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# Beyond the Demonstration of Procedures in YouTube-Style Math Videos


## Abstract

Despite the tremendous growth in online mathematics videos for K-12 students, there is surprising uniformity in the expository mode of presentation and the procedural nature of the content. We sought to locate and analyze alternative online mathematics videos that focus on the development of concepts and/or feature student dialogue. The result of our research is a taxonomy of approaches to dialogue and to conceptual development in online math videos that are aimed at elementary and secondary school learners. Our goal was to identify sources of inspiration for future video development efforts, as well as gaps to be addressed. Additionally this research can be used to help the field develop effective ways to communicate about online videos and to make distinctions that are important for research, teaching, and development.


## Introduction

"Math instruction has gone viral" (Boston Globe, 2011), as evidenced by the proliferation of online instructional math videos available on YouTube (www.youtube.com), from non-profit organizations such as the Khan Academy (https://www.khanacademy.org/), and through textbook publishers such as Pearson's MyMathLab (www.pearsonmylabandmastering.com/northamerica/mymathlab/). Online math videos have many positive affordances. They allow students to control of their rate of movement through material and to replay or skip sections based on their personal understanding (Lin & Michko, 2010). Advanced mathematical topics are accessible before students would be exposed to them in public schools (Thompson, 2011). Finally, mathematics learning is within reach at anytime from anywhere, by virtue of the accessibility of the Internet and portable devices (Khan, 2012).

However, despite the ever-burgeoning number of math videos available online across a wide range of topics, there is surprising uniformity in the mode of presentation and the nature of the content (Hopper, 2001). Specifically, "talking hands" or "heads" demonstrate step-by-step procedures using traditional pedagogical approaches (Bowers, Passentino, & Connors, 2012). Critics have noted an overwhelming emphasis on procedural skills, unquestioning close alignment to traditional math curricula, and missed opportunities to address common student difficulties that experienced teachers could anticipate or that research in mathematics education has identified (Danielson & Goldenberg, 2012; Noguchi; 2012; Talbert, 2012).

Even a casual online search of math videos leaves one wondering why childrens' voices are largely missing and why there isn't a greater focus on meaning making and the development of important mathematical ideas.  A number of excellent videos do show students engaged in problem solving and explaining their reasoning (e.g., Annenberg Learner's *Insights into Algebra*), but these videos were filmed to expose teachers to different images of mathematics classrooms, rather than to facilitate students learning from the videos. Consequently, we wanted to develop videos that would insert a new voice into the discussion about what's possible in video-based online mathematics learning. Before doing so, we undertook a systematic review of online math videos that include conceptual elements and/or involve student dialogue. The product of our analysis (presented in the Results section) is a set of categories of ways in which alternative video-based models develop mathematical concepts (versus procedures), as well as different ways in which they make use of learner-centered dialogue (versus exposition).

Our goal was to identify sources of inspiration for future video development efforts, as well as gaps to be addressed. Additionally the resulting taxonomy can be used to help the field develop effective ways to communicate about the online videos and to make distinctions that are important for research, teaching, and development.

## Theoretical Framework

National reform documents and mathematics education research have long maintained the importance of both conceptual understanding and procedural fluency (Hiebert & Lefevre,1986; National Council of Teachers of Mathematics, 2000; National Governors

Association Center for Best Practices, 2010; Star, 2005). However, the definitions of these constructs do not share universal agreement. For example, in standards documents *concept* can sometimes refer to "topic" (as in the "concept of exponential functions"), or it can mean "category" (e.g., when "understanding the concept of linear and non-linear functions" refers to the ability to accurately classify functions into these two categories). To frame our identification of math videos with conceptual content, we characterize "concept" broadly to include any or all of the following aspects of mathematical understanding, which can be leveraged productively in students' mathematical development:

- *meanings*, which refer to one's interpretation of situations, arithmetic operations, representations, and symbols (Voigt, 1994);
- *images*, which refer not only to mental pictures but also denote the internalization of objects via one's actions on them, the anticipation of an outcome of an action performed on an object, or a thought experiment (Thompson, 1996);
- *ideas*, which can include the result of forming relationships between such objects, operating on them, and coordinating mental actions (Hackenberg, 2007);
- *connections*, which include a network of links across representations, ideas, and referents in situations (Hiebert & Lefevre,1986);
- *ways of comprehending a situation*, which may involve noticing some mathematical features and paying less attention to others, as well as elaborating characteristics of a mathematical event or object (Marton, Runesson, & Tsui, 2004); and
- *explanations* regarding why particular procedures work (Skemp, 1976).

Our analysis of non-expository videos is framed by the tenet that dialogue is central to learners' enculturation into forms of academic argumentation, and it mediates certain types of thinking through exposure to the language of more capable others (Vygotsky, 1978). Indeed research suggests that students who watch dialogues tend to model their own language after what they have observed (Mayes, Dineen, McKendree, & Lee, 2001). Participating in dialogue vicariously can facilitate learning by bringing in multiple points of view, supporting the process of perspective-taking, and seeing productive reasoning modeled by other students (Wegerif, 2007).

We borrow from (Alrø and Skovsmose, 2002) to define dialogue as a conversation among two or more people that involves the quality of inquiry, meaning that there is an interaction that aims to generate new meaning or to open up different ways of experiencing things. This is in contrast to univocal discourse (or exposition) in which one-way communication is used to convey or transmit information (Truxaw & De Franco (2007). However, we had great difficulty locating online math videos that included dialogue (among children or between a teacher and students) and that were produced for student learning (rather than for teachers). Thus, we introduce the term "children's voices" to refer more broadly to any substantive contributions (in an online video) from a child or stand-in for a child (e.g., an animated character).

## Methods

The goal of our research review was to first locate online videos that included childrens' voices or that was conceptually oriented (based on the definitions presented in the previous section), and then to categorize the nature of the approaches. It was not our intent to establish the relative frequency of procedural/expository math videos versus conceptual/dialogic videos; we took as a given the predominance of the former (Bowers et al., 2012). Thus, we excluded from the review the vast majority of expository, skills-based videos found on YouTube. We also employed two additional inclusion criteria. First, the videos had to support use by K-12 children to learn mathematics. Thus, we excluded classroom videos produced for teacher professional development or for learning by undergraduates. Second, the digital material had to feature videos. Thus, we excluded sites with simulations, games or applets alone.

We began searching for videos in locations that were most likely to include reform-oriented instructional approaches (and thus focus on conceptual development or include children voices). First, we searched the following digital repositories: *HippoCampus* (http://hippocampus.org/); the *National Science Digital Library* (http://nsdl.org/); *PBS Learning Media* (www.pbslearningmedia.org/); *Merlot* (www.merlot.org); *MathFlix* (http://mathflix.luc.edu/); NROC (http://thenrocproject.org/#/); *TeacherTube* (www.teachertube.com), and *iTunes U* (www.apple.com/education/ipad/itunes-u/). In a literature review of published research articles, one can use key word searches in relevant journals. However, the online repositories typically afforded searches only by "mathematics" and grade level rather than search criteria such as "non-expository" or "conceptual." Second, we searched all NSF DRK-12 awards using the search term "video."

Finally, we opened up the search more broadly to include videos posted on YouTube or YouTube channels. To make our search more tractable, we selected one topic per grade level band—fractions at the elementary school level, proportions at the middle school level, and quadratic functions at the high school level. These topics were chosen because of their importance mathematically and their complexity conceptually.

We analyzed videos from the identified projects across two themes (following Bowen, 2006)—the use of childrens' voices and the approach to conceptual development. Using *open coding* from grounded theory (Strauss, 1987), we induced categories of the different ways in which childrens' voices are used. Next, we induced categories of approaches to developing concepts.

## Results Part 1:
## Categories of the Use of Childrens' Voices in Math Videos

We found very few examples of the use of childrens' voices in math videos that were created for learners in Grades K-12. The two major approaches are described below: (a) children in traditional roles; and (b) children represented by animated characters resolving dilemmas.

### Children in Traditional Roles

In this category of video, children take on the role of either the teacher or the students in the narrative of a typical traditional mathematics classroom. For example, *Children Teach Maths* (http://www.youtube.com/watch? v=e9HH924XeHM) is a series of short videos on YouTube, 10-15 minutes each in duration. In each video, a pair of elementary school children stand near an easel holding a large pad of paper on which a series of diagrams and arithmetic calculations were prepared prior to filming (see Figure 1). The children take turns playing the role of a traditional math teacher by carrying out a procedurally-oriented mathematics script. We are not told whether the teacher prepared the script or if the children created it as a result of listening to how the teacher taught the lesson, but the script does not appear to contain the type of ideas that arise from childrens' struggles and engagement with mathematics.

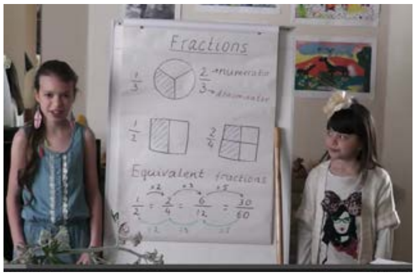


*Figure 1.*Screen from *Children Teach Maths*

Another example of videos that fall in this category are those found on *Mathtrain.TV* (www.mathtrain.tv), which contains a set of videos from the *Kids Teaching Kids* project at Lincoln Middle School in Santa Monica, California. The short videos (typically 2-3 minutes) are tablet-driven tutorials that demonstrate a variety of procedures (such as how to add fractions with unlike denominators). The only difference between then and typical “talking hands” videos is that the voice narrating each video is that of a child or pair of children taking turns.

Finally, in the *MathFlix* (http://mathflix.luc.edu/) videos, children also play a central role, but it is as the student responder in an interaction with an instructor that maintains the initiation-response-evaluation (IRE) interaction pattern typical of traditional math classrooms (Franke, Kazemi, & Battey, 2007). The MathFLIX web site contains over 1000 short (4-7 minute) math videos demonstrating procedures across a wide variety of topics from the K-12 curriculum. The videos were excerpted from a cable access television show in Chicago in which students call a televised phone number and work through a different math concept with a teacher who leads the demonstration of a procedure.

**Children Represented by Animated Characters Resolving Dilemmas**

In this category of video, children are represented by animated characters who work together to resolve a mathematical dilemma. Characters interact with each other and sometimes express misconceptions or confusion. For example, PBS Learning Media hosts a web site containing a set of short video clips (with support materials) excerpted from the animated math show *Cyberchase* (http://www.pbslearningmedia.org/collection/cyb/). In the videos a team of kids called the CyberSquad use math to outsmart a villain in a digital universe called Cyberspace. In a segment from the video called “Crumpets Recipe” (http://ca.pbslearningmedia.org/resource/vtl07.math.number.fra.crumpetrec/crumpets-recipe/), the CyberSquad "kids" face the dilemma of how to double a recipe for Cosmic Crumpets. During the discussion, confusion is expressed over the fraction $\frac{6}{4}$ that results from doubling $\frac{3}{4}$. As one of the characters puts it, "How can you have a fraction that is bigger on the top than the bottom?" The confusion gets resolved when one of the characters says she doesn't know how a fraction can look "top heavy" like $\frac{6}{4}$, but she does knows that $\frac{3}{4}$ means 3 one-fourth cups. This allows the "children" to then interpret $\frac{6}{4}$ meaningfullly as 6 one-fourth cups.


*Figure 2.* Screen from *Cyberchase*

Another example of videos from this category are the short animations found in the *MathSnacks Project* ([http://www.mathsnack.com/)](http://www.mathsnack.com/). This web site contains short animations and games, geared at middle school students, which one can view/play on a computer, iphone, ipod or ipad. Like the *Cyberchase* videos, each animation has a story line and captures a fanciful situation that is likely to appeal to children. For example, in a fictional game called Atlantean Dodgeball, each team starts out with 1000 players. As they lose large numbers of players (e.g., leaving the two teams with 500 vs 480 players at one point and 9 vs 2 at a later point), some confusion is expressed over what really matters in this situation—an additive comparison or a multiplicative one. Animated characters share different ideas as they interact.

**Discussion**

Across both categories, the videos were largely scripted by adults. One exception are *MathFlix* videos, because of the student callers. However, their input was constrained to short answers that fit into the teacher-guided narrative. We were unable to find instances of students discussing their concerns or confusions. For example, in one of the proportions videos it is clear that the editors cut out mistakes the child made in answering questions.

The most promising models of the use of childrens' voices relied on animated characters, in which sources of confusion were presented. However, the scripts likely miss the type authentic student confusion that adults have trouble anticipating. By expanding our search to undergraduate science videos, we found two projects that expanded beyond interactions of animated characters to feature human learners' dialogue. The *Veritasium Project* (https://www.youtube.com/user/1veritasium) features man-on-the-street interviews in which real people express their common misconceptions about a range of physics topics. However, the misconceptions are resolved, not by the learners but through an explanation provided by the interviewer often with the aid of some experiment or physical materials. The *Interactive Video Vignettes for Biology Project* (under construction but described at http://resourcecenters2015.videohall.com/posters/518) presents scripted dialogues between scientists tackling biological dilemmas.

These examples point to a gap that could potentially be quite useful if filled. There seems to be a need for unscripted videos in which real kids resolve their own dilemmas, argue for and against particular ways of reasoning, and convey sources of confusion that are difficult for adults to anticipate. In other words, alternative videos are needed that capture student dialogue in the sense defined in the Theoretical framework section, as a conversation among two or more people that involves the quality of inquiry. Videos that highlight student dialogue may also depict learners as wonderers and generators of ideas, as well as making explicit the struggle and persistence that is part of authentic mathematical practice.

## Results Part 2:
## Categories of the Nature of Conceptual Development in Math Videos

Our analysis of online math videos with conceptual content revealed six different categories: (a) visual representations; (b) stated ideas; (c) real world contexts; (d) precision of language, (e) using patterns to explain why; and (f) using mathematical properties to explain why. Each of the following approaches is described and illustrated below.

### Visual Representations

Videos in this category use animation, highlighting, color, and illustrations to bring to life the following aspects of a concept—mathematical connections, meanings, and imagery. For example, in the video "Bad Date" from the *MathSnacks Project* ([http://mathsnacks.com/baddate-en.html)](http://mathsnacks.com/baddate-en.html) a woman tells her friend that she went on a bad date in which the man spoke 175 words to her 25 words. To illustrate the ratio relationship of 7 to 1, a set of 1 utterance from the woman and 7 utterances from the man is animated via the use of "word bubbles" (see Figure 3). The set of 1 word bubble from the woman and 7 from the man is then repeated. The video medium is particularly well-suited to the creation of such dynamic imagery. Something that is quite difficult


*Figure 3.* Screens from *MathSnacks*

to accomplish on a whiteboard in a classroom (e.g., continuing the use of word bubbles until there are 25 from the woman and 175 from the man) is easy to accomplish on video. Such a visual representation could aid student in developing the concept of the meaning of a ratio as a composed unit (Lamon, 1995), in which two quantities (here man-words and woman-words) are composed or joined together to form a new unit (a ratio). Evidence for the formation of a *composed unit* is often seen when a student iterates (repeats) or partitions (breaks apart into equally-sized sections) a composed unit, thus preserving the multiplicative relationship present in the ratio (Lobato & Ellis, 2010).

**Stated Ideas**

This category emerged from our effort to capture the approach of the *WhyU Project* (http://whyu.org), whose aim is to "give insight into the concepts on which the rules of mathematics are based" rather than "focusing on procedural problem solving." To accomplish this, a collection of mathematical ideas and relationships are stated verbally and often briefly illustrated. For example, in a lesson on linear functions, the following ideas are stated:

- Taking any linear equation of the form y = mx and adding a constant b to the right side shifts the graph vertically by b units (see Figure 4).
- If we double the horizontal change, the vertical change will also be doubled for a line.

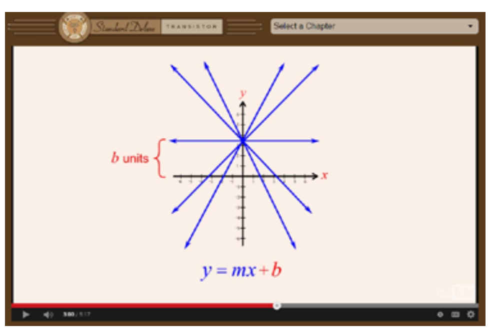


*Figure 4.* Screen from the *WhyU Project*

**Real World Contexts**

Videos in this category consist of real world contexts from which problems can be posed online, in classrooms, or explored in accompanying simulations. For example, in the video "The Incredible Shrinking Dollar" from the *Three-Act Math Series* by Dan Meyer (http://mrmeyer.com/ threeacts/ shrinkingdollar/), Dan photocopies a dollar bill at 75% its original size and then repeats the process multiple times (see Figure 5). The question is then posed: If Dan shrinks the dollar nine times like this, how big will it be? Experiencing a context via video versus reading a textbook problem, can aid in the comprehension and meaning-making of a problem situation. However, it's important to note that these are videos *for* instruction, rather than videos *of* instruction, meaning that classroom supports are necessary for learners' conceptual development using the digital material.

*Figure 5.* Screen from the *Three-Act Math Series*

Another example of videos that fall in this category are those from the *Scale City Project* (http://www.ket.org/scalecity/), which is geared toward helping middle school students develop an understanding of proportional reasoning. There are seven "roadside stops" in the journey to "Scale City," each of which presents a narrated short video that explores some real world context in which proportional reasoning can be used. After each video there is an interactive simulation, which provides students with the opportunity to develop concepts such as a ratio as a multiplicative comparison (Lobato & Ellis, 2010).

**Precision of Language**

In this category, a teacher develops a mathematical idea through careful use of precise language. As an example, consider the videos provided by the *Art of Problem Solving* (AoPS), which is an organization that aims to provide instructional resources for mathematically ambitious students. In addition to offering fee-based online courses for students and a line of textbooks, AoPS also offers hundreds of free videos featuring founder Richard Rusczyk (http://www.artofproblemsolving.com/videos). As an example of the role of precise language in concept development, consider the video in which the instructor develops the meaning of slope as a rate of change (rather than only as a calculation like "rise divided by run," which is much more common in YouTube videos; see Ani, 2012). The instructor frequently talks about slope as a relationship between the change in y and the change in x.  He consistently uses language like "For every 2 that y changes, x changes 3."

**Using Patterns to Explain Why**

In this category and the next, the videos focus on the same aspect of conceptual development, namely an explanation for why some procedures work or why some property holds. However, the source for the explanation differs across the two categories. In this category, the instructors in the videos appeal to the need to maintain consistency in a number pattern. For example, in the YouTube video https://www.youtube.com/watch?v=uD7JRdAzKP8, the teacher explains why the product of two negative numbers is positive by continuing a numerical pattern that begins with the products of positive numbers and ends with a product of two negative numbers (see Figure 6).

In a similar fashion, number patterns can be used to explain why any number raised to the zero power is equal to 1. For example, see https://www.youtube.com/watch?v=9svqGWwyN8Q from *Mr. T's Videos*, a collection of short tutorials created by a math teacher who also has a blog called Teaching and Learning Math (http://teachingandlearningmath.blogspot.com).

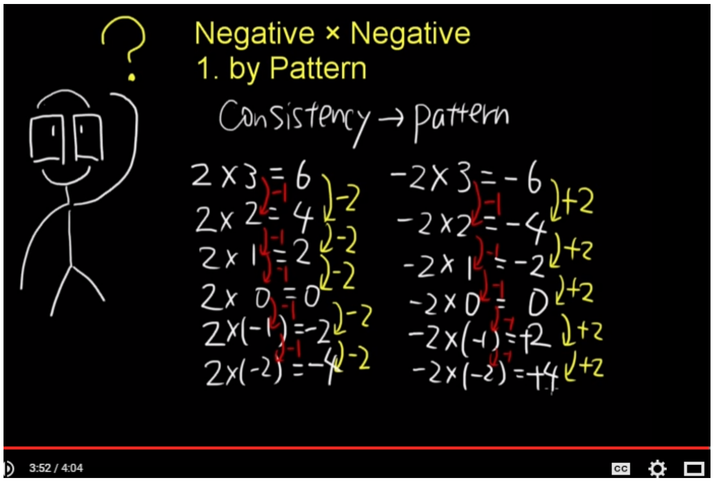


*Figure 6*. Screen from https://www.youtube.com/watch?v=uD7JRdAzKP8

**Using Mathematical Properties to Explain Why**

This category is similar to the previous one in terms of the aspect of conceptual development that is targeted. However, the explanations in this category rely, not on number patterns, but on mathematical logic using properties and identities. For example, the project *Thinking Mathematically!* (http://www.jamestanton.com/) by Dr. James Tanton (Mathematician in Residence at the Mathematical Association of America) has a video (see https://www.youtube.com/watch?v=eV6iYvd4KS0) that explains why the product of two negative numbers is positive by appealing to mathematical properties, such as the commutative and distributive properties, as well as the zero property of multiplication (see Figure 7). Similar explanations can be found on *WhyU* (see https://www.youtube.com/watch?v=NtJy8uQVN7w) and a number ofYouTube videos (e.g., https://www.youtube.com/watch?v=hLm5lRxt1rE and http://www.youtube.com/watch?v=p_PZIGPYlME).

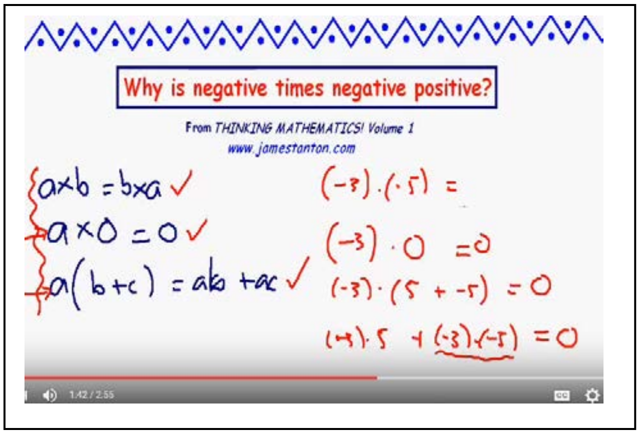


*Figure 7*. Screen from *Thinking Mathematics!*

**Discussion**

Despite the proliferation of procedurally-oriented math videos, our research review revealed some ways in which concepts are developed in online videos. In particular, we found the use of visual representations a source of inspiration, especially in that it made such good use of the dynamic medium of film. We wondered if this approach could be combined with the use of real children in dialogue and extended to visually highlight and illustrate the ideas that *children* raise in the videos.

The issue of the role of childrens' thinking emerged in additional ways. For example, the *WhyU Project* videos are strikingly explicit about being driven by the goal to help learners develop insights and ideas rather than being driven by skill development. However, there is not much of a sense of students' conceptual difficulties or a psychology of mathematics that considers how children think productively about different ideas. The approach is driven by a expert view of the relatedness of ideas rather than starting with what we know about students' reasoning. This is true for many of the videos in the other categories as well. In contrast, the videos from the *Math Snacks Project* seem to consider, for example, how children form ratios as opposed to simply presenting a textbook definition of a ratio (e.g., as a comparison of two numbers using division, usually expressed in fraction form). One shortcoming of the latter approach is that simply writing $\frac{a}{b}$ or $a \div b$ does not ensure that a student has actually formed a ratio between $a$ and $b$. Thus, there are opportunities to expand on the approaches reviewed here by highlighting what is known about students' reasoning and their conceptual difficulties.

## Conclusion

The taxonomy that emerged from this research can help teachers, video-developers and researchers identify and extend conceptual and dialogic components of existing digital resources and point to gaps in which additional models will be useful. As a result, we launched Project MathTalk (www.mathtalk.org) using several design principles that emerged from our review of the currently available online math videos.

First, we created unscripted videos that each feature a pair of secondary school students engaged in dialogue (see Figure 8a). The learners interact with each other (and with a teacher who is facing them but who is off-camera; only her voice is heard, in an effort to keep the focus on the students' mathematics). The students raise and resolve confusions that we did not anticipate, even as researchers familiar with the mathematics education literature. The students justify and explain their reasoning, elucidate their own comprehension of mathematical situations, and argue for and against various misconceptions and alternative strategies. Second, the focus of our videos is on the development of mathematical meanings and interpretations, which are connected to a model of how students learn particular content over time. Finally, we use annotations and animation to highlight key ideas expressed by students in the videos (see Figure 8b). We do not believe that the taxonomy is complete or that our videos will be the final word on alternative approaches to online math videos. Instead, we hope this project can inspire developers of video-based tools to re-imagine the possibilities for online learning in mathematics and in other content domains.

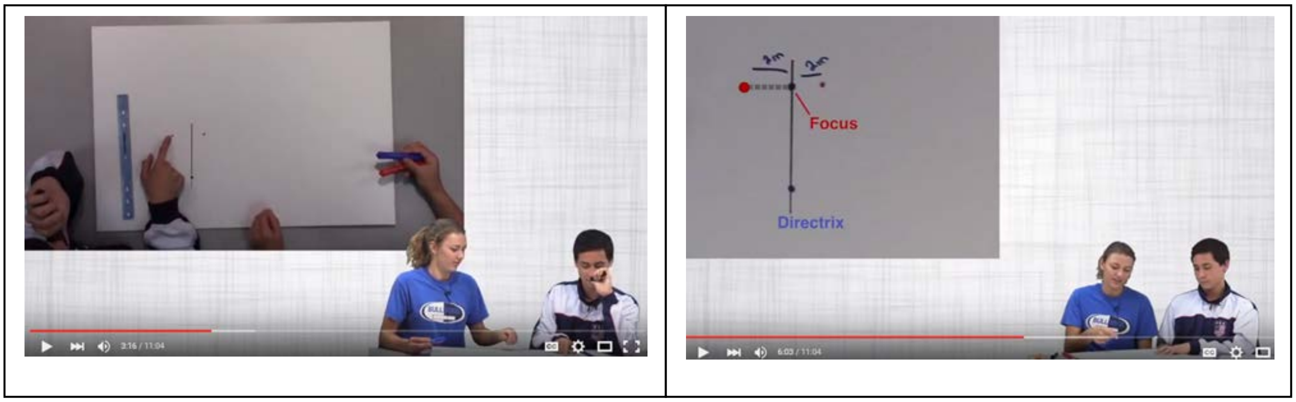


*Figure 8*. Screens from *MathTalk.org*

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