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Paper Title: Rethinking Elementary Pre-service Teachers' Addition Strategies
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Abstract

This paper will focus on an exploratory study about how elementary pre-service teachers (PSTs) communicated their mathematical thinking while solving carefully selected, multi-digit, addition problems. The numbers in the task were purposefully chosen to elicit both PSTs' procedural and conceptual knowledge. The study seeks to demonstrate that although many PSTs used the standard algorithm, some simultaneously used unprompted conceptual strategies based on their own prior mathematical knowledge and experiences. The paper will discuss the implications of this study for mathematics teacher educators to leverage the nuanced ways that PSTs' communicate their mathematical thinking while developing PSTs' pedagogical content knowledge.

Rethinking Elementary Pre-service Teachers' Addition Strategies

Introduction and Rationale

Pre-service teachers (PSTs) bring their prior experiences and knowledge about mathematics to their preparation programs that inform their views of teaching and learning mathematics (Hammerness et al., 2005). Because children may bring diverse ways of knowing and doing mathematics to their classrooms (Philipp, 2008), PSTs need to be prepared to recognize the many ways that children can solve a mathematics problem. Yet there is research that has found that often PSTs are not prepared to consider the multifaceted ways that children can communicate their mathematical thinking in ways that differ from PSTs' own understanding of mathematics (sometimes a procedurally-driven perspective of mathematics) (Ambrose, 2004). One way to help PSTs develop their mathematical knowledge for teaching (Hill, Ball, & Schilling, 2008) is for PSTs to engage in discourse about their procedural problem solving strategies while attending to the conceptual underpinnings of those strategies (Rowland, 1995; Herbel-Eisenmann, Wagner & Cortez, 2010). Furthermore, PSTs should also consider how purposeful number choice for the problems may ultimately pose to their students can also elicit various procedural and conceptual strategies (Land et al., 2015). According to Rowland (1995) it is only through discourse that uncertainty of PSTs' (or children's) mathematical thinking can be identified, acknowledged, and rectified. Therefore, this study investigated how PSTs communicated their mathematical thinking when they solved six two-digit addition problems. The addition problems, written both horizontally and vertically, were purposefully crafted with numbers that could elicit PSTs' procedural and conceptual strategies. We were interested in how the mathematical task posed informed the different strategies used by the PSTs. We asked the following questions:

1. When PSTs' are given a multi-digit addition problem, what kind of solution strategies do PSTs provide?
2. Do the different presentations of a problem, horizontally versus vertically, elicit different types of knowledge about solving the task (if at all)?

This study aims to better understand the connections between *conceptual* and *procedural* knowledge about multi-digit addition as defined by Hiebert and Lefevre (1986). According to them, conceptual knowledge is characterized by the presence of relationships between individual facts, forming a network of knowledge (p. 3-4). The facet of procedural knowledge that we focus on in this study is made up of algorithms for solving mathematical tasks (Hiebert and Lefebre, 1986, p. 6). Investigating multi-digit addition allowed resarchers to observe a great deal of coordination between a conceptual view of numbers that are being added (leveraging the particular characteristics of the numbers given to make calculation easier) and procedural view of the operation of addition (viewing the addition problem as a set of steps that need to be carried out in a particular order, regardless of the numbers involved). As proposed by Hiebert and Lefevre, we analyzed our data with the view that conceptual and procedural knowledge are related.

One important distinction that should be discussed is the difference between *procedural knowledge* and *procedural fluency*. The latter is discussed by the authors of *Adding It Up* (Kilpatrick, Swafford, and Findell, 2001) and, in our view, refers to students' abilities to choose, appropriately utilize, and successfully complete particular procedures. From that perspective, this study investigated PSTs' procedural *fluency* with multi-digit addition by analyzing how they coordinated their procedural and conceptual *knowledge* in order to solve the given problems.

Methodology

This exploratory qualitative study with descriptive statistics took place at a large university in the southern United States, during the spring 2015 semester. Data was collected from 14 students enrolled in a randomly selected introductory mathematics course for elementary education majors. Participants were paired in order to facilitate dialogue and to obtain video recordings. All transcribed data from the structured interactions and the videotaped classroom lesson and attendant discussions were coded for specific content markers, as is usual in content analysis (Cohen, Manion & Morrison, 2007).

The data sources consisted of PSTs' responses to a set of two-digit addition problems in which the numbers were purposefully chosen. These problems fell into three distinct types: 1) problems where the ones place value summed to ten (e.g. 29 + 31), 2) problems where both of the ones place value digit were close to ten and where only the ones place would need regrouping (e.g. 47+38), and 3) problems where both the ones place value and tens place value would need regrouping (e.g. 68+73). Three problems were positioned horizontally, and three problems were positioned vertically, as we hypothesized that different orientations might elicit different types of solution strategies, some procedural and some more conceptual. PSTs were then asked to solve the problems by verbally communicating their solution to their partner without writing anything down but reading the problem that was written visually on an index card; PSTs videotaped each other's strategies using iPads. The PSTs also completed a written debrief of their strategies after each group of (vertical or horizontal) group of three problems.

Data Analysis

The authors first reviewed the transcripts and categorized the solution strategies into three broad categories, as informed by various research that describes different strategies for solving similar problems (Van de Walle, Karp, Bay-Williams, 2013): of (1) Purposeful Redistribution, (2) Adding by Place Value, and (3) Digit Language, which included traditional algorithmic thinking and the process of vertically stacking the numbers on top of each other. Table 1 describes these three solution strategies with examples from our data.

Solution Strategy Category	Description	Example
Purposeful Redistribution	The process of regrouping a number by any place value so that it is easier to add	For the problem 21+39: "Well, I got 21 plus 39. What I would do is carry the 1 to this number [points at 39] so it would be 40 so 40 plus 20 is 60."
Adding by Place Value	The process of adding a number by its place value and regrouping when necessary	For the problem 12+ 28: "So I have 12 plus 28. And that would be 40...8 plus 2 is 10. 20 plus 10 is 30. So 30 plus 10 is 40. And that last 10 is what I got from 8 plus 2. So the answer is 40."
Digit Language	The process of adding only by considering the numbers as individual digits and disregarding any place value.	"I have 28 plus 19. And I know that 8 plus 9 is 17. Carry the 1. And 2 plus 1 is 3 plus 1 is 47."

Table 1. Solution Strategies with Descriptions and Examples.

Secondary coding of the data included evidence of using derived facts (Carpenter et al., 1999). Examples of derived facts could be when students use landmark numbers such as 10 (e.g., 7+3 is 10 so 7+4=11) or doubles (e.g., I know 7+8 because 7+7 is 14 and one more is 15). Students could have also used counting up or counting down in order to solve the problem (e.g. 7+5 could be solved by counting up from my starting number five units until I got to 12.) The students could have also produced the answer by using automatic facts with no observable evidence of how an answer was derived. Also, it was noted when students appeared to be mirroring or replicating a strategy used by their partner or by themselves in the previous problem. Once initial coding was completed, data was examined for patterns based on number choice within problems, common strategies across problems, or correlations based on the visual representation of the problem (vertical vs. horizontal).

After the initial and secondary coding of the data, we then re-categorized the strategies into three larger themes: *algorithmic-algorithmic*, *algorithmic-conceptual*, and *conceptual-conceptual*, where the first word characterizes the approach the student took to the entire procedure of adding the two given multi-digit numbers and the second word characterizes their approach to the sub-procedures within the overarching strategy that they used. The overarching strategies were coded as *algorithmic* if they followed the standard American algorithm for adding numbers. Hallmarks of this strategy included PSTs' mentally rearranging the numbers so that they could visualize a horizontal problem as vertically in order to satisfy what they explicitly perceived was the unavoidable first step of the standard algorithm for addition. *Conceptual* overarching strategies dealt with each multi-digit addend as a coherent value and leveraged the particular characteristics of the number to make calculation easier.

Within these overarching strategies, however, we saw an interesting range of sub-procedures. *Sub-procedures* were the strategies used by the PST to, for example, add the ones digits while operating within the overarching strategy of the standard algorithm for addition. These sub-procedures, embedded within the PSTs' overarching solution strategies, displayed a variety of algorithmic and conceptual thinking, as we will discuss below. Similarly, these sub-procedures were categorized as either *algorithmic* or *conceptual*. Table 2 details our overarching/sub-procedure coding scheme for coding *algorithmic-algorithmic* and *algorithmic-conceptual* solution strategies. It should be noted that our data did not show any examples of *conceptual-conceptual* strategies, but we hypothesize that those strategies exist, and that PSTs are capable of using them. It may be that the way the problems were posed did not elicit those types of strategies.

Code	**Description**	**Example in the Data**	**Code**
Algorithmic-algorithmic	Solutions that relied on the standard algorithm for multi-digit addition as an overarching organizing principle for the solution and then relied on recalled facts (for example) to complete the attendant sub-procedures.	29 + 38. 9 plus 8 is 17. Carry the 1. 2 plus 3 is 5 plus the 1 is 6. So the answer is 67.	Digit Language
Algorithmic-conceptual	Solutions that relied on the standard algorithm for multi-digit addition, but utilized a conceptual strategy (such as a derived fact) to complete the sub-procedures.	"28 + 19 and [unclear, thinks while looking at front of card] okay so alright so I'll just take away 1 from 28 so I'd be stuck with 27 and then I would put the one over here [points to 19] to make 20 so because I have 27 plus 20 it would 47 so I would have 47"	Purposeful Redistribution, Derived Fact, Landmark Number
		"12 + 28. And that would be 40. Because uhm, kinda like the same method that she used uhm 8 plus 2 is 10. 20 plus 10 is 30. So 30 plus 10 is 40. And that last 10 is what I got from 8 plus 2. So the answer is 40."	Adding by Place Value
		"57 + 64. 4 plus 4 is 8. And you have three left over from 7. Hold on. I am confused. 7 plus 4 is... 4 plus... 7 plus 4. 4 plus 4 is 8. You have three left over from the 7 which makes it 9, 10, 11 (counting up to 11). 6 plus 5, 6 plus 6 is 12, you subtract the 1, which makes it 11. And you carry the 1 which makes it somethin'.... 7, 8,9, 10, 11. 111."	Digit Language, Counting Up Strategy and Doubles

Table 2. Overarching/Sub-Procedure Coding Scheme.

Neither the content coding nor the discourse coding was analyzed statistically, beyond the accounting of the frequency of specific codes. The focus of this small exploratory study is to investigate instances of PSTs' thinking about solving arithmetic problems, not to make generalizable statements about the content of those thoughts.

Findings

The majority of the PSTs strategies fell into the *algorithmic-conceptual* category. For example, the use of Digit Language, which was another hallmark of an overarching traditional algorithm-based strategy, often made way for conceptual sub-procedures. In fact, the PSTs’ solution strategies overwhelmingly were coded as Digit Language with the use of automatic facts and explicitly evoked the standard algorithm for multi-digit addition. Within this overarching algorithmic strategy, some PSTs also leveraged other conceptual justifications and strategies to make sense of their summations. For example, one student used Digit Language with a doubling strategy to add numbers in the ones place: “...I have 29 plus 38 and I know that nine plus eight is 17 because if I do eight and eight that’s 16 and then one more, that’s 17.” This suggests that although PSTs may use solution strategies that mimic the standard algorithm, they also simultaneously leveraged other conceptual strategies that were appropriate given the number choice.

In a second finding, the orientation of the problems (horizontal versus vertical) did not elicit different strategies even though visual display and number choice were purposeful for eliciting both conceptual and algorithmic strategies. Originally we hypothesized that the horizontally displayed problems might elicit more diverse solution strategies with conceptual justifications, but this turned out to not be the case for the students in our study. Analysis of the PSTs' discourse suggested that their cognitive load (Sweller, 1994) was greater when solving the horizontal problems because nearly all of them resorted to mentally repositioning these problems into a vertical format in order to solve the problem more easily (e.g., "...My first card is 18 plus 22. That [vertical position] makes it easier that they're on top of each other so I know 8 plus 2 is 10 so i put the 0 [points below the ones places] and carry the one and then 2 plus 1 is 3 plus another is 4 so the answer is 40.")

Finally, we originally hypothesized that problems when the ones places summed to ten (e.g., 12+28, 29+31) might elicit more conceptual overarching strategies such as Adding By Place Value, but that was not the case; for the most part, the PSTs solved the problems by using Digit Language. We also noticed that the highest incident of derived fact strategies appeared when the PSTs solved problems where both ones digits were close to ten (e.g., 48+37) and only the ones place summation needed regrouping (e.g., 63+78).

Next Steps

Research in mathematics education at times frames elementary PSTs' mathematical knowledge to be somewhat limited in its sophistication and depth, especially with respect their diversity of strategies. However, our initial findings suggest that PSTs used various conceptual strategies even while focusing on the numbers as individual digits and while using the standard American algorithm. Furthermore, the PSTs who used "big picture" conceptual strategies such as Purposeful Redistribution or derived facts with doubles to solve the problems did this based on their own prior mathematical knowledge and experiences. Therefore, the results from this study highlight another opportunity for mathematics teacher educators to leverage PSTs' prior understandings in order to further develop their pedagogical and specialized content knowledge of mathematics.

The next steps for this exploratory study are three fold. First, we acknowledge the limited sample size of our study and would extend this study to include more participants and other operations such as subtraction. Second, we would replicate this study in both content and methods courses in order to examine the ways that PSTs might solve these problems given what they learn about children's mathematical thinking and typical solution strategies. Finally, future iterations of this study might employ the commognitive analytic framework (Sfard, 2008) to examine how the PSTs communicate their thinking in pairs and potentially mirror each other's strategies.

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