2016 NCTM RESEARCH CONFERENCE
April 11-13 • San Francisco

Paper Title: Strengthening Prospective Elementary Teachers' Knowledge of Divisibility: An Interventional Study

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Session Title: A Classroom Intervention: Strengthening Prospective Elementary School Teachers' Knowledge of Divisibility
Session Type: Brief Research Report
Presentation Date: April 12, 2016
Presentation Location: San Francisco, California

Strengthening Prospective Elementary Teachers' Knowledge of Divisibility
An Interventional Study

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Introduction

Question: which of the numbers in the set $\{3, 7, 15, 22\}$ are factors of the number $N = 2^3 \times 3^2 \times 5$? Which are not factors of *N?* How do you know? One route to a solution is to compute the whole number value *N* and then to test factor candidates 3, 7, 15 and 22 via trial division. If a number in the list evenly divides *N* then the number has demonstrated its identity as a factor. If the number in the list fails to evenly divide *N* then the number has demonstrated that it is not a factor. Using this method, one can readily answer the first two questions: 3 and 15 are factors of *N* and 7 and 22 are not. The answer to the second question is just this: because 3 and 15 evenly divide *N* they are factors, and, because 7 and 22 do not evenly divide *N* they are not factors.

A less frequently taught method compares the prime decomposition of *N* to the prime decomposition of the factor candidates. If the prime decomposition of *N* includes the entire prime decomposition of the factor candidate, the number is a factor. If the prime decomposition does not include the entire prime decomposition of the factor candidate, the number is not a factor. For example, consider the factor candidate 15; because $15 = 3 \times 5$ and $N = 2^3 \times 3^2 \times 5 = (3 \times 5)(2^3 \times 3)$ we can conclude that 15 is a factor of *N*. In contrast, consider 22; since the prime decomposition of 22 is $22 = 2 \times 11$ and the prime decomposition of *N* does not include *both* 2 and 11, 22 is not a factor of *N*. Using this method, one can determine the same answer presented above to the first question: 3 and 15 are factors of *N*, 7 and 22 are not. The answer to the second question, however, is very different from that presented above. Here, "how do I know?" rests upon a much richer set of prerequisites: the commutative and associative properties of multiplication, rules of exponents, and, perhaps most importantly, an understanding of the uniqueness of prime decomposition guaranteed by the Fundamental Theorem of Arithmetic (FTA).

Research has shown that preservice elementary school teachers (PSTs) show a preference for answering this question using the first method: trial division. This result is supported by the authors' anecdotal experience teaching mathematics courses aimed at the preparation of teachers, as well as prior research on PSTs' work with prime factorization (Zazkis & Campbell, 1996a, 1996b; Zazkis, 1998). This preference for trial division points towards a need for intervention in the mathematical preparation of teachers where a richer view of the multiplicative structure of number, one that makes use of the FTA, would benefit teachers and their students. Responding to this need, the authors devised an intervention consisting of a sequence of cognitively demanding tasks and studied the impact of participation in the intervention on aspiring teachers' mathematical understanding of divisibility - particularly their use of unique prime factorization as a tool for identifying factors.

Theoretical Perspective

Constructivist theory frames the study described herein (Piaget, 1970; von Glasersfeld, 1987, 1995). Individuals make sense of mathematical ideas by interacting with their environment in ways that lead to more organized mental structures. These interactions follow Piaget's notion of *assimilation* and *accommodation* (Piaget, 1970). An individual makes sense of mathematical concepts by attempting to fit, or assimilate, his mathematical experiences into his currently held structures, or understandings; if those experiences do not fit, the individual may work to modify, or accommodate, her structures in order to make sense of new experiences and develop new understandings.

In order to help students develop deeper mathematical understanding, research suggests using cognitively demanding tasks during classroom instruction (Stein, Grover, & Henningsen, 1996; Tarr, Reys, Reys, Chavez, Shih, & Osterlind, 2008). Cognitive demand represents the thinking processes needed to solve a mathematical task. Tasks that are high in cognitive demand reflect challenging problems that require connecting procedures to underlying concepts and justifying one's thinking; tasks low in cognitive demand focus on executing familiar procedures without requiring connections to concepts or providing reasoning (Stein, Smith, Henningsen, & Silver, 2000).

Prior research has shown that PSTs display a limited ability to make use of unique prime factorization guaranteed by the FTA in the analysis of factors (Zazkis & Campbell, 1996a). For example, when asked to find the factors of a number in prime-factored form, PSTs relied heavily on trial division instead of prime factorization (Zazkis, 1998). Success with calculation-dependent schemes (i.e. trial division) may actually impede participants' motivation to make use of prime factorization thereby inhibiting the establishment of stronger schemas of multiplication and division concepts (Brown, Thomas, and Tolias, 2002).

While research has called for classroom interventions that emphasize flexible reasoning with prime factorization, few studies have examined the efficacy of these types of interventions. One study found that PSTs' ability to identify factors and non-factors using prime factorization significantly improved following a three week number theory unit (Feldman, 2012). Another used an applet of visual arrays where PSTs explored factors and multiples. These researchers found that *visualization* and *experimentation* fostered new understandings of the multiplicative structure of natural numbers (Sinclair, Zazkis, & Lilhedahl, 2004; Liljedahl, Sinclair, & Zazkis, 2006).

Method

This study was conducted at two large universities with students enrolled in mathematics content courses aimed at the preparation of elementary school teachers (n = 69). Students in these courses were both pre-service and in-service teachers seeking a teaching credential: pre-credentialed teachers (PCTs). Pre- and post-tests were administered before and after an intervention consisting of a sequence of three cognitively demanding lessons. The intervention was aimed at strengthening participants' understanding of a number's unique prime factorization in determining divisibility properties. The main research questions that guided the study were as follows:

1. Does the intervention support a growth in PCTs' understandings of the multiplicative structure of number provided by unique prime factorization?
2. Does the intervention promote PCTs' use of unique prime factorization in the analysis of factors across four subtypes: prime factors, prime non-factors, composite factors, and composite non-factors?
3. Does the intervention promote PCTs' ability to use a number's prime decomposition to construct an entire list of the number's factors?

Each lesson began with a sequence of cognitively demanding tasks. In these tasks, PCTs were asked to connect procedures with underlying concepts, to make and test their own conjectures, and to provide reasoning for their ideas (Stein, Smith, Henningsen, & Silver, 2000). The particular sequence of tasks adopted was meant to gradually build PCTs' understanding of a mathematical concept or procedure. After completing a series of tasks, PCTs were asked to engage in group discussion – to evaluate and synthesize their thinking regarding the key mathematical idea developed by the tasks preceding the discussion (for more details see Chapin, Feldman, Salinas, & Callis, under review).

Lesson 1 focused on promoting PCTs' understanding of the Fundamental Theorem of Arithmetic (FTA). PCTs constructed different factor trees for the number 72 and were asked to discuss how the factor trees were similar and how they were different (see Figure 1). PCTs then used the factor trees to write 72 as a product of prime numbers. They then were asked to explore three different factorizations of the number 720 – only one of which was a prime factorization. This activity was meant to impress upon PCTs the non-uniqueness of factorizations versus the uniqueness of prime factorization.

PCTs were asked to complete homework assignment 1 following the first lesson. In homework 1, they were given a 10-by-10 gridded array numbered from 1 to 100 (see Figure 2) and asked to fill in each gridded square with the number's prime factorization. PCTs were instructed to use "any method" to complete the task and were encouraged to find shortcuts and make use of patterns. After completing the grid-filling activity, PCTs were asked to explain their process and to describe any patterns they noticed. Finally, PCTs were presented with a formal statement of the FTA and asked to restate the theorem using their own words and to explain its meaning using a few examples.

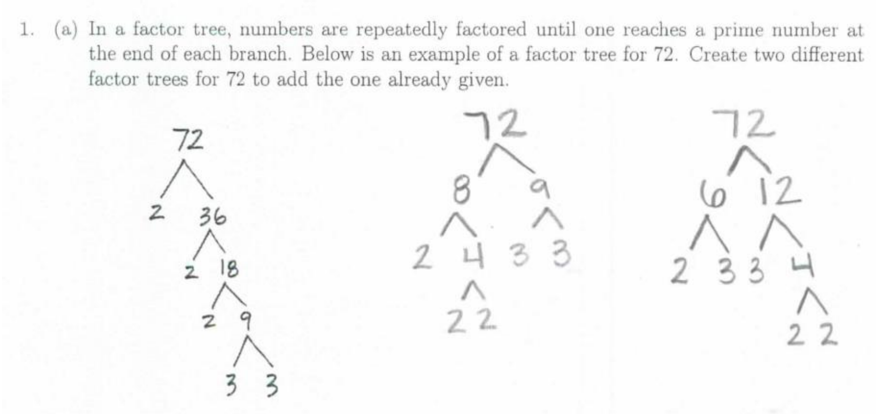

Figure 1: Examining different factor trees

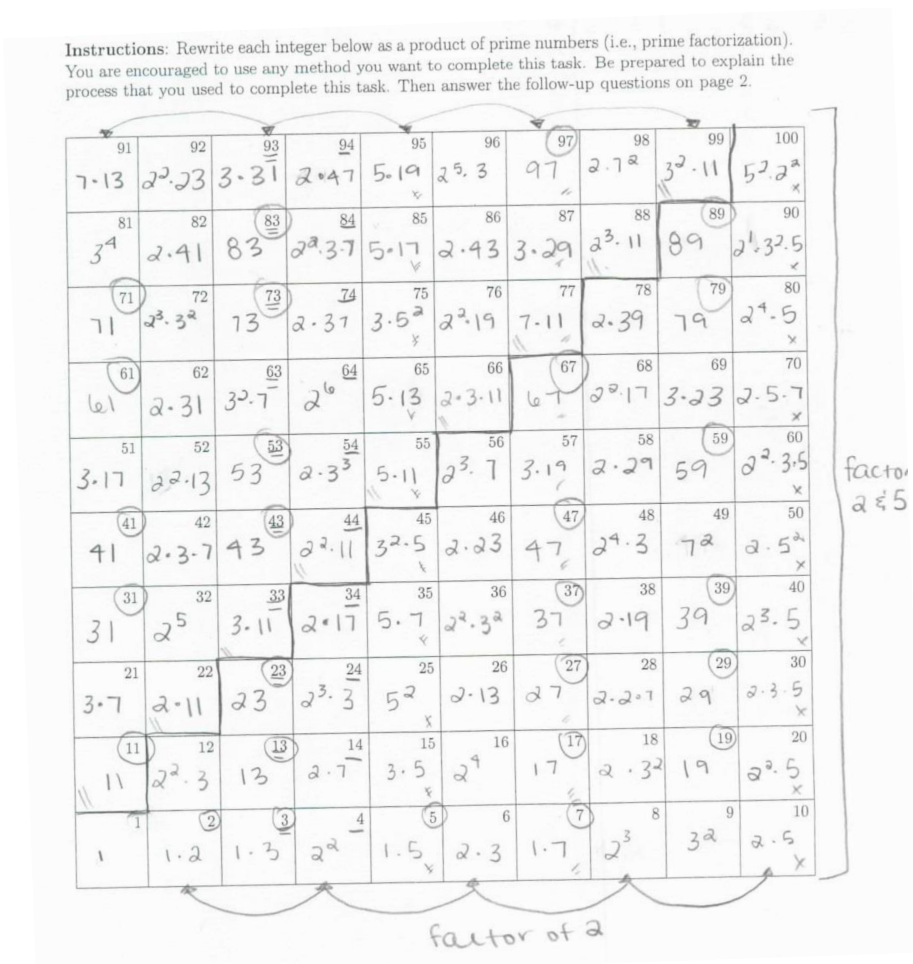

*Figure 2***:** Homework 1 array of prime decompositions

Lesson 2 used the prime factorization arrays that PCTs constructed in the first homework to investigate the relationship between a number and its factors when these numbers are written in prime-factored form. For example, one group was assigned to investigate the factors of the number 90 (see Figure 3). After writing the factors in base-ten and prime-factored form, PCTs were asked to discuss how the prime factorization of their number is related to the prime factorizations of its factors. A series of tasks that involved finding the factors of numbers written in prime-factored-form followed. The lesson concluded by presenting PCTs with a "traditional" definition of factor (e.g., A is a factor of N if $N = A \times B$, where A and B are counting numbers) and then asked them to provide a "new" definition of factor incorporating prime factorization.

Factors	Prime Factorizations	Factors	Prime Factorizations
90	$2 \cdot 3^2 \cdot 5$	9	$3 \cdot 3$
45	$3^2 5$	6	$2 \cdot 3$
30	$2 \cdot 3 \cdot 5$	5	5
18	$2 \cdot 3^2$	3	3
15	$3 \cdot 5$	2	2
10	$2 \cdot 5$	1	1

Figure 3: Example lesson 2

Homework 2 followed lesson 2. The homework activity asked PCTs to apply and extend their knowledge of the role of prime numbers in predicting the multiplicative structure of numbers. The homework included a variety of challenging questions including several hypothetical "questions from the classroom" (see Table 1).

In lesson 3 we asked PCTs to investigate how a number's prime factored form predicts the number of factors that the number possesses (i.e., $p_1^{n_1} \times p_2^{n_2} \times p_3^{n_3} \times \ldots \times p_m^{n_m}$ has $(n_1 + 1) \times$

$(n_2 + 1) \times (n_3 + 1) \times ... \times (n_m + 1)$ factors). PCTs were asked to search for numbers that possess exactly 2 factors, then to search for numbers that possess 3 factors, and so on for 4, 5 and 6 factors (adapted from Teppo, 2002). Next, PCTs were asked to find the number of factors of the number 2^2, then to find the number of factors of the number 2^3, followed by 2^4 and finally 2^n. This question was meant to help PCTs recognize that 2^n has $n+1$ factors. To extend this idea to the more complex setting where two primes and their powers must be coordinated, PCTs next investigated the number of factors for each number in the sequence $2^3 \cdot 3$, $2^3 \cdot 3^2$, $2^3 \cdot 3^3$, and finally $2^3 \cdot 3^n$. PCTs were challenged to apply their new knowledge to order the numbers 50, 51, 52, 53, 54 and 55 from least to greatest according to the number of factors that each number possesses. Finally PCTs were asked to summarize what they had learned in the lesson by describing what information a number's prime factorization provides in predicting the *number* of factors it possesses.

Table 1
A Sample of Homework 2 Questions

#	Prompt
1	Consider the number $N = 2^5 \cdot 3^2 \cdot 7 \cdot 11$ Without calculating the value of N, decide if each of the following is a factor of N. Explain each choice briefly. a) 3 b) 6 c) 18 d) 30 e) 35 f) 40
2	If a 5th grader states, "Larger numbers have more factors than smaller numbers," What numbers will you give him to investigate? Why those numbers?
4a	If both 3 and 5 are factors of a number x, must 15 be a factor of x? Why or why not?
4b	If both 4 and 6 are factors of a number x, must 24 be a factor of x? Why or why not?
5	Jennifer and Billie are having an argument. Both have constructed a factor tree for 132. Jennifer claims the factor trees are the same. Billie claims that the factor trees are different. What, do you suppose, is the source of their confusion? Give examples. How would you resolve their argument?

Data Sources

Table 2 shows the four question types administered on both pre- and post-tests. Question 1 assesses PCTs' abilities to make use of a number's prime factorization to determine their ability to analyze factor candidates across four subtypes explored in previous research: prime factor, prime non-factor, composite factor, and composite non-factor (see Zazkis & Campbell, 1996a). Composite factors of form $p \cdot q^2$ were also included to assess whether the "complexity of composition" plays a role in PCTs' ability to identify it as a factor.

Table 2

Assessment Question Types for Pre- and Post-Test Analysis of PCTs

Question	Prompt
1	Consider the number $N = 3^2 \cdot 5^4 \cdot 11 \cdot 17^3$. Without calculating the value of *N*, determine whether each of the following is a factor of N. Justify each briefly. a) 5 b) 19 c) 15 d) 21 e) 75
2a	List all of the factors of 225. Show how you found all of them.
2b	List all of the factors of $5^2 \times 7^2$. Show how you found all of them.
3	What is the smallest positive integer that has the first ten counting numbers, 1 through 10, as its factors? Show or explain your work so that others can follow your logic. *Note: you may leave your answer in factored form.*

Questions 2a and 2b were formulated to assess whether PCTs could construct factor-lists for a number and whether this ability is dependent upon the number's representation in either base-ten or prime-factored form. Note that both numbers have the same prime factored form $p^2 \cdot q^2$ and possess 9 factors each. In particular, the authors meant to assess whether PCTs increasingly take advantage of prime factored form, when provided, to create factor lists as a result of the intervention.

Finally, question 3 probes PCTs ability to construct a minimal integer given its factors. This question measures PCTs' ability to reverse their application of the FTA. Instead of analyzing the factor structure of a given number, PCTs are asked to construct a minimal number given its factors. This procedure was never explicitly addressed in the intervention, making question 3 a measure of PCTs' ability to apply their emerging knowledge to novel settings. The results of performance on this question are not discussed here, but, will be the subject of future analysis by the authors.

Both authors scored all questions using a researcher-developed rubric. Inter-rater reliability was established (82.5% agreement on 21.7% of the data set); discrepancies were resolved via discussion and rubric clarification until 100% agreement was achieved.

Results

Results of a paired sample *t*-test showed a statistically significant increase participants' performance as measured by the pre- and post-test scores (pre-test M=8.81, SD=4.40; post-test M=17.78, SD=4.97; $t(68)=-13.88$, $p<0.05$). This result indicates that the intervention is associated with an increase in PCTs' understandings of the multiplicative structure of number as provided by unique prime factorization (i.e. the FTA).

Individual item scores revealed that the intervention promoted PCTs' use of unique prime factorization in the analysis of factors across the four subtypes: prime factors, prime non-factors, composite factors, and composite non-factors (see Table 3). The results of the pre-test demonstrate that prior to intervention PCTs are more proficient at identifying factors than identifying non-factors, and, they more easily discern prime factors than composite factors. This result replicates similar findings found in the literature (i.e. Zazkis & Campbell, 1996b)

Table 3
Mean Scores for Questions 1a-1e

Question	Factor Type Given	Pre-Test	Post-Test
1a	Prime factor	62.3%	83.3%*
1b	Prime non-factor	46.4%	81.2%*
1c	Composite factor of form $p \cdot q$	46.4%	78.3%*
1d	Composite non-factor of form $p \cdot q$	31.9%	70.3%*
1e	Composite factor of form $p \cdot q^2$	38.4%	75.4%*

* Indicates a statistically significant difference from pre-test score ($\alpha = 0.05$)

More importantly, *differences* in success rates across all factor subtypes diminished as a result of the intervention. Participants initially showed a large difference in their abilities to identify prime (62.3%) versus composite (46.4%) factors. Following the intervention, however, these success rates were much closer (83.3% and 78.3%, respectively). This result was maintained even when participants were faced with a more challenging composite factor, as is the case in question 1e (75.4%).

Differences in success rates identifying factors versus non-factors also diminished. Initially, participants were more proficient at identifying prime factors (62.3%) than prime non-factors (46.4%) but were nearly equally proficient following the intervention (83.3% and 81.2%, respectively). Differences in success rates among composite factors versus composite non-factors (Questions 1c, 1d, 1e) also diminished, albeit not as dramatically as for prime factors and non-factors. This result is not surprising given that identifying composite factors and non-factors is known to be more challenging for PCTs (Zazkis & Campbell, 1996b).

Analysis of question 2 revealed a significant improvement in participants' ability to use both whole number and prime-factored forms to construct factors (see Table 4). Contrary to prior research, participants' success rate constructing a number's factors using prime-factored forms reached almost the same rate (69.6%) as when using base-ten forms (72.5%). Given that participants only received credit for question 2b if they correctly used prime combinations to construct factors, their marked improvement demonstrates an increased willingness to consider a number's prime-factored form as useful in the analysis of factors.

Table 4
Mean scores for Questions 2a, 2b

Question	Pre-Test	Post-Test
2a	40.9%	72.5%*
2b	25.5%	69.6%*

* Indicates a statistically significant difference from pre-test score ($\alpha = 0.05$)

Conclusion

The study results reveal significant improvements in participants' abilities to solve divisibility problems across a variety of factor subtypes. More encouraging is that the ability to use prime-factored forms nearly rivals the use of traditional trial division. Given the benefits of prime factorization as a conceptually rich tool for understanding divisibility, the intervention shows promise in expanding PCTs' conceptual understanding of the multiplicative structure of number made transparent by prime decomposition.

As such, the intervention responds to national reports and previous research that calls attention to the importance of developing teachers' conceptual understanding of the mathematics they are tasked to teach to their students in grades K-12 (Conference Board of the Mathematical Sciences, 2001, 2012; Greenberg & Walsh, 2008; Ma, 1999). The urgency of the situation is well-summarized by Greenberg and Walsh, "Aspiring elementary teachers must begin to acquire a deep conceptual knowledge of the mathematics that they will one day need to teach, moving well beyond mere procedural understanding" (2008, p. 2). The learning outcomes we have captured and documented here are a demonstration that the intervention is a promising step in this direction.

This result is somewhat surprising. Although instruction on the Fundamental Theorem of Arithmetic, primes, composites and factors is commonly included curricula presented to aspiring teachers, teachers have consistently shown little awareness of the connections between these topics (i.e. Zazkis and Campbell, 1996a; Zazkis, 199b; Zazkis, 1998). Plausible explanations for PCTs' growth in understanding can only be conjectured and are the focus of our continued work in this area. One possible explanation lies in the activity required in homework 1. Here PCTs were required to construct a gridded array of the numbers 1-100 in *dual representation* – in both base-ten and prime-factored-form (see figure 2). Many PCTs reported making discoveries that, collectively, seemed to awaken them to the importance of prime numbers in predicting divisibility properties (i.e. all the multiples of 10 have a 2 and a 5 in their prime decomposition; there is a 2 in the prime decomposition of every number in even columns; to find the prime decomposition of 36 we can eliminate a 2 in the prime decomposition of 72; etc.). Certainly this activity is a rich task that presents *opportunities for making connections* between numbers and their factors when all are written in *both* base-ten and prime-factored-form. Indeed, many researchers have alluded to the fact that PCTs exhibit a preference to compute when a number is presented in its prime-factored-form. It may be the case that the construction of the gridded array initiates an abstraction whereby a number is disassociated from its base-ten representation and is more generally conceived of as a quantity that can be represented in multiple ways – some more "transparent" than others in terms of understanding divisibility (i.e. Zazkis and Godowsky, 2001). Qualitative research to investigate this question is needed.

Another possible explanation for the marked improvements may be attributed to the intervention's consistent and pressing instructions which asked PCTs to *attend to prime factorization*. This repeated mandate, we conjecture, forced PCTs to accommodate unique prime factorization into their prior conceptual structures associated with the terms factors, multiples, primes and composites. Further analysis is necessary, however, to determine the particular strategies and understandings that participants developed as a result of this aspect of the intervention.

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