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Paper Title: The Connections between Repeating Decimals and Fractions: A Case Study of Elementary PSTs' Understanding

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Elementary teachers need a deep understanding of the mathematics they will teach in order to teach it effectively (Ball & Bass, 2003; Morris, Hiebert, & Spitzer, 2009). However, research also suggests that, in general, elementary preservice teachers (PSTs) lack a deep understanding of the various areas of number and notation encountered in elementary school mathematics (Cramer & Lesh, 1988; McClain, 2003; Steiner, 2009; Thanheiser, et al., 2013; Tobias, 2009). Part of developing a deep understanding of mathematics is understanding the ways in which mathematical concepts are interconnected and build on one another (Ball, Hill, & Bass, 2005). The understanding of number and notation that is necessary for elementary teachers entails knowing about whole, rational, and real numbers, their related notations, *and the connections between them*. However, much of the current research that is devoted to examining methods of improving PSTs' understanding of number concentrates on improving their understanding in only one of these areas, particularly place value related to whole numbers *or* fractions. The problem is that attending separately to whole and rational numbers does not support PSTs in understanding why the whole numbers are a subset of the rationals, how fraction and decimal notation are related to whole number notation, or why *all* fractions but only *some* decimals denote rational quantities. There is also very little research that addresses PSTs' understanding of decimal notation, despite the fact that decimal notation bridges whole, rational, and real numbers and is often poorly understood by PSTs (Kastberg & Morton, 2014).

One challenge with supporting PSTs in developing this understanding is time. Finding ways to develop PSTs' understanding of multiple areas within the domain simultaneously would be useful for preservice course design. Another challenge is the fact that PSTs' many years of experience working with these numbers. Coursework intended to deepen preservice elementary teachers' understanding of number and numeration must take into account their prior learning experiences and how those will impact their learning.

Purpose and Research Questions

The purpose of this study was to gain a deeper understanding of the learning needs of PSTs with regards to number and numeration in order to support the development of curriculum that can effectively and efficiently meet those needs. The research questions guiding this study were: (a) In what ways do PSTs understand the *connections* among whole and rational numbers, and fraction, decimal, and whole number notation before and after participating in a number and numeration unit of study designed to highlight these connections?; and (b) In what ways do PSTs use representations as tools for thinking as they reason about these connections before and after the unit? Due to space limitations, this report will focus on the results related to the first question only.

Theoretical Frameworks

Two theoretical frameworks guided this study. First was the idea that translations between multiple representations of a mathematical concept are a way to both build and assess conceptual understanding of that idea (Lesh, Post, & Behr, 1987; NCTM, 2014). Second was the notion of *bridging tools* (Abrahamson & Wilensky, 2007; Abrahamson, 2004, 2006; Fuson & Abrahamson, 2005). For this study, a bridging tool was interpreted as a non-symbolic representation that may be used to support students in making connections between representations of a concept that are difficult to connect directly. For instance, a bridging tool can be used to help students make connections between the notations "⅓" and "0.333…," since the infinite decimal makes it difficult to directly connect these notations symbolically. Bridging tools provide a way for learners to reconcile competing interpretations of a concept inherent in various representations and therefore support the development of a rich understanding of mathematics. The way bridging tools were used in this study is described further in the section below describing key activities from the instructional unit.

Methods

This study was designed as a descriptive case study (Yin, 2014) to document the ways that preservice teachers understood the *connections* among fractions and decimals, and between those notations and the sets of rational numbers.

Participants

Data came from 32 undergraduate, prospective elementary teachers enrolled in the first of two required mathematics content courses at a large, Midwestern university. Three of the participants were male, the rest female. The majority of the students (20 out of 32) were juniors, and the median age of the students was 21. For the highest math class taken, one had taken courses beyond calculus, fifteen of the students had taken calculus, three had taken AP Stats, four had taken pre-calculus, and eight had taken College Algebra.

Study Design

The study took place during the number and numeration unit, a sixteen-class unit co-designed and co-taught by the researcher and the regular instructor for the course. This unit had been iteratively tested and re-designed with a focus on creating opportunities for students to deepen their understanding of whole and rational numbers and also to explore the connections among them. Data were collected from pre- and posttests designed by the researcher, from written artifacts produced during the instructional unit including classwork and homework, from field notes taken by the researcher, and from one-on-one interviews conducted with eight students from the course. A purposive selection strategy (Merriam, 2009) was used to select interview participants with relatively weak and relatively strong initial content understandings. Four interview participants, Eva,[1] Nina, Willa, and Korey, were selected because they displayed multiple misunderstandings on the pretest. An additional four students, Soren, Jo, Andie, and Mei, were selected because they gave mostly correct responses on the pretest. Of these eight students, only Willa and Korey had not taken calculus.

The pre- and posttest questions were designed by the researcher to elicit preservice teachers' understandings of the connections among fractions, decimals, and the set of rational numbers. The researcher designed the pre- and posttest based on a review of the literature and with feedback from the instructor in the course. The researcher also designed and conducted all interviews. The researcher and instructor for the course acted as co-designers and co-instructors for the instructional unit.

[1] All names are pseudonyms.

The pretest was given on the first day of the course. Students were given approximately twenty-five minutes to complete the pretest. They were told that the pretest would not count as a graded assignment. They were encouraged to do their best work and told that the questions were designed based on the types of knowledge they would need for teaching. The posttest was given on the last day of the seventeen-day unit. It served as the final exam for the number and numeration component of the course. This test was not timed, but was designed to take approximately one hour to complete. Students were told in advance that they could stay and finish the test after the regular class period ended if they desired to do so. The majority of participants finished within the allotted one-hour-and-fifteen minutes of class time, and all others finished within an additional thirty minutes. Three participants were absent on the day of the posttest and took it at another time, all within one week from the missed class. All tests were proctored by the instructor for the course.

Summary of Key Activities from the Instructional Unit

Approximately five class days during the instructional unit were devoted to exploring the relationship among fractions and decimals and the sets of rational and irrational numbers in various ways. Two of these days were devoted to exploring different ways that *a* loaves of bread could be shared by *b* people and how those different ways of sharing the bread connected to various symbolic notations, including the standard fraction and decimal notations. These activities were called the *Breaking Bread* activities. An example of two different ways that three loaves of bread could be shared by four people so that the answer relates to either standard fraction or decimal notation is shown in Figure 1.

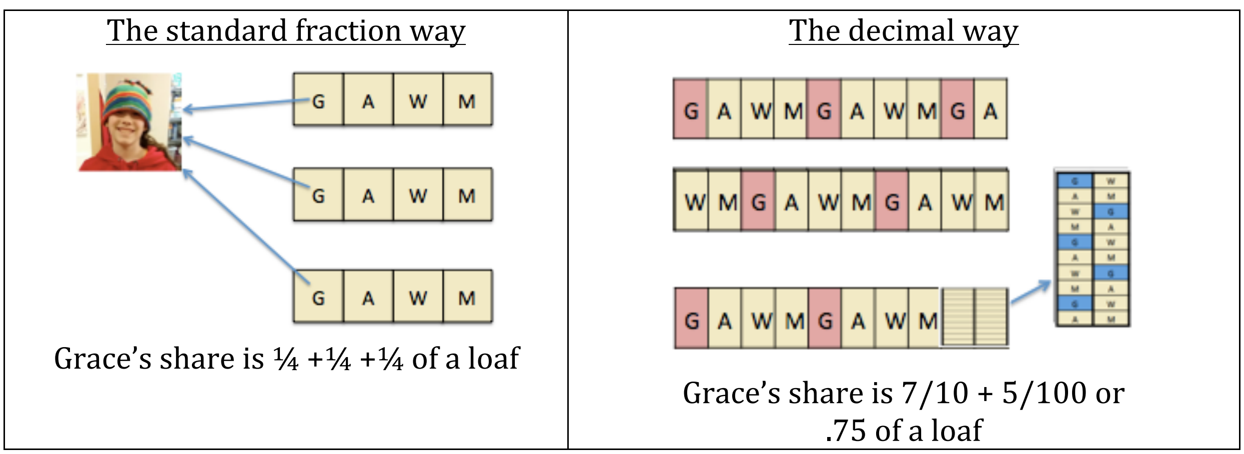


Figure 1. An example of the *Breaking Bread* activity showing how three loaves of bread could be shared by four people so that the answer could be denoted by a fraction (left) or decimal (right).

In this study, the *Breaking Bread* model was considered to be a bridging tool for two reasons. First, it supported students in understanding the equivalence of a fraction and decimal representation as they may be viewed as different ways of partitioning and sharing bread so each person gets the same amount. Second, it supported students in seeing the connection among fractions, repeating and terminating decimals, and the set of rational numbers because the process of partitioning the bread in the "decimal way" transparently results in shares that may be denoted by either a repeating or terminating decimal while the "fraction way" transparently results in shares that may be denoted by a standard fraction (see Figure 1).

In addition to the *Breaking Bread* activities, two days were devoted to activities involving the number line and fractions and decimals. One day was devoted number line activities that involved locating fractions and rational decimals on a number line using partitioning. This day also included an exploration of the idea that $0.\overline{9}$ is located at the same spot on the number line as "1" and thus the two are considered equivalent. The second day was devoted to activities related to locating pi on a number line and how irrational numbers such as pi differ from rationals in how they may be located by a process of partitioning. Finally, one class day was spent looking at how the prime factorization of a fraction related to the decimal notation.

Data Analysis Procedures

A case study design was employed in this study in order to gain deeper insights into preservice elementary teachers' understandings of the connections between the various aspects of number and numeration related to rational numbers. The analysis for this research was conducted using the transcriptions of the interviews, the pre- and posttests completed by the students, and field notes taken by the researcher during the implementation of the instructional unit. Using a constant comparative method of analysis (Strauss & Corbin, 1998), data were analyzed for patterns in student responses to tasks designed to elicit or support students in making connections between the number types. Data were also coded using *a priori* codes from Lesh, Post, and Behr (1987) for the ways that PSTs used multiple representations in their work. All data for the pretest and first interview were coded prior to coding the data for the posttest and second interview.

Results

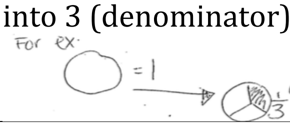
This report focuses on data gathered on a set of related questions from the pre- and posttests that targeted the preservice teachers' understandings of the relationship between a fraction and a repeating decimal. In this section, results from the ways the whole class responded to pretest Question 4, posttest Question 5, and posttest Bonus Questions 2 and 3 will be described. Information from the interviews will be added as appropriate.

Pretest Question 4: Explain Why ⅓=0.333…

Pretest Question 4 asked: "Tell how you would help a student understand why ⅓ = 0.333… when written as a decimal." The purpose of this question was to see how preservice teachers described the relationship between a fraction and a terminating decimal. The context, "Tell how you would help a student understand," was used in order to prompt students to give the clearest description this relationship possible. Table 1 summarizes the preservice teachers' responses to this question.

Table 1

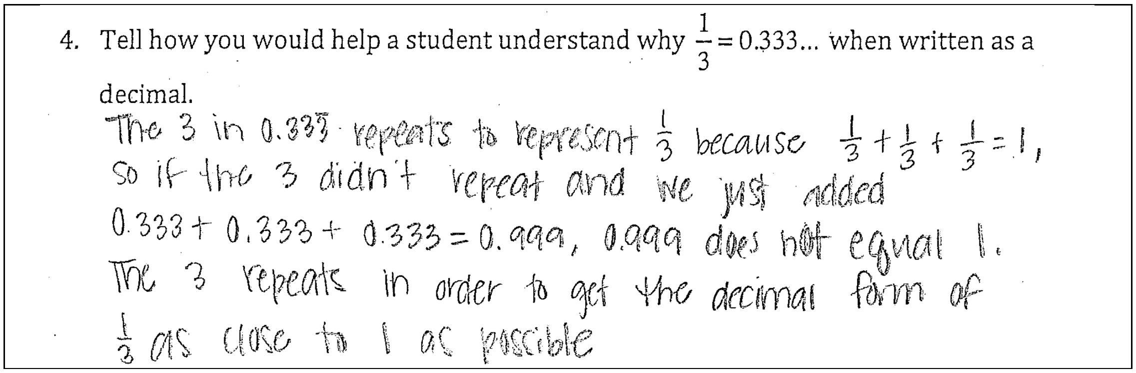
Summary of Ways the Whole Class Explained why ⅓ = 0.333… on the Pretest (N=32)

Type of response	Number (%)	Example[a]
Division (1 ÷ 3)	9 (28%)	"I would explain that when you are converting a fraction to a decimal, you divide the top number by the bottom. 1÷3=.33."
Division (100 ÷ 3)	2 (6%)	"'That's just how it is.' Kidding, I would try to use a visual, for example, to show that when you split 100 into 3 parts, there's a little leftover that you need to divide evenly."
Division and picture or story (all circle models)	7 (22%)	"Explain that the 1 on top is the numerator and shade one part and explain that we are splitting it into 3 (denominator) equally and one piece is 0.333." 
Three thirds equals 1 whole (repeated addition)	7 (22%)	"The 3 in 0.33$\overline{3}$ repeats to represent ⅓ because ⅓ +⅓+⅓=1, so if the 3 didn't repeat and we just added 0.333+0.333+0.333=0.999, 0.999 does not equal 1. The 3 repeats in order to get the decimal form of ⅓ as close to 1 as possible."
Other	3 (9%)	"I would show him or her by making the denominator 100 and showing what happens to the numerator."
Blank/ "Don't know"	4 (13%)	"I'm not sure how to explain this yet."

Note. Minor corrections to spelling, grammar, and/or punctuation were made to improve readability. All numbers are presented exactly as written by the students.
[a]Student initials (pseudonyms) are shown in parentheses.

As shown in Table 1, a total of eleven people in the class relied solely on division to explain why ⅓ = 0.333…. Two of these first converted the problem to 100÷3. Seven people relied primarily on division but also included a picture or story to illustrate. In all cases, the picture or related to a circle model and showed only the fraction ⅓, not why ⅓ equals $0.\overline{3}$ (see picture in Table 1 for an example). Together, this means that 18 of the 32 students (56%) used division as their only viable method to describe the equivalence of ⅓ and $0.\overline{3}$.

The next most common strategy used to show why ⅓ is equal to $0.\overline{3}$ was a repeated addition strategy. A total of seven students used some version of the idea that since three groups of one-third equal one, three groups of $0.\overline{3}$ would also equal one. This line of reasoning would lead to the true, but generally misunderstood (Dubinsky, Arnon, & Weller, 2013), statement that $0.\overline{9}$=1. Importantly, no student who used this strategy directly claimed that $0.\overline{9}$=1. Four students simply stated that three groups of $0.\overline{3}$ would equal one, rather than $0.\overline{9}$. Two students argued that it the total would be close, but not equal, to one (see Figure 2 for an example).


4. Tell how you would help a student understand why $\frac{1}{3} = 0.333\ldots$ when written as a decimal.

The 3 in 0.333 repeats to represent $\frac{1}{3}$ because $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$, so if the 3 didn't repeat and we just added 0.333 + 0.333 + 0.333 = 0.999, 0.999 does not equal 1. The 3 repeats in order to get the decimal form of $\frac{1}{3}$ as close to 1 as possible

Figure 2. A preservice teacher's explanation for why ⅓ = $0.\overline{3}$ using repeated addition.

The remaining student, the interview participant Soren, gave a partial response to this question. First, he noted that "⅓ = $^3/_9$," and then he listed the decimal equivalents to $^1/_9$, $^2/_9$, $^3/_9$, and $^4/_9$. Clearly, if Soren had continued his pattern, he would have ended with the line "$^9/_9 = 0.\overline{999}$." The idea that $0.\overline{9}$ equals one is a difficult concept for many mathematics students to accept (Dubinsky et al., 2013; Richman, 1999), so it was possible that he had abandoned his pattern because he was troubled by the idea that $1=0.\overline{9}$. He verified that this was indeed the case during the first interview when he was asked about where he thought $0.\overline{9}$ should go on a number line. The following conversation took place shortly after he explained that $0.\overline{6}$ would be located at the same point as ⅔ on the number line.

I: "What about this one, point nine repeating. Where would you put that?"
S: "Um (…) Oh, it's (…) oh man. (…) Now, I'm reconsidering."
I: "What's making you, what's tripping you up?"
S: "Well, I mean, I just, well I guess not many of the other ones. So if point nine nine repeated infinitely it would equal one. Hypothetically. Theoretically. So it could be one is a whole number and an integer. Point nine nine isn't I would say, but they're the same thing. So, okay. But I don't know if that's a real number necessarily. Because, well yeah it's a real number."
I: "Where would it go on the number line?"
S: "Really really really close to one. Or at one, depending on/"
I: "Where does point six repeating go on a number line?"
S: "Um (…) at two-thirds."
I: "Okay, where does point nine repeating go on a number line?"
S: "So it would go a one by that logic."
I: "Okay."
S: "Um."
I: "What's the problem?"
S: "Well, it's not one."
I: "Why not?"

S: "It's infinitely close to one. But it's not one."

As this conversation demonstrates, Soren saw the logic in the argument that $1=0.\overline{9}$, but also believed that the two were not actually equal. Soren's struggles with this equivalence demonstrate the difficulties with the approach of trying to use repeated addition to argue that since three thirds equal one, three groups of $0.\overline{3}$ are also equal to one.

Finally, three students from the class suggested unclear methods for showing that $\frac{1}{3}=0.\overline{3}$. For instance, one student wrote, "I would explain it to them that we notate ⅓=.333… because the 3's continue on." An additional four students left the problem blank or wrote, "I don't know." Thus, the majority of students (56%) suggested using division to show that ⅓ equals $0.\overline{3}$. A substantial minority (22%) suggested using repeated addition of ⅓ and $0.\overline{3}$, with no explanation of how they would respond to the implied idea that $0.\overline{9}=1$. The remaining 22% offered no strategy for explaining this equivalence.

Summary of findings for pretest Question 4. Two themes emerged from the preservice teachers' responses to this question on the pretest. First, *the relationship between fractions and decimals was poorly understood* by the students. No student offered a clear explanation of the equivalence of ⅓ and $0.\overline{3}$. The overwhelming majority of students stated that they would *use the division algorithm to explain this equivalence* to children. The division algorithm is opaque in that it hides the role of the powers of ten in the decimal representation of one-third. Second, *decimals were poorly understood.* In particular, there was *a lack of use of accurate models for decimals*. One of the indicators of decimal understanding is the ability to connect decimal symbols with pictorial representations (Cramer et al., 2015), but in this problem, no preservice teacher represented, or even attempted to represent, the decimal $0.\overline{3}$ in any non-symbolic way. In fact, the few students who attempted to use a non-symbolic representation to show this equivalence all chose a circle model and represented only the fraction ⅓. Clearly, preservice teachers must understand how choose appropriate representations for decimals if they are to teach this topic meaningfully to elementary students.

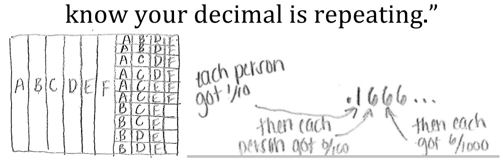
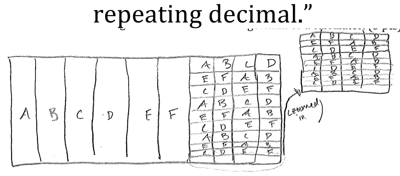
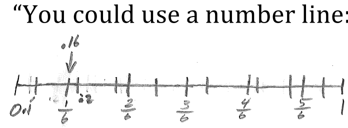
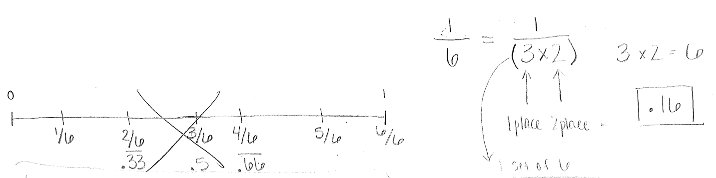
Posttest Question 5: Find ⅙ as a Decimal

A similar question to pretest Question 4 was posed on the posttest. Posttest Question 5 asked, "Show/explain how you could help a student find the decimal representation of the fraction ⅙ without using the standard division algorithm or a calculator." The wording, "without using the standard division algorithm," was used on the posttest in order to document if and in what ways students were able to describe this relationship other than by using division after the unit.

As shown in Table 2, the majority of the class (24 students, 75%) used the *Breaking Bread* story and/or picture to correctly solve this posttest problem. These students all drew a picture of a rectangle divided into tenths and showed six tenths being given out. Then they divided the remaining four tenths of the rectangle into hundredths and showed each person receiving 6 hundredths. The remaining four hundredths were then sometimes shown as being divided into thousandths, with each person receiving six thousandths and four thousandths remaining, and sometimes students stopped showing the partitioning with the hundredths. In all cases, however, students showed and/or described that the fact that four pieces (hundredths, thousandths) remained meant that the process would continue indefinitely. In other words, they transparently explained why ⅙ written as a decimal is *repeating*. See Table 2 for an example.

Table 2

*Summary of Ways the Whole Class Showed How to Find ⅙ as a Decimal on the Posttest (*N*= 32)*

Type of response	Number	Example
Rectangle divided into tenths, hundredths, etc.	24	"You have 1 loaf of bread that is shared among 6 people. Decimal law states that you have to make cuts into tenths. Each person gets 1/10 and you have 4/10 remaining. You partition the 4/10 into tenths, which would actually represent hundredths. Then each person would receive 6/100. There are still 4/100 leftover so you would keep repeating this pattern over and over. There will always be 4/(some power of 10) leftover so you know your decimal is repeating." 
Rectangle shown, no decimal answer given	2	"One way to explain is to represent 1 loaf of bread for 6 people. Since we're using decimals, you divide the loaf into 10 pieces. Each person gets 6 piece, you are left with 4, which you further partition into 100ths. After assigning these pieces to 6 people, the student will realize that every will initially get 1 piece and then, you will constantly be partitioning because you be left with 4 pieces for 6 people and need to keep cutting hence the repeating decimal." 
Number line	3	"You could use a number line:  "First partition the number line of 0→1 into sixths. Then partition it into tenths (decimals). Having located ⅙, now we see that $\frac{1}{6} = 0.\overline{16}$."
Other	3	

Note. The question prompted students to show how to find the decimal without using the standard division algorithm.

Of the seven students who did not correctly solve the problem on the posttest, two students in class created a correct picture for this situation but did not write the corresponding decimal related to ⅙. An additional three students stated that they would use a number line to show how to find the decimal representation of ⅙. All three students who used this method stated that ⅙=$0.1\overline{6}$ and that they would locate ⅙ and $0.1\overline{6}$ at the same point on the number line. In other words, these students showed how they could model the *equivalence* of these two notations, but not why the two were equal.

Finally, three students took a purely symbolic approach to this problem. One student showed the standard division algorithm on her page as her description of how to find the decimal form of ⅙. A second student showed a number line crossed out and the fraction ⅙ written in its prime factorization form "$^{1}/_{(3x2)}$" (see her example in Table 2). The third student, Korey, was an interview participant. In her response on the posttest, Korey wrote, "⅙=0.166…," then stated that she was unable to explain this without the standard division algorithm ("standard deviation"). After completing her test, she approached the researcher about this problem and she and the researcher discussed how to use the *Breaking Bread* story and picture to solve the problem. Early in the second interview, she stated that she felt more confident about finding the decimal form of a fraction without division after the conversation following the posttest.

> I like the fraction to decimal thing. Like with the "How do you get the one-sixth, how do you explain/ how does it go into a decimal?" I liked that. That kind of stuck with me, especially after the test, and you were like, and I was like, "Oh yeah, I forgot, that's how we're supposed to do it!" And it really stuck with me. (Korey, interview 2)

Korey was then asked to show how she would find one-sixth as decimal using the *Breaking Bread* context.

K: "So, this is how I think of it. The one on top means how many of something you have. So we've been doing bread so I'll do bread [drawing]. So then you have one loaf of bread. And the number on bottom, the denominator is how many you need to share it with the one. Oh and then it's also in tenths because it's in its one place so it's in the tenths. Yeah, that's it [drawing]. So then I split it into six [drawing]. So first I split it into three and then I split into two."

I: "So thirds and then halves make sixths?"

K: "Yeah. Oh wait, is that how/?

I: "How much would each person get right now?"

K: "Each person would get, if I was splitting into six, each person would get one."

I: "Everyone would get one. And what's the name of those parts?"

K: "Um, one tenth."

I: "One?"

K: "Oh, one sixth!"

After Korey partitioned her rectangle into sixths, a discussion ensued of how her picture related to the fraction notation ⅙, and that decimal notation required her to create tenths, hundredths, thousandths, and so on. The following exchange then occurred.

K: "Oh! Oh! That's what I forgot. Okay. So if it's in tenths. So then get one piece of bread because that's still the same, you split in tenths, tenths."
I: "Because you need tenths if you want decimals."
K: "Let's pretend it's even." [referencing her drawing]
I: "That's fine."
K: "So then each person would get one. And I know you guys do a, b, and c, but I/" [referring to how we labeled each person's share of bread in pictures in class]
I: "You do whatever you want."
K: "One, two, three, four, five, six [writing on drawing]. So then each person gets (…) point one because each person gets one of a tenth?"
I: "Exactly. And that's what that says. It says 'one tenth.'" [pointing to decimal]
K: "And then these you have to split. So there's only four left, so then you have to split this stuff into tenths again?"
I: "Right, because you want to be able to talk about these things in terms of decimals, and decimals always have tens, hundreds/ tenths, hundredths, thousandths."
[Six turns omitted, Korey describing how she is partitioning]
K: "Okay so then they get one, two, three, four, five, six."
I: "So everyone gets six, and what was the name of those little pieces?"
K: "Six, these ones are hundreds."
I: "How come?"
K: "Because if you split it all the way across, it turns into hundreds."
I: "There you go. And that let's you write it as a decimal //That's why you want to do that."
K: "Yes. And then there's four left over, so then again you'd have to split and split and split, then you can tell its/ there would always be four left over."

Note that in the above exchange, Korey referred to the pieces as "hundreds" rather than "hundred*th*s," indicating that her language use with regard to decimals was still fragile. However, also note that she was making clear connections between the picture, the decimal notation, and the reason why the decimal notation was repeating. Thus, the nature of Korey's understanding of the relationship between fractions and repeating decimals after the unit was fragile, but based on the idea that both fractions and decimals were notations that depicted *partitioning*. Furthermore, her understanding of decimal numeration after the unit was based on an understanding that decimal notation is related to partitions of powers of ten, a much more mathematically sound understanding of decimal numeration than she showed early in the unit.

Summary of findings from responses to posttest Question 5. Two related themes emerged from the data related to the ways the preservice teachers explained how to find ⅙ as a decimal on the posttest. First, there was *widespread evidence of understanding of the relationship between the fraction ⅙ and the decimal* $0.1\overline{6}$*.* On the posttest, the majority of the students in the course *clearly described the division process as partitioning* a given unit into tenths, hundredths, and so on in order to find the decimal form of the fraction ⅙. Furthermore, one of the interviewees, Korey, who was among the minority of students who did not describe the division process as partitioning on this question on the posttest wasable to do so during their second interviews (with some support in the form of questioning from the interviewer).

Second, there was *widespread evidence of decimal understandings* in the preservice teachers' descriptions of how to find the decimal for ⅙*.* There was widespread use of *precise mathematical language* to describe the decimal $0.1\overline{6}$ as one-*tenth*, six-*hundredths*, six-*thousandths*, and so on. In fact, many students actually wrote these words out in their descriptions. Notably, two of the four interviewees (Nina and Willa) gave very clear description of the decimal places as "tenths," "hundredths," and so on in their responses. This was notable because both women were unable to use place value understandings to translate terminating decimals to a fraction during the first interview. Additionally, there was *widespread use of an accurate model for the decimal* in students' responses. The ability to make this division process transparent for students along with the ability to accurately model and describe decimals using precise mathematical language are important skills for elementary teachers to develop.

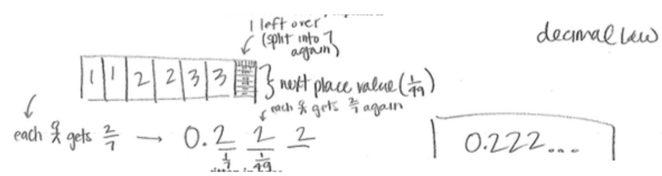
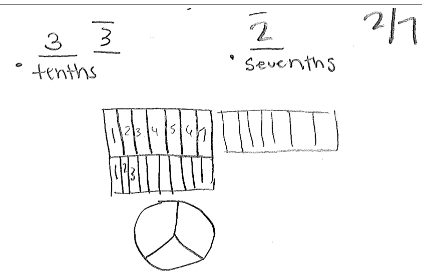
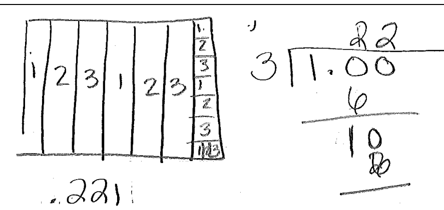
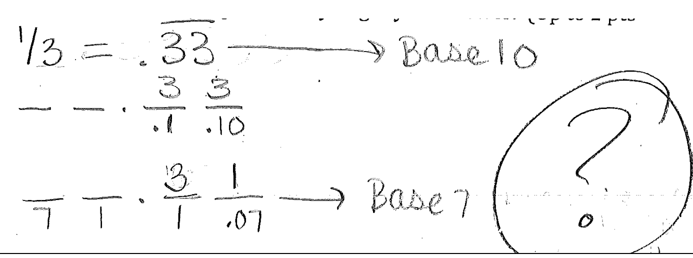
Posttest Bonus Questions 2 and 3: Find ⅓ as a Decimal in Base Seven and Explain How It Relates to 2/9 in Base Ten

Posttest Bonus Question 2 asked, "Write ⅓ as a decimal in base-7. Show/explain how you got your answer." Posttest Bonus Question 3 asked, "Explain why ⅓ written in base-7 decimals is similar to 2/9 in base-10 decimals." These questions were related to posttest Question 4 which asked students to find ⅙ as a decimal as they asked students to generalize the process used to find ⅙ as a decimal in base ten and to make connections between finding decimals in base ten and finding decimals in other bases. The concept of generalizing the fraction-decimal relationship had been only briefly touched upon in a whole class discussion during the final ten minutes of one class period. Knowing if students can generalize concepts is a way to measure depth of understanding, but since this content had only been briefly included in the unit, the instructor for the course and the researcher agreed that making such questions required on the unit test would be unfair to students. Thus, these questions were included as optional "bonus" questions (worth up to two percentage points each). As a result, not all students responded to these questions.

As shown in Table 3, for posttest Bonus Question 3, a total of thirteen students in the course (41%) were able to determine that one-third as a "decimal" in base seven would be denoted $0.\overline{2}_7$. Ten of these students used the *Breaking Bread* story and/or picture to show how they found their answer. An additional three students drew a picture showing a rectangle partitioned in sevenths with pieces being given out, but were not able to correctly determine the base seven "decimal" using their picture.

Table 3

*Summary of Ways the Whole Class Found ⅓ as a Decimal in Base 7 on the Posttest (*N*=32)*

Type of response	Number	Typical explanation
⅓ = $0.\overline{2}$ in base 7 with picture clearly related to answer	10	
⅓ = $0.\overline{2}$ in base 7, unclear how answer found	3	
Drew fully or partially correct picture, wrote decimal incorrectly	3	
Incorrect, symbolic work only	6	
Blank[a]	10	

Note. Minor corrections to spelling and grammar were made to improve readability. All numbers are written exactly as the student wrote them.

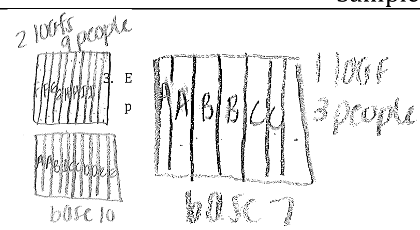
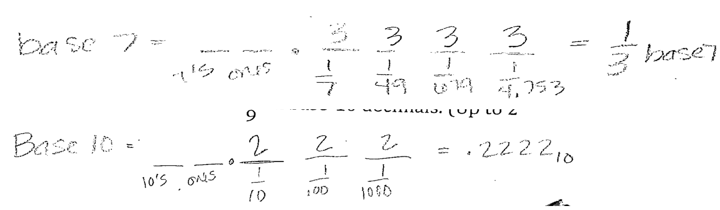
Three students in the class correctly found that the "decimal" would be $0.\overline{2}_7$ but gave an unclear explanation for how they determined their answer. Of these, only one student, Soren, showed purely symbolic work. In particular, he wrote, " "⅓=$^2/_7$+$^2/_{49}$+$^2/_{343}$" with an arrow pointing to it that said, "Calculator, sorry." The other two drew pictures but their pictures were not clearly related to the decimal notation they found (for example, see work sample for "⅓=$0.\overline{2}_7$, unclear how answer found" in Table 3). In addition to Soren, six other students also showed purely symbolic work but all six of these students failed to find one-third as a "decimal" in base seven. Thus, Soren's ability to find the "decimal" in base seven without using a story or picture as a tool for thinking was unique in the class.

Notably, during his second interview, Soren was unable to describe how he used his calculator to find the decimal for ⅓ in base seven. However, he was able to use the *Breaking Bread* model during the interview to show why ⅓ would be the repeating decimal $0.\overline{2}_7$.

Posttest Bonus Question 3 was related to Bonus Question 2, asking students to make a connection between 2/9 as a decimal in base ten and ⅓ as a decimal in base seven. Clearly, this question was most appropriate for students who were able to find that ⅓ as a decimal in base seven was $0.\overline{2}_7$. As shown in Table 4, nine students in the class noted the structural similarity between finding $^2/_9$ as a decimal in base ten and ⅓ as a decimal in base seven. In particular, all nine noted that both would be denoted "$0.\overline{2}$" in their respective bases because the process of partitioning and "sharing the bread" would lead to analogous situations. Notably, all nine also used the *Breaking Bread* strategy to find ⅓ in base seven (Bonus Question 2). One additional student noted that both would be denoted $0.\overline{2}$ but did not explain why this was the case. Six students noted that both would be repeating decimals but gave no further explanation of their relationship.

Table 4

*Summary of Ways the Whole Class Explained Why ⅓ in Base 7 is Similar to $^2/_9$ in Base 10 on Posttest (*N*=32)*

Type of response	Number	Sample explanation
Both $^1/_3$ and $^2/_9$ would be written as $0.\overline{2}$ in base 7 and 10 respectively with clear explanation	9	 "It is similar because each person either gets 2/10 or 2/7 of bread and there is always a remainder that repeats and everyone always ends up receiving 2 of that remainder, causing both decimals to be $.\overline{222}$."
Both are $0.\overline{2}$, no explanation	1	"It is. $\overline{2}$ for both of them."
Description of both decimals as repeating without direct reference to both being written as $0.\overline{2}$	6	 "These numbers are similar because they are both repeating rational numbers."
Other (incorrect)	2	"They are similar in value to one another because base 10 has larger numbers because it is power of tens, and in base 7 the numbers are smaller in value because the powers of 7 are smaller than 10."
Blank[a]	14	

Note. Minor corrections to spelling and grammar were made to improve readability. All numbers are written exactly as the student wrote them.

Summary of findings related to posttest Bonus Questions 2 and 3. Two related themes emerged from the data related to the preservice teachers' responses to Bonus Questions 2 and 3 on the posttest. Although this content was only briefly discussed for approximately ten minutes during one class period and was not otherwise supported in the course, a sizeable minority of students in the course (thirteen of thirty-two students, 41%) were able to find that ⅓ as a decimal in base seven would be $0.\overline{2}_7$. The fact that so many students were able to *generalize their understanding of the fraction-decimal relationship to non-base-ten situations* supports the idea that there was *widespread evidence of understanding of the relationship between fractions and decimals* on the posttest. The ability to generalize within the domain of number is an important part of developing "a profound understanding of fundamental mathematics" (Ma, 1999).

Of the thirteen students who found ⅓ as a decimal in base seven, ten used the same *Breaking Bread* story and picture that was used by the majority of students when finding ⅙ as a decimal (in base ten) on posttest Question 6. Moreover, nine of the ten students who used this model were able to explain why ⅓ as a decimal in base seven and $^2/_9$ as a decimal in base ten would both be denoted $0.\overline{2}$ in their respective bases. Thus, the second theme to emerge from this data was that *non-symbolic representations supported students in making connections between fractions and decimals and in generalizing decimal understandings.* That it was, in fact, the non-symbolic representations that were supporting their thinking was validated during several of the final interviews. For instance, both Soren and Willa used the model to find and make sense of why the decimal was denoted $0.\overline{2}_7$ during their interviews. Additionally, both Jo and Andie stated outright that it was the context and model that had helped them to solve this problem. In fact, Andie stated that she did not know how to think of it otherwise, suggesting that she was truly using the model as a tool for thinking about this situation.

Discussion and Implications

This study adds to the limited body of research that directly addresses preservice teachers' understanding of the connections between fractions and decimals, and their connection to the sets of rational and real numbers (Amato, 2005, 2006; LeSage, 2011; Sinclair, Liljedahl, & Zazkis, 2006). This study also extends prior work on bridging tools (Abrahamson & Wilensky, 2007; Abrahamson, 2006; Fuson & Abrahamson, 2005) by documenting how the *Breaking Bread* model was used as a bridging tool to promote understanding of the connections between fraction and decimal notation and their relationship to the concepts of *partitioning* and *measure.* In doing so, this study addresses Kastberg and Morton's (2014) call for more research on how to develop preservice teachers' understandings *productively* and *efficiently* by using activity sequences that develop more than one concept at a time (p. 329). In this section, two general implications for the design of curricula in mathematics content courses for preservice elementary teachers that arose out of this study will be presented. Recommendations for related future research will be made as well.

The first implication to arise from this study was related to the bridging tools used in the instructional unit. Over the past two decades, there have been multiple and continued calls for content coursework that supports preservice teachers in better understanding fractions, decimals, place value, and the operations of multiplication and division (Chick, Baker, Pham, & Cheng, 2006; Kastberg & Morton, 2014; Kilpatrick, Swafford, & Findell, 2001; Mewborn, 2001; Olanoff, 2011; Post, Harel, Behr, & Lesh, 1991; Tatto & Senk, 2011; Thanheiser, 2014). The fact that the overwhelming majority of preservice teachers were able to successfully use both the *Breaking Bread* model as a tool for connecting fractions and decimals is therefore a promising finding since this tool has the potential to simultaneously support preservice teachers in developing understandings in multiple areas. "Fair sharing" situations, like the one used in the *Breaking Bread* tool, are useful for developing fraction understandings in elementary students (Empson & Levi, 2011). Extending such situations to include fair sharing "the decimal way"—that is, by partitioning so the pieces created may always be expressed using decimal notation—can build meaning for the decimal notation and for why a given fraction and decimal are equivalent. Moreover, the partitioning process necessary to produce shares that may be notated by a decimal foregrounds the role that the powers of ten play in decimal notation and therefore meaningfully models place value. This process also naturally and meaningfully models the steps of the standard division algorithm.

The second implication to arise from this study was related to the preservice teachers' decimal understandings. In this study, preservice teachers' decimal understandings emerged as a linchpin that held, or failed to hold, many interrelated ideas together. In their review of the research on preservice teachers' understanding of decimals, Kastberg and Morton (2014) argued that more research is needed that investigates how preservice teachers develop decimal understandings and how teacher educators can support that development. One issue with deepening preservice teachers' knowledge of decimals is that they are familiar territory and preservice teachers do not always know that they lack the understanding necessary to teach decimals meaningfully to students. The results of this study suggest that taking a connected approach to developing fraction and decimal understandings could be a productive approach as asking students to make connections between fractions and decimals revealed many areas of limited or inaccurate decimal understandings. Furthermore, activities that ask students to make connections between fractions and *non-base-ten* decimals have the advantage of making the familiar strategy of using the division algorithm to convert a fraction to a decimal untenable. Converting to non-base-ten decimals forces students to think about the role that partitioning by powers of the base plays in decimal notation. The fact that a large minority of students in this study were able to figure out how to write a fraction as a non-base-ten decimal (and a repeating one, no less!) despite having only limited exposure to the idea in the unit suggests that such activities could be viable for use teacher education coursework. One area of possible future research would be to use the *Breaking Bread* activity with the extension to non-base-ten decimals with other groups of preservice teachers and examine how it supports, or fails to support, their understandings of base-ten decimals. Another possibility for future research would be to examine more generally how taking a connected approach to developing fraction and decimal understanding impacts learners' understandings in the separate areas of fractions and decimals.

Conclusion

This case study documented ways that one class of preservice elementary teachers worked to describe the relationship between a fraction and a repeating decimal before and after their participation in the number and numeration unit in their teacher education coursework. Initially, the preservice teachers in this study were limited in the models they could use to explain why ⅓ is equal to $0.\overline{3}$, a common decimal-fraction equivalence. In particular, most relied on the long division algorithm to explain this equivalence, an opaque algorithm that hides the role of the powers of ten in the decimal representation. Moreover, those that did try to use a pictorial or contextual model all represented only the fraction ⅓; none chose a model that showed tenths, hundredths, thousandths, and so on. However, following the instructional unit, the majority of preservice teachers clearly showed and described why the fraction ⅙ would be written as the repeating decimal $0.1\overline{6}$. The models and words they used in their explanations showed a clear understanding of the repeating decimal $0.1\overline{6}$ as *one tenth*, *six hundredths, six thousandths*, and so on. They also clearly described why the decimal was repeating. This ability to use models and accurate decimal language to describe a decimal is important for the work of teaching elementary mathematics.

In the instructional unit related to this study, this understanding was developed by employing the *Breaking Bread* model as a bridging tool in order to help support the preservice teachers in not only understanding fraction and decimal notation separately, but also in understanding how and why fractions and repeating and terminating decimals are all related to the notion of equal partitioning (and therefore to the set of rational numbers). The widespread success of the preservice teachers in utilizing the *Breaking Bread* model on the posttest suggests that using such a model may be an effective and efficient way to help preservice teachers make sense of the connections between number types and/or notations.

This study was designed with the intention of supporting the development of curriculum to build mathematical knowledge for teaching of preservice elementary teachers. Developing a *coherent* and *connected* understanding of number and notation as it relates to the whole numbers, fractions, and decimals is important for the work of elementary teaching and must be supported by mathematics content coursework. This study suggests that preservice elementary teachers may be well-served by activity sequences that intentionally develop their understandings of multiple aspects of number and notation simultaneously. Such learning sequences can potentially promote understandings of the individual aspects of number and notation that need to be addressed during teacher education coursework, while at the same time, helping make explicit the ways that these concepts are connected mathematically.

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