Generic Use of Examples for Proving

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NCTM Research Conference
San Francisco, April 2016

Introduction

There is a consensus that students at all grade levels have difficulties understanding and constructing mathematical proofs (e.g., Harel & Sowder, 2007; Healy & Hoyles, 2000; Knuth, 2002). In particular, students tend to rely on empirical evidence as a way of establishing the truth-value of a mathematical statement (Harel & Sowder, 2007). From a pedagogical viewpoint, it is important to emphasize the explanatory aspects of a mathematical proof (Hanna & Jahnke, 1996; Harel, 2013). This is not always possible or easy, as many valid proofs do not provide an explanatory element as to why the proven statement is in-fact true. Additionally, examples that merely verify that a certain claim is true do not necessarily provide insight as to why the claim is true.

Our study was conducted from the perspective that examples can and should be used in a way that provides an explanation why a claim is true (or false), and that such practice is pedagogically sound. We extend Mason and Pimm's (1984) notion of a *generic-example* as "the carrier of the general" (ibid, p. 287) and adapt Leron and Zaslavsky's (2013) notion of *generic proving* as a means to help students "engage with the main ideas of the complete proof in an intuitive and familiar context, temporarily suspending the formidable issues of full generality, formalism and symbolism." (ibid, p. 27). This approach can be seen as a bridge for students, as it may help them move from empirical views and misleading intuition to an understanding of *why* a statement is true (or false). Although the idea of a generic proof has been addressed by several researchers (Balacheff, 1988; Mason & Pimm, 1984; Rowland, 1998, 2001), there is little research on the feasability of using generic proving as a pedagogical tool for developing an understanding a proof. In this study, we aim to examine the extent to which students are able to capture the main ideas of a proof by using examples generically.

The Study

The study reported in this paper is part of a larger study on the roles of examples in learning to prove (NSF Grant, DRL-1220623, PI – Eric Knuth, co-PIs – Amy Ellis, Orit Zaslavsky). In the larger study, we looked at several other categories of example use for conjecturing and proving, including the source of the examples that are used (distinguishing between student-generated and researcher-provided), and examine the productivity of example-use in terms of developing a proof, a deductive argument, or a sound justification that can lead to a proof. In this paper, we focus on examples that are *provided* by the researchers. These cases can be viewed as "provisioning of examples"

(Zaslavksy, 2014). The provided examples that we consider here are examples that the researchers considered *generic examples*, in the sense of having the potential to convey the main idea(s) of the proof.

We aim at answering the question: How can the provisioning of examples be helpful for students in gaining insights into the main idea(s) of a proof? More specifically, our research questions are: 1. To what extent do students treat examples generically?; and 2. How productive are students in using the (generic) examples that are provided to them?

Data Collection

This study was based on individual task-based interviews with 12 middle school (MS) students, 17 high school (HS) students, and 10 undergraduate (UG) students. The interviews lasted approximately 1 hour and were comprised of a series of tasks in which participants were given the opportunity to conjecture and prove.

The interview protocol for middle and high school participants each included 8 tasks total, and the protocol for undergraduate participants included 7 tasks total. Some tasks were shared across all participant populations (with a few differences). Depending on the task and how the student was approaching it, the interviewer often prompted students to use examples, in addition to using examples in order to illustrate what the conjecture was about. In three of the tasks (two shared tasks and one task for MS and HS students only) the interviewer used generic-examples as prompts to help the student come up with a valid explanation or proof (i.e., this was considered provisioning of examples). In this paper, we focus on students' responses to these three tasks.

Due to time constraints and to some modifications in the interview protocols that resulted from reviewing the first few interviews, not all students engaged in all the tasks that were included in the interview protocols.

In the first task (directed to MS and HS students only) students were asked to form a conjecture and explain why their conjecture is true. In the second task students were given a conjecture and were asked to determine whether it is always true and to provide an explanation to why it is (or is not) always true. In the third task students were given a mathematical rule that was true and were asked to explain why this rule always works. The second and third tasks were shared across all participant populations. We present the main part of each task and the examples that the interviewer provided. Full versions of the tasks are given in the Appendix.

<u>Task 1</u>: Numbers with odd number of factors

The interviewer illustrated what a factor is by a specific example 30, providing all its factors in a "generic" way, as a product of two factors: (1,30), (2,15), (3,10), (5,6) and asked the student to find all the factors of 18 and 25. Then the main question was posed: Can you think of a conjecture about numbers that have an odd number of factors?

The conjecture: Any perfect square has an odd number of divisors;

The main idea of one way of proving the conjecture: For any natural number n, we can find all its divisors by listing pairs of divisors like in the case of 30 (above). This process ends when the divisors start repeating themselves. If n is a perfect square, that is $n = k^2$, it ends when you reach k. At that point, the pair of divisors are k and k, that is – this adds only one distinct divisor. So for a perfect square the list comprises of a finite number of pairs plus one more divisor, that is, an odd number of divisors.

Task 2: The sum of consecutive numbers

Students were asked whether they thought the following conjecture was true and why: "If you add any number of consecutive integers together, the sum will be a multiple of however many numbers you added up." In the MS and HS protocols, the interviewer first asked the student whether they thought the conjecture is true for any five consecutive integers. This prompt was not given in the UG protocol.

The interviewer offered a prompt when the student was "stuck", and used the example of the sum of $5 + 6 + 7 + 8 + 9$, by presenting it in the following "generic" form:

$$(7 - 2) + (7 - 1) + 7 + (7 + 1) + (7 + 2).$$

The conjecture: The sum of n consecutive integers is divisible by n if and only if n is odd.

The main idea of one way of proving the conjecture: We can represent the sum of any odd number of integers as:

$$(a - k) + [a - (k + 1)] + \cdots + (a - 1) + a + (a + 1) \ldots + [a + (k - 1)] + (a + k)$$

The n of the conjecture is equal to $2k + 1$ using this representation. Since we are dealing with an odd number of numbers, there must be a "middle" number a. This way of representing the sum allows us to make use of symmetrical considerations around the middle number. In particular, the sum of pairs of numbers that are located an equal distance from a is equivalent to $2a$. As we are adding k number of these pairs to the middle number a, this sum is equal to $(2k + 1)a$. Since $n = 2k + 1$, the sum is divisible by n. The added value of using symmetry to prove that the conjecture is true for all odd integers is that this consideration also provides insight into proving that the conjecture is not true for any even n, as it is enough to note that an even number of numbers has no middle number.

Task 3: The divisibility-by-3 Rule

The students were first asked to decide, using the divisibility-by-3 rule, whether 852 is divisible by 3, and to explain why this rule works for 852. Then they were given the following representation of 852 that can be considered a "generic proof", and were asked whether this representation helps them understand why the rule works:

$$852 = (8 \cdot 100) + (5 \cdot 10) + 2$$
$$= [8 \cdot (99 + 1)] + [5 \cdot (9 + 1)] + 2$$
$$= [(8 \cdot 99) + 8] + [(5 \cdot 9) + 5] + 2$$
$$= (8 \cdot 99) + (5 \cdot 9) + (8 + 5 + 2).$$

The conjecture: A natural number is divisible by 3 if and only if the sum of its digits is divisible by 3.

The main idea of one way of proving the conjecture: We operate using the base-10 system, so the digits of any natural number represent a power of 10. The main idea of this proof is based on observing that $10^k - 1$ is a number string containing k number of 9's (for example, $10^4 - 1 = 9{,}999$) and therefore this number will always be divisible by 3 (since it is always divisible by 9, and 3 is a factor of 9). When a natural number is decomposed using decimal representation as the sum of powers of 10, any arbitrary term can be viewed as $a \cdot (10^k)$, where a represents one of the digits. We can decompose this term as $a \cdot (10^k - 1) + a$. Since the product $a \cdot (10^k - 1)$ is divisible by 3, this sum will be divisible by 3 if and only if a is divisible by 3. Each term in the decimal representation of our natural number can be decomposed in a similar manner as the sum of a number that is divisible by 3 and one of the digits. Therefore, the natural number can be decomposed as two separate sums: a sum of numbers that are *de facto* divisible by 3, and the sum of digits. It follows that the natural number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Data Analysis

The unit of analysis was either a task or a sub-task. We term this unit a *case.* We coded a total of 241 cases, 72 of which included examples provided by the researcher, which we consider generic. In this paper we focus on these 72 cases.

Each case was classified according to two categories:

1. Generic Treatment of Example(s)
2. Contribution of Example(s) to Proving

Although we consider the provided examples to be generic examples, not all students treated the examples as such. The first category deals with the student's treatment of the example: Did the student treat the example *generically* or *specifically*? *Generically* refers to using the example to gain insight to main idea(s) of the proof or treating the particular as general. *Specifically* refers to any non-generic treatment, which, according to our findings, included use of examples for "Illustration and/or Verification".

The second category aims at answering the question: Was the use of example(s) productive for proving? That is, were there indicators that working with the example(s) led the student to gain insights into some aspects of the key ideas of the proof? Here we have three sub-categories:

- Productive for Constructing a deductive argument (PC): the student either used example(s) productively towards constructing a general argument, or the proof was partially or fully based on ideas the student observed from the example.
- Productive for Understanding a general argument (PU): the student was unable to construct a proof or a deductive argument, but was able to understand a given argument or explanation using example(s).
- Non-Productive for Proving (NP): the student used example(s) non-productively and did not construct or exhibit understanding of a deductive argument.

We coded 5 of the 72 cases as *indecisive*, when the role of the examples in the provided argument was unclear, or when it was unclear whether the student's argument could qualify as deductive.

Below are a few cases that convey what generic and non-generic use of examples may look like. We present these cases in order to exemplify what we consider productive and non-productive uses of examples towards constructing a proof or a deductive argument.

Task 1: Numbers with odd number of factors

In the following excerpt a 7th grade student, Ibeliz, used the examples generically to form the correct conjecture that perfect squares have odd number of factors and to explain why her conjecture was true. It seems that she captured the main idea of the proof:

> *"If you square a number then it's going to have an odd number [of factors]. So, like for [25], if you squared 5 it's 25, so it's gonna have an odd number of factors because by squaring 5 you just do 5 times 5, so that only counts as 1 not 2 factors."*

Below is the work of a 10th grade student, Amelia. At this point, the interviewer (I) provided the generic example for the factors of 30 and Amelia calculated the factors of 18, 25, and 16 and wrote their number of factors. At this point Amelia is not certain about the conjecture:

> I: "*So do you think you could come up with any conjectures on which numbers have an odd number of factors?*"
>
> Amelia: "*Umm maybe like perfect square numbers like 25 was the only one that had an odd number of factors oh except this one oh wait never mind because 36 is also a perfect square. Umm I don't know [Student pauses] hmmm yeah I'm not really sure actually…*"
>
> To verify her conjecture, Amelia tests the number of factors of 9 and 49.
>
> I: "*Do you think there's a way to prove that it works for that the conjecture that every perfect square has an odd number of factors is true?*"

Amelia: "*Umm well so far every perfect square we've done has an odd amount of factors so I think yeah it's probably true.*"

I: "*How do you think we could prove it?*"

Amelia: "*If we did every perfect square. I guess.*"

Amelia did not treat the provided example generically and was unable to provide any explanation as to why perfect squares must have odd number of factors. She did not see the logical necessity of this conjecture.

Task 2: The sum of consecutive numbers

In the following excerpt a 9th grade student, Richard, worked on the consecutive numbers conjecture using examples of his own. He did this mainly out of his own initiative. Up to now, he proved the conjecture is true for 1, 3, 5, 7, 9, and 11 consecutive numbers and concluded that the conjecture is true for all odds. Richard also proved that the conjecture is false for 4 consecutive numbers (through providing a counter-example), but was unable to form a conjecture regarding even numbers. The interviewer presents him with an example (generic), and he immediately sees through it and is able to develop a deductive argument to support the claim that the conjecture is true for *any* odd number of consecutive numbers and is false for *any* even number of consecutive numbers.

"*Because it's clear that (7-1) and (7+1) cancel. Uh, (7-1)+(7+1) equals 2 times 7. (7-2)+(7+2) equals 2 times 7. And that gives you a total of five 7s, and these five 7s- well, 5 times 7 divided by 5 is- is 7. Well I mean, 5 times 7 is a multiple of 5. And also, that helps you understand why, um, it only works for odd numbers, because if you have 5, 6, 7, and 8, there's no real middle number.*"

The excerpt below is from the work of an undergraduate student, Daniel. Before the interviewer presented the generic example, Daniel provided empirical arguments and didn't give any reason to why the conjecture works for odd numbers and doesn't work for even numbers. Following the interviewer's prompt, in which Daniel was given a generic representation and told that it helped another (hypothetical) student understand the conjecture, Daniel said:

"*That helped her understand it better?* (7-2)+(7-1)+7+(7+1)+(7+2) *is just another way of rewriting* 5+6+7+8+9*. I don't understand how it helped her explain it like this.*"

It seems that Daniel did not see the general within the example. He was unable to use the provided example productively and to come up with an argument explaining why the conjecture is true for odd numbers and false for even numbers.

Task 3: The divisibility-by-3 Rule

In the following excerpt an undergraduate student, Yaritza, is unsure about how to prove the rule on her own. The interviewer presents her with a (generic) example, and she immediately identifies the key idea of the proof and is able to explain why it works:

"Oh yeah, this makes sense… if you break it down into multiples of 99 and multiples of 9 those are all divisible by 3, and then plus the original three digits in the number [student circles (8 + 5 + 2)] those were divisible by 3 to begin with so the breakdown really explains like why it works."

Yaritza treats the example generically, and is able to generalize it for 4-digit numbers and to explain why it works for any number.

The excerpt below is also from the work of Daniel (an undergraduate student), who did not use the provided example generically and was unable to provide an argument as to why the rule works. Daniel's response to the (generic) example provided by the interviewer is as follows:

"Let's see... 800... 5 times 10, 50. It helps me explain why the rule works, but it doesn't- that only works- this only works for 1 distinct case, 852. It only works for- it only works for 852. How would I show that it works for 1000 or 3000 or a million? How to show that it works? To show that it's not divisible by 3 or it is divisible by 3? Or n can be equal to anything? That's my reasoning."

Daniel then tries to find a "general formula that works" but is unable to provide any argument as to why the rule works.

Findings

The findings in Table 1 include all cases that were coded either as productive or as non-productive (excluding the 5 indecisive cases). There were a total of 67 cases, 23 MS, 31 HS, and 13 UG. Of the 67 cases, 49 (73%) included productive use of examples for proving.

Table 1: The Interplay Between use of Provided Examples and Proving

	Productive for Constructing a Deductive Argument	Productive for Understanding a Deductive Argument	Non-Productive for Proving	Total
MS	14	2	7	23
HS	15	7	9	31
UG	11	0	2	13
Total	40	9	18	67

In terms of productivity, middle school and high school students performed similarly, as 69% of MS cases and 70% of the HS cases exhibited productive use of examples for proving, while the undergraduate students exhibited considerably more productive use of examples (85%) when provided with a (generic) example.

<u>Task 1 (MS)</u>: Out of 10 MS students who worked on this task, 8 students used the examples generically to support their reasoning and 7 of these students were able to form

a correct conjecture and to identify the main idea of the proof based on examples (one case was indecisive).

Task 2 (HS): At first, all 17 HS students conjectured that the sum of any five consecutive numbers is divisible by 5, mostly based on empirical evidence. 13 of them were not able to provide a valid explanation why it worked (not even for the case of 5 consecutive numbers). All but the first two HS students received the generic examples, as the prompt was added to the protocol after reviewing the first two interviews. For 11 of the 15 students, the example was useful, and they all used it generically to explain both why the conjecture works for odd numbers and why it doesn't work for even numbers.

Task 3 (UG): 8 out of 10 UG students were unable to explain why the rule works, 7 of them received the generic proof (one student didn't receive the generic proof due to time constraints). For 6 of these students, the generic proof was helpful. Evidence for this was that they were able to identify the main idea of the proof and to generalize how it would work for numbers with more than three digits. The remaining student did not see through the decimal representation of 852 and did not understand why the rule works.

Conclusion

Our findings show that the majority of the students were able to use the generic examples provided by the researcher generically, to capture the main ideas of the proofs. These findings support Leron & Zaslavsky's (2013) theoretical paper, and indicate that provisioning of examples that potentially can be treated generically has great potential in terms of helping students move towards more deductive arguments and eventually formal proof.

It is important to note that not all students were able to see the potential generic nature of these examples, which supports prior research findings (Bills et al., 2006; Zaslavsky & Zodik, 2007). As Mason and Pimm (1984) note, there is often a mismatch between the teacher's intention and what the students "see" in an example. However, results of our study show that careful selection and presentation of (numerical) examples have the potential to serve as a strong pedagogical tool for teachers in creating activities for students to engage in with the goal of capturing key ideas of proof and proving. These findings have significant implications for teaching, as similar choices and uses of examples can be incorporated within classroom contexts from early grades and up.

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Appendix

Task 1: Numbers with odd number of factors

This question involves the factors of a number. For example, 4 is a factor of 80 because 4 goes into 80 evenly. But, 7 is not a factor of 80 because 7 does not go into 80 evenly.

I want to show you my method for finding all of the factors of 30.

(1, 30)   (2, 15)   (3, 10)   (5, 6)

How many factors does 30 have?

Can you try listing the factors of 18? 25?

Can you think of a conjecture about numbers that have an odd number of factors?

- <u>If student is able to provide a conjecture:</u> Ask student whether s/he thinks the conjecture will always be true and why.
- <u>If student is unable to provide a conjecture</u>: State that Anita has a conjecture that any perfect square will have an odd number of factors.

Here is how Anita explained why 36, which is a perfect square, has an odd number of factors: The factors of 36 are (1, 36), (2, 18), (3, 12), (4, 9), and (6, 6), and I noticed that the number 6 is repeated twice. So counting the factors, I see that there are 9 factors.

Can you prove that Anita's conjecture is true for all perfect squares?

- <u>If student cannot prove the conjecture for all perfect squares</u>:
    - Can you show that the conjecture works with another perfect square the same way I did for 36?
    - Do you think you can now you prove that this conjecture works for all perfect squares?

Task 2 (MS/HS Version): The sum of consecutive numbers

This question involves consecutive numbers. For example, 2, 3 and 4 are consecutive numbers, but 2, 3, and 8 are not consecutive numbers.

Tyson came up with a conjecture about consecutive whole numbers that states: If you add any number of consecutive whole numbers together, the sum will be a multiple of however many numbers you added up.

Can you give an example of how the conjecture works if you use five consecutive whole numbers?

- If student is unable to give an example, provide the following example:

For instance, we could pick the following five consecutive numbers, 13, 14, 15, 16, and 17, and if we add the numbers together we get 75: 13+ 14 + 15 + 16 + 17 = 75. And 75 is a multiple of 5, since 75 = 5 x 15.

- Suggest that the student try it with another set of five consecutive numbers.

Do you think the conjecture will be true for any five consecutive numbers? Why?

Tyson thinks that the conjecture will always be true no matter how many consecutive numbers you use or which consecutive numbers you choose. So he thinks that if you add any 3 consecutive numbers, the answer will be a multiple of 3, or if you add any 6 consecutive numbers, the answer will be a multiple of 6, and so on.

Do you think the conjecture is true for any set of consecutive numbers, not just when you pick five consecutive numbers? Why?

- <u>If student comes up with only true cases:</u>
    - o Ask again if the student thinks that the conjecture is true for any set of consecutive numbers.
    - o Provide the following counterexample: Let's try the conjecture with four consecutive numbers, for instance, 3, 4, 5, and 6. If we add the numbers together we get 18, 3 + 4 + 5 + 6 = 18, but 18 is not a multiple of 4.

      Do you still think the conjecture is true for any set of consecutive numbers? Why?

      Does this example prove that the conjecture is not true? Why?

      When will the conjecture be true? How do you know?
- <u>If student comes up with some false cases (in addition to any true cases):</u>

  Do you think Tyson's conjecture is true? Why?

  Does your example prove that Tyson's conjecture is not true? Why?

  When will the conjecture be true? How do you know?

Task 2 (UG Version): The sum of consecutive numbers

Tyson examined the following rule: if you add k consecutive numbers together, then the sum will be divisible by k. He came up with the conjecture that this rule holds for any $k > 2$.

What do you think of Tyson's conjecture? Why?

- <u>If student thinks the conjecture is true:</u>
    - o Can you find a case for which Tyson's rule doesn't work?

Do you still think the conjecture is true? Why?

- If/When student finds counterexample:
    - Use the student's example of an even *k* and ask: Can you find a case of [*k*] consecutive numbers for which Tyson's rule does work?

      Is there an even value of k for which you can find k consecutive numbers whose sum is divisible by k? How come?

    - <u>If the student thinks it is not possible to find *k* consecutive numbers for which this works (and uses only examples):</u>
        - How do you know this will never work? Why can't there exist a set of *k* consecutive numbers, for a specific even value of *k*, whose sum is divisible by *k*, that you just didn't try?
        - How sure are you?
        - In what other ways can you determine that for sure?

You just established that the rule doesn't always hold for k>2.

<u>If student has found cases of it both working/not working</u>***:*** ***For which values of k does the rule work? How do you know?***	<u>If student has only found counterexamples</u>***:*** ***Can you find cases for which Tyson's rule does work? Why or why not?***

- <u>If student has trouble coming up with a general argument or provides a different general argument:</u>
    - Use one of the student's examples for an odd *k* and replacing the bracketed items with the student's example:
      Another student had an idea of how to explain it. For the [*five*] consecutive numbers [*5, 6, 7, 8, 9*], she decided to write the sum as

      $[(7-2)+(7-1)+7+(7+1)+(7+2)]$.

      How do you think that helped her see why the rule is true for any [five] consecutive numbers?

      Would this argument work for a different number of consecutive numbers? How come?
      Does this help you determine why the rule doesn't work for even numbers?

Task 3 (MS/HS Version): The divisibility-by-3 Rule

Have you ever heard of the divisibility by 3 rule? The rule is that you add up all of the digits in a number, and if that sum is a multiple of 3, then the original number is divisible by 3. And if the sum is not a multiple of 3, then the original number is not divisible by 3. ***Can you give me an example of this rule?***

- If the student can't give an example:
  - For instance, let's take the number 231. If we add up its digits, 2 + 3 + 1, we get 6, and 6 a multiple of 3. So that means 231 is evenly divisible by 3 (231 ÷ 3 = 77).
  - Let's try another number, 572. If we add up its digits, 5 + 7 + 2, we get 14, and 14 is not a multiple of 3. So in this case that means 572 is not evenly divisible by 3 (572 ÷ 3 = 190.666…).
  - Now ask the student to give an example to make sure s/he understands the rule.

Without actually dividing, do you think 57 is divisible by 3? Why?

Here is one way to explain why the divisibility by 3 rule works for 57:

$$57 = (5 \cdot 10) + (7 \cdot 1)$$
$$= 5 \cdot (9 + 1) + (7 \cdot 1)$$
$$= (5 \cdot 9) + (5 \cdot 1) + (7 \cdot 1)$$
$$= (5 \cdot 9) + (5 + 7) \cdot 1$$

Do you think that this rule works for all 2-digit numbers? Why?

- If student cannot prove the rule for all 2-digit numbers:
  - Ask if student can show that the rule works with another 2-digit number the same way that was done for 57?
  - Do you think you can now you prove that this rule works for all 2-digit numbers?

Do you think that this rule works for all 3-digit numbers or 4-digit numbers or any number of digit numbers? Why?

Task 3 (UG Version): The divisibility-by-3 Rule

Are you familiar with the divisibility by 3 rule? The rule is that you add up all of the digits in a number, and if that sum is a multiple of 3, then the original number is divisible by 3. And if the sum is not a multiple of 3, then the original number is not divisible by 3.

Using this rule, do you think 852 is divisible by 3? Why?

Can you explain why this rule works?

- If student cannot explain why this rule works:
  Here is the work that a student wrote down while explaining why the divisibility by 3 rule works for 852:

$$852 = (8 \cdot 100) + (5 \cdot 10) + 2$$
$$= [8 \cdot (99 + 1)] + [5 \cdot (9 + 1)] + 2$$
$$= [(8 \cdot 99) + 8] + [(5 \cdot 9) + 5] + 2$$
$$= (8 \cdot 99) + (5 \cdot 9) + (8 + 5 + 2)$$

Does this help you understand how the rule works? Why or why not?

Do you think that this explanation works for all 3-digit numbers? Why?

- <u>If the student cannot see through the generic example for all 3-digit numbers</u>:
    - Can you show that the rule works with another 3-digit number the same way that was done for 852?
    - Do you think you can now explain how this rule works for all 3-digit numbers?

Do you think that this explanation works for any number? Why?