Dan's diagram of the fish tank and fish (b) Craig's diagram of the fish and tank

Figure 10. Dan and Craig's initial diagrams of the cubic fish tank and fish.

In the second teaching episode on Task A, Dan worked alone. To encourage measuring activities, I asked him to describe the locations of each fish to another person who is trying to make a replica of the fish tank. I prepared a frame of a cube as the model of the other person's tank that was the same size as the cubic tank, made out of wired straws. This frame was used for me or the students to enact their instructions. At the beginning, Dan heavily relied on the perceptual imagery of the tank. For example, he said that he would tell the person to "put the purple [fish] in the corner." It required a lot of questioning (e.g., Using the information you found, what would the other person do in this tank? Would that be enough information for the other person in locating the fish?) and pushing him to use his ruler to move forward. Dan finally started to consider measurements using spatial elements of the tank (e.g., face, vertex, edge) as spatial references. However, the distances he chose to measure were inconsistent in that the references he selected were different for each fish. Sometimes the distances he measured were horizontal/vertical and sometimes diagonal, not fully accounting for all three dimensions. For example, he measured the depth of each fish from the top of the water and then claimed that the purple fish was 1 inch straight from the corner of the tank. After I enacted Dan's instructions using the frame of the cube, Dan gradually came to consider distances. At the beginning of the third teaching episode on Task A, he summarized his activities as “first, I measured how far each one was from the top. And then after I got that, I started to see how far they were from the sides of the tank.” In considering which sides, at the beginning he selected faces that were closest to the fish and later claimed that he would have to measure the distances from each face.

In a previous task where I asked the students to locate a point in two-dimensional space, Dan used the idea of longitude and latitude in locating the point. So, I asked him whether he could use a similar idea in the cubic tank case to see if he would assimilate the three-dimensional situation to one where he could coordinate horizontal and vertical distances. At the beginning, he explained that he couldn’t: “The only way I know to use longitude and latitude is with the map. But this is three-dimensional. Latitude and longitude [inaudible]. It’s like bird’s eye view. Or at least in my mind for using latitude and longitude.” However, towards the end of the third teaching episode, Dan revisited this idea of longitude, latitude without any prompting: “Now I get when he [Craig] was saying it’s like the latitude and longitude lines [see *Figure 10 (b)*], cuz from the top and then the sides come to one point.” This to me is an indication that Dan has started to decenter his perspective from one vantage point to another in considering the location of the fish, which allowed him to become aware of the coordination of the vertical and horizontal distances he measured. Unfortunately, Dan was absent in our last episode with the cubic tank and thus I did not get further data on his locating activities in terms of considering the movement of one fish to another.

On the other hand, in the third teaching episode, Craig considered “length, width, depth” of each fish (Craig was absent in the second teaching episode). To elaborate, Craig noticed the visible lines of layers of the gelatin in the tank and considered each layer as a unit for measuring the depth of each fish. Then, Craig noticed the sticker label on one of the side faces of the tank and consistently measured the distance from the sticker face into the tank and the distance from the left edge towards the right edge of the sticker face. For example, in order to locate the purple fish, he first identified that the fish was in the fourth layer (depth); then, measured that the purple fish was four inches from the sticker face into the tank and one and one-fourths of an inch from left to right of the sticker face (*Figure 11*). Craig referred to these measurements as the length and width.



(a) width *(b)* legth *(c)* written instructions for the other person

Figure 11. Craig’s measuring activities in locating the purple fish.

When finding the measurements which he referred to as the length and the width, often Craig stood up to look down onto the tank to measure the distances he wanted to measure. As such, Craig frequently shifted his perspective back and forth from the top of the tank and the side of the tank. From his consistent way of first identifying the layer of the fish and then finding the distance from the sticker face and the distance from left to right along the sticker face, I infer that

Craig has partitioned the cubic tank horizontally into square cross-sections, then took each two-dimensional square and superimposed a set of horizontal and vertical lines onto it, re-enacting his longitude and latitude concept. He superimposed the horizontal and vertical lines such that a set of perpendicular lines lined up with the edges of the square which corresponded to that of the sticker face and the one adjacent to it on the left. Using the set of perpendicular lines and point of intersection as his frame of reference, Craig constructed two sets of horizontal and vertical lines that would each pass the fish in the tank. As a result of utilizing this rectangular frame of reference, Craig was able to coordinate the length and the width of the position of each sea creature (*Figure 12*).

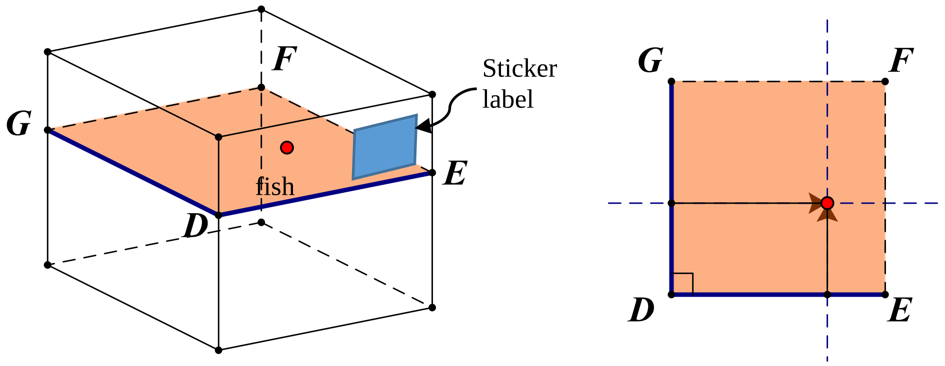


Figure 12. Craig's rectangular frame of reference imposed onto the second face of the tank.

Although similar to Kaylee's locating activities, I consider Craig's coordination of measurements different from Kaylee's. The difference was most apparent when each student was asked to explain the movement of one fish to another. In Kaylee's case, she used her system of coordinated measurements and measured the distance between the two fish along the length, width, and height dimensions she identified earlier. In other words, the length, width, and height measurements for each fish were tightly coupled (Saldanha & Thompson, 1998) and thus when demonstrating the movement from one fish to another, she considered the change in each measurement between the two fish. On the other hand, when it came to describing the motion of one fish to another, Craig first identified the layers each fish were in, thus finding the amount of vertical movement along the "depth" dimension that was needed. Then, once the fish were in the same layer, he explained that one fish would swim straight, diagonally, to the other. The breakdown of the movement in all three dimensions was lacking; Craig did not use his system like Kaylee did. Therefore, I consider Craig's coordinate system to involve a sequential coordination of two-dimensional rectangular frames of reference and measurements; in contrast, I consider Kaylee's coordinate system to be a multiplicative structure of three-dimensional rectangular frame of reference.

Task B: Insertion and coordination of units within frames of reference

When reasoning about the cubic blocks of various sizes, Kaylee consistently *decomposed* the blocks into layers of squares[1], as shown in *Figure 13*. This decomposition was confirmed in

[1] By layers of squares, I refer to the layer of the n×n×n cubic block shaped as a square that consists of n×n unit cubes. For example, see Figure 15, which illustrates the four layers of squares each consisting of 4×4 unit cubes.

her own verbal explanations. For example, when explaining how she thought about the 3×3×3 cubic block, Kaylee said that "there would be three layers and nine squares, that's how it's set up". When talking about the 4×4×4 cubic block she explained, "Well, I was like, mentally separating these into four squares like, there's four of them [tapping at each layer in the cubic block in front of her; the layers she referred to are illustrated in *Figure 14*]." As such, enacting her disembedding operation, Kaylee was able to partition the cubic block into square layers while aware that each layer was a part of the cubic block.

(a) 2×2×2 cubic block *(b)* 3×3×3 cubic block *(c)* 4×4×4 cubic block

Figure 13. Kaylee's sketch of her reasoning about the cubic blocks.

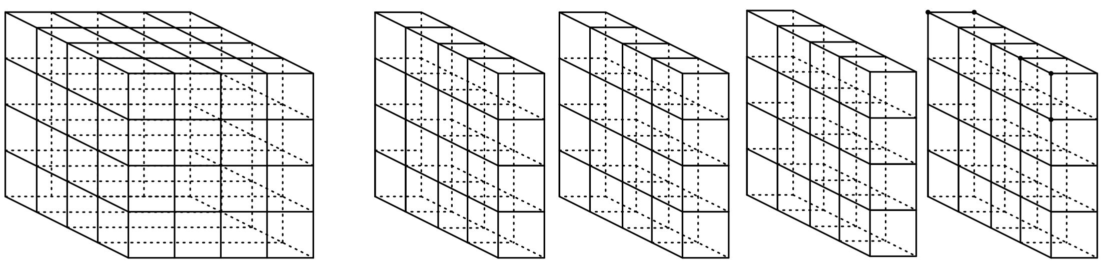
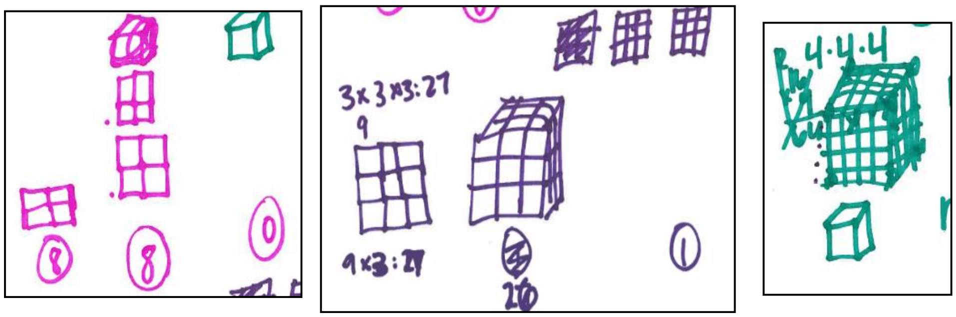


Figure 14. The four layers made of four-by-four unit cubes.

Utilizing her decomposition of the cubic block, in order to find the total number of unit cubes in each block, Kaylee consistently multiplied the number of unit cubes in the square layer by the number of layers. Therefore, I inferred that Kaylee has inserted the number of unit cubes along the length and width dimensions to obtain the unit cubes in each layer and inserted the layers into each of the units along the third dimension—height. This insertion of layers into the third dimension was analogous to that in her construction of her rectangular frames of reference in the cubic tank (see *Figure 9*). That is, through disembedding, Kaylee held the square layer as a unit structure, she translated and inserted the square layer into each unit cube along the third dimension resulting in a multiplicative structure. From this I infer that she enacted her rectangular frame of reference coordinating scheme in decomposing and coordinating layers within the cubic block which produced a unit (unit cubes) of unit (unit squares) of units (cubic block), i.e., a three levels of units structure. This mental decomposition of the cubic block was also utilized in finding the number of unit cubes that would be unpainted. After decomposing the cubic blocks, Kaylee used her disembedding operation, which allowed her to take each layer as a given, aware of it being a part of the whole of the cubic block and coordinated each of the square layers with each face of the cubic block. Mentally rotating her perspective to various positions surrounding the cubic blocks, Kaylee was able to re-compose the de-composed layers of the cubic block to coordinate which of the unit squares in her 3×3 square layers corresponded to which face of the cubic block. The unit squares that are shaded in *Figure 13 (b)* and *Figure 13 (c)* are the ones she concluded painted. Utilizing these operations, Kaylee was able to anticipate the interior of the cubic block without having to physically take the cubic block apart.

In contrast to Kaylee, Morgan had a more difficult time anticipating the interior of the cubic block without taking it apart. For the 2×2×2 cubic block, Morgan easily found the number of unit cubes that composed the block and those that were painted but had a difficult time sketching a cube on paper. After seeing Kaylee's sketch of a cube, she recalled how to sketch one and used that for the 3×3×3, and 4×4×4 cubic blocks (*Figure 16*).



2×2×2 cubic block 3×3×3 cubic block 4×4×4 cubic block

Figure 15. Morgan's sketch of her reasoning about the cubic blocks.

When I asked Morgan to explain how she was reasoning with the 3×3×3 cubic block, she explained that she was using a similar idea with Kaylee's in that she was thinking about layers of the block. However, from careful analysis of the videos, I found that when she referred to layers, she was pointing to the three faces that were visible in her sketch. The layers were not the three 3×3 square layers when put together formed the cubic block; they were the 3×3 square faces that were each of one of three adjacent faces, as shown in *Figure 16*.

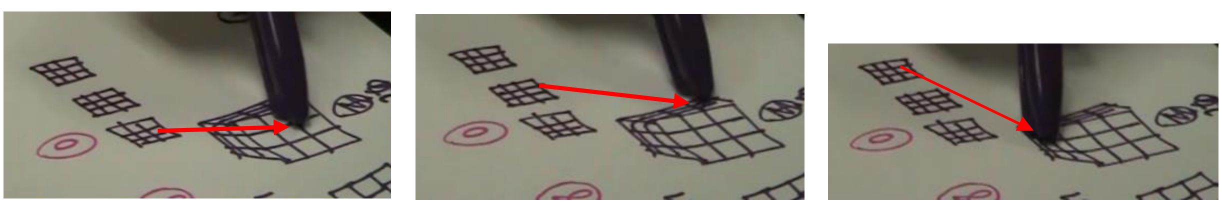
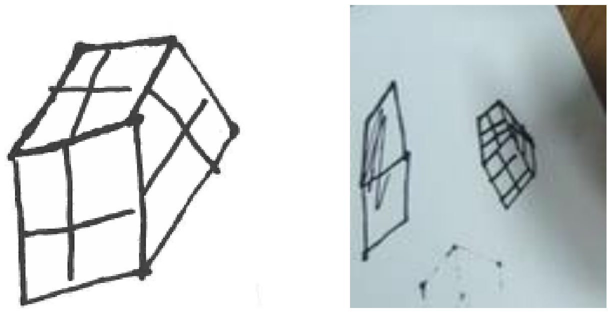


Figure 16. The three layers Morgan was referring to in her sketch.

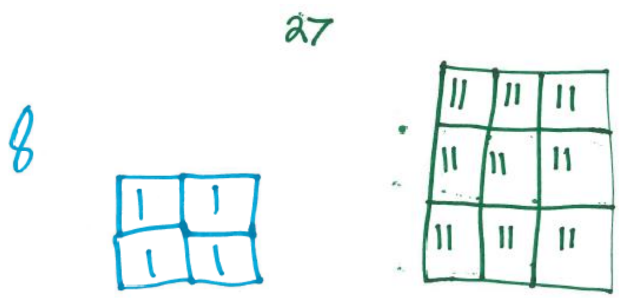
Overall, Morgan accounted for the more perceptual elements of the cubic blocks, especially the ones that she could see. Although Morgan showed powerful ways of thinking when she could see the blocks, her re-presentation of the cubic block was more of a perceptual copy of the cubic block and not one that was a result of mental rotations, de-compositions, and re-compositions of the cubic blocks.

Dan and Craig engaged in a minor variation of Task B in that they first constructed the 2×2×2 and 3×3×3 cubic blocks. After building these blocks, I covered the blocks and asked Dan and Craig to re-present the cubic blocks in illustrations and to find the total number of unit cubes contained in each cubic block. Dan's and Craig's sketches are shown in *Figures 17* and *18*, respectively. When I asked them to first draw a sketch of the 2×2×2 cubic block, Dan drew a frame of a cube and then partitioned each face into fours (*Figure 17 (a)*). He said there were 8 cubes in total and said there were four on each side. Dan's ability to draw a cubic shape on paper allowed him to count the number of cubes. On the other hand, Craig said he "forgot how to draw a cube" and was struggling to draw a three-dimensional figure on paper. He said there would be 8 cubes but "forgot how he found it." Craig's difficulty to draw a cubic shape resulted in him sitting for a long while but when asked the number of cubes it didn't take him long to respond that there were eight cubes. So, while Dan's sketch provided a visual representation that allowed him to figuratively count the number of cubes, Craig could determine the number of unit cubes despite his difficulties in producing a perceptual copy of the cubic block. He was able to mentally re-present the unit cubes composing the 2×2×2 cubic block.



(a) 2×2×2 cubic block *(b)* 3×3×3 cubic block

Figure 17. Dan's sketch of the cubic blocks.



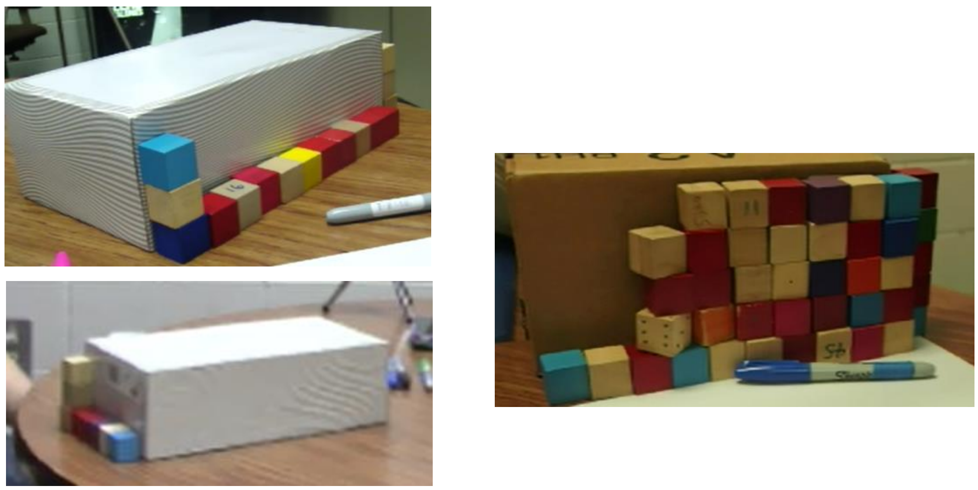
(a) 2×2×2 cubic block *(b)* 3×3×3 cubic block

Figure 18. Craig's sketch of the cubic blocks.

When I asked them to draw a picture illustrating the 3×3×3 cubic block, this difference became more apparent. Dan started his sketch with the frame of the cube (just like he did for the 2×2×2 cubic block) and then added a vertical line segment through the middle of the front facing face. Then he drew a perpendicular looking line to that line segment and another one below that resulting in a 2×3 face. He completed his sketch by adding the same inscription into each face (*Figure 17 (b)*). Then, Dan relied on the sketch in counting the number of unit cubes contained in the 3×3×3 cubic block. Therefore, his sketch of a 2×3×3 cubic block did not give him the number he was desiring to obtain. Moreover, given the sketch he made, Dan still counted nine unit cubes, which was only one more unit cube than the number of unit cubes contained in the 2×2×2 cubic block. His organization of the spatial three-dimensional figure was a figurative one in that the counting act was not systematically obtained through the visualization. On the other hand, Craig explained that he could not remember how to draw a cube so he sketched the cubic block as if he is viewing it from one side and then made tally marks to account for the number of cubes behind each cube on the face he sketched (*Figure 18 (b)*). Although unconventional, Craig was able to come up with a representation that was a reflection of his spatial organization of the cube, and thus allowed him to further operate on the cubic blocks. He made a similar illustration for the 2×2×2 cubic block (*Figure 18 (a)*) and used his illustrations to find the total number of unit cubes contained in each cubic block.

Task C: Insertion and coordination of units within frames of reference (Crag and Dan)

To further investigate the differences in Craig and Dan's insertion and coordination of units within frames of reference, I asked them to determine the total number of unit cubes needed in order to fill in a given shoe box. *Figure 19* illustrates the different ways these students engaged in this task.



(a) Craig's shoebox (b) Dan's shoebox

Figure 19. Craig and Dan's shoeboxes.

As shown in *Figure 19 (a)*, Craig aligned unit cubes along each dimension of the box and counted the number of cubes along each edge. He then multiplied the height (3 unit cubes), width (11 unit cubes) and the length (6 unit cubes) to obtain the total number of unit cubes needed to fill in the box. When I asked him to explain why he multiplied the three numbers, Craig explained, "Three times eleven, I got thirty-three. So, that makes this whole wall…And then I had six on this wall…So, since there's six here, then I would take six of these walls to fill up this whole box." When I asked him to demonstrate this in activity, he picked up each of the six unit cubes along the length saying that a wall would go in each one. Analogous to his locating activities in the cubic tank, Craig has disembedded and partitioned the box into layers of rectangles and counted the number of "walls" he needed to fill in the entire box. In other similar tasks in which I asked him to find the total number of unit cubes that would fill a rectangular prism, Craig flexibly changed the "wall" and the third dimension in which he inserted the wall into. From this, I infer that Craig has utilized his rectangular frames of reference in coordinating and inserting units along three spatial dimensions.

On the other hand, as shown in *Figure 19 (b)*, Dan covered up the majority of one face of the shoe box to find out the number of unit cubes needed to cover up that face. Then, he lined unit cubes along the adjacent face to find out the number of unit cubes needed to cover up that face. He then explained that he would double each number of unit cubes and then add them all together. Essentially, Dan was finding a portion of what I view as the surface area of the shoe box. In other similar tasks, Dan was limited in coordinating the third dimension with the two-dimensional areas. I claim that his lack of disembedding, and not having constructed a frame of reference that he could utilize in decomposing the show box limited his ability to anticipate the interior of the shoe box.

Conclusions and Implications

In summary, throughout the tasks, Kaylee consistently demonstrated behaviors indicating mental rotation and decentering from one perspective to another, taking multiple positions surrounding the spatial objects. Kaylee was aware of the individual perspectives and the object that would be viewed from that perspective. Thus, Kaylee was able to select spatial elements of the object to superimpose and anchor her rectangular frames of reference. Further, she enacted her rectangular frame of reference scheme to coordinate measurements or insert units within each spatial dimension. Using her coordination of frames of reference and disembedding and partitioning operations, Kaylee was able to mentally decompose, re-present, and anticipate (Piaget & Inhelder, 1967) the interior of the spatial objects in the absence of their perceptual availability. In other words, Kaylee has constructed re-presentations of space that she could reason upon reflectively; further, using her re-presentations of space she was able to coordinate units within three spatial dimensions. As a result, Kaylee produced multiple three level of units structures. Her construction of powerful re-presentations of the space are supported by her immediate activities when presented with new situations and abilities to anticipate and carry out these activities without readily available perceptual material. On the other hand, Craig and Morgan demonstrated the use of many of the operations that Kaylee used in activity. There were many instances where they sat in deep thought staring at the objects for relatively long periods of time or often needed the perceptual material in reasoning. Although, in activity, Craig and Morgan were able to sequentially coordinate two dimensions to reason within three spatial dimensions. In contrast to the other three students, Dan reasoned primarily in two, but not three, dimensions. He did not enact the mental operations that the other three students did in decomposing and recomposing the spatial objects. His spatial objects were more perceptual than a re-presentation of the space on which he could operate on.

As summarized above, this report illustrates differences in four ninth-grade students' reasoning within frames of reference in spatial contexts and their constructions of coordinate systems in organizing space. Among the four students, Kaylee was the only one who demonstrated reasoning with three levels of units as given in the initial interviews. As shown in the results I presented above, Kaylee was the only student who I considered to have coordinated three spatial dimensions without any perceptual imagery of the spatial objects or enactment of physical activities on the spatial objects. Craig and Morgan both demonstrated reasoning with three levels of units in activity in the initial interviews. As shown in the results I presented above, they were able to coordinate two spatial dimensions sequentially to reason with three spatial dimensions. In order to coordinate three spatial dimensions, they needed some perceptual imagery or physical activities carried out on spatial objects. Dan demonstrated reasoning with two levels of units and as shown in the results, he primarily reasoned with two spatial dimensions. Considering the students' levels of units in operating I inferred from the initial interviews, I propose that the mental operations that produce three levels of units are necessary for constructing frames of references and coordinating measurements (i.e., constructing coordinate systems) within the frames of reference in three-dimensional space. Further investigation on whether the development of three levels of units precedes the ability to coordinate three spatial dimensions, or the coordination of three spatial dimensions precedes the development of three levels of units, or whether the development occurs concurrently is needed.

In *Principles to Actions*, the use and connection of mathematical representations is outlined as one of the eight Mathematics Teaching Practices considered important in every mathematics lesson (National Council of Teachers of Mathematics, 2014). As such, in curriculum and classroom instruction, we use mathematical representations and expect students to make connections between them. In school curricula, however, coordinate systems seem to be taken for granted. For example, consider the introduction to the conventional Cartesian coordinate plane in the Common Core State Standards for Mathematics (CCSSM), in fifth grade Geometry (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010):

> Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate). (p. 38)

After this introduction to a coordinate plane, students are expected to make use of the coordinate plane as a tool to investigate other mathematical ideas in several domains throughout various grade levels. All four students reported in this paper were ninth-grade students who already had experience in using two orthogonal axes to form the Cartesian coordinate plane in school and thus it seems natural that they superimpose a Cartesian-like structure of grids in these spatial tasks. However, even when they superimposed a Cartesian-like grid structure onto the spaces they were trying to organize, the ways of reasoning within frames of reference and consequently the coordination of measurements differed between the students. Considering the fact that high school students are expected to use coordinate systems as a ground for reasoning with various mathematical concepts throughout school mathematics, lack of powerful ways of reasoning within frames of references and coordinating measurements could become a limiting factor for students developing powerful mathematical concepts and connections we expect students to make.

Joshua, Musgrave, Hatfield, and Thompson (2015) proposed a framework of conceptualizing a frame of reference in terms of quantitative reasoning (Thompson, 2011). In their framework, Joshua et al. discussed the mental actions that are involved in coordinating or combining multiple frames of reference and suggested that students' ability to think about measures within a frame of reference supports students in algebraic thinking. For some of the ninth grade students who participated in this study, coordinating three-dimensional space and constructing re-presentations of perceptual space into representational space was a challenging task. Given the difficulty that they had with a relatively perceptual case, imagine how challenging it could be for a student to make meaning of quantitative relationships represented on an abstracted structure. The findings of this study may provide explanations for why students have difficulty constructing and understanding graphs of functions (e.g., shape thinking; Moore & Thompson, 2015). If students' sense-making of graphical representations are not taken into account, the use of these representations could be nothing more than the imposition of mathematical conventions. In turn, these graphical representations might not bring forth the re-presentations of the desired or intended mathematical concepts (such as those stated in the CCSSM). The use of coordinate systems in high school curricula needs to take students' constructions of coordinate systems into account and how students use their coordinate systems in the construction of mathematical concepts and ways of reasoning. As such, this study provides a building block for future investigations into students' constructions and uses of coordinate systems in relation to quantitative and algebraic reasoning.

References

DiSessa, A. A., Hammer, D., Sherin, B., & Kolpakowski, T. (1991). Inventing graphing: Meta-representational expertise in children. *The Journal of Mathematical Behavior*, *10*(2), 117-160.

Hackenberg, A. J. (2010). Students' Reasoning With Reversible Multiplicative Relationships. *Cognition & Instruction*, *28*(4), 383-432. doi:10.1080/07370008.2010.511565

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In S. Wagner, C. Kieran (Eds.) , *Research issues in the learning and teaching of algebra, Vol. 4* (pp. 60-86). Hillsdale, NJ Reston, VA EnglandUS: Lawrence Erlbaum Associates, Inc.

Joshua, S., Musgrave, S., Hatfield, N., & Thompson, P. W. (2015). Conceptualizing and reasoning with frames of reference. In T. Fukawa-Connelly, N. E. Infante, K. Keene & M. Zandieh (Eds.), Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education, pp. 31-44. Pittsburgh, PA: RUME.

Kaput, J. J. (1987). Representation systems and mathematics. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 19–26). Hillsdale, NJ: Lawrence Erlbaum.

Leinhardt, G., Zaslavsky, O., & Stein, M. (1990). Functions, Graphs, and Graphing: Tasks, Learning, and Teaching. *Review Of Educational Research*, *60*(1), 1–64.

Lloyd, G., Beckmann, S., & Cooney, T. (2010). *Developing Essential Understanding of Functions for Teaching Mathematics in Grades 9-12*. National Council of Teachers of Mathematics.

Moore, K. C. (2014). Signals, symbols, and representational activity. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic Algebraic Students: Emerging Models of Students' Algebraic Knowing*, WISDOMe Monographs (Vol. 4, pp. 211–235). Laramie, WY: University of Wyoming.

Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behavior, 32*(3), 461–473.

Moritz, J. (2003). Constructing coordinate graphs: Representing corresponding ordered values with variation in two-dimensional space. *Mathematics Education Research Journal*, *15*(3), 226–251.

National Council of Teachers of Mathematics. (2014). *Principles to Actions: Ensuring Mathematics Success for All*. Reston, VA: Author.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). Common core state standards for mathematics. Washington, DC: Author.

Oertman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C.

Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27–42). Washington, DC: Mathematical Association of America.
Piaget, J., & Inhelder, B. (1967). The child's conception of space (FJ Langdon & JL Lunzer, Trans.). *New York*.
Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America*. Raleigh, NC: North Carolina State University. Abstracted from http://www.patthompson.net/Publications.html (pp. 1-7).
Schwarz, B. B., & Hershkowitz, R. (1999). Prototypes: Brakes or Levers in Learning the Function Concept? The Role of Computer Tools. *Journal For Research In Mathematics Education*, (4), 362. doi:10.2307/749706
Steffe, L. P. (1991). Operations that generate quantity. *Learning and Individual Differences*, *3*(1), 61–82.
Steffe, L. P (2012). On children's construction of quantification. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning in Mathematics and Science Education: Papers from an international STEM Research Symposium*, WISDOMe Monographs (Vol. 3, pp. 143-148). Laramie, WY: University of Wyoming Press.
Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillsdale, NJ: Erlbaum.
Steffe, L. P. & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.
Steffe, L. P., Liss II, D. R., & Lee, H. Y. (2014). On the operations that generate intensive quantity. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebraic students* (Vol. 4). Laramie, WY: University of Wyoming Press.
Tillema, E. S. (2012). Relating one and two-dimensional quantities: Students' multiplicative reasoning in combinatorial and spatial contexts. In R. Mayes, R. Bonillia, L. L. Hatfield & S. Belbase (Eds.), *Quantitative reasoning: Current state of understanding* WISDOMe Monographs (Vol. 2, pp. 143-148). Laramie, WY: University of Wyoming Press.
Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Scheonfeld, & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 1* (Issues in Mathematics Education Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.
von Glasersfeld, E. (1980). The concept of equilibration in a constructivist theory of knowledge. In F. Benseler, P. M. Heijl, & W. K. Kock (Eds.), *Autopoiesis, communication, and society*. (pp. 75–85). Frankfurt : Campus.
von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. *Studies in Mathematics Education Series: 6*. Washington, D. C.: Falmer Press, Taylor & Francis Inc.