Multiplication warm-up

- Biggest product? x
- Using five different digits?
- Visual? Numeric?
- Explain how you know your answer is the best.

A Math Teachers' Circle: Functions, Algebra, and Symmetry

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A Teachers' Circle

- Interested participants
 - Usually K-12 teachers
- Problem-solving sessions
 - Focused on a topic or theme
- Led by a mathematician
- Interactive, participatory
- For the teachers, not directly for their students

Problem vs Exercise

- Exercise: already know how to do.
- Problem: don't know what to do.
- Algorithms and computations are almost always exercises.
- Know which tools to use, but not how to put them together? That's somewhere in between, perhaps.
- Learn the problem solving salute!
 - Our first strategy: Work backwards.

Problem Posing

- Common Core: "Students consider analogous problems" as a problemsolving technique.
- Creating your own problems also makes you more interested in solving them!
- Basic tool of problem posing: variations on a theme. "Add a knob"

First input	Second input	Output
2	8	26
1	4	9
4	1	9
8	2	26
0	0	0
0	8	8
3	6	27
3	2	11
5	10	65

First input	Second input	Output
1	1	3
4	7	39
-1	1	-1
2	1	5
4	5	29
2	5	17
100	1,000,000	101,000,100
10	100	1110
10	20	230

- A machine takes in two cards with numbers written on them, and produces a new card with ...
- The product plus the sum?
- One more than the first card, times the second, and then plus the first? (A-SSE2)

$$f(x,y) = xy + x + y?$$
 (F-IF 1)

• You have 100 cards, labeled 1 through 100. What questions do you want to ask?

- What numbers can be outputs?
- What happens if you keep using the machine?
- How many cards do you end up with?
- What numbers might be on the final card?
- What other questions should I have anticipated?

- Where are the knobs?
- What are easier problems, not too scary, that you can work on?
- Does working backwards help?
- What other strategies can we come up with?
- What do we know about what happens?

- Turn down 100 to something easier.
- Replace "product plus sum" with something easier.
- Obvious result: *monovariant* (Be willing to state the simple stuff!)
- Less obvious strategies: symmetry? Invariant?

- If we have just "sum", there's only one answer, which is the total.
- Why? How do we know the answer is *invariant* no matter how we rearrange our use of the machine?
- So, now suddenly we *care* about whether the "product plus sum" operation is commutative and associative. (3.0A5, 6.EE3)

- xy + x + y = yx + y + x
- How about if we put in x and y, and then that with z, versus if we put in x with the result of y and z?
- (xy + x + y)z + (xy + x + y) + z =xyz + xz + yz + xy + x + y + z
- And now we don't even need to check the other. *Symmetry*!
- So now what do we know?

• Invariant! There is only one possible

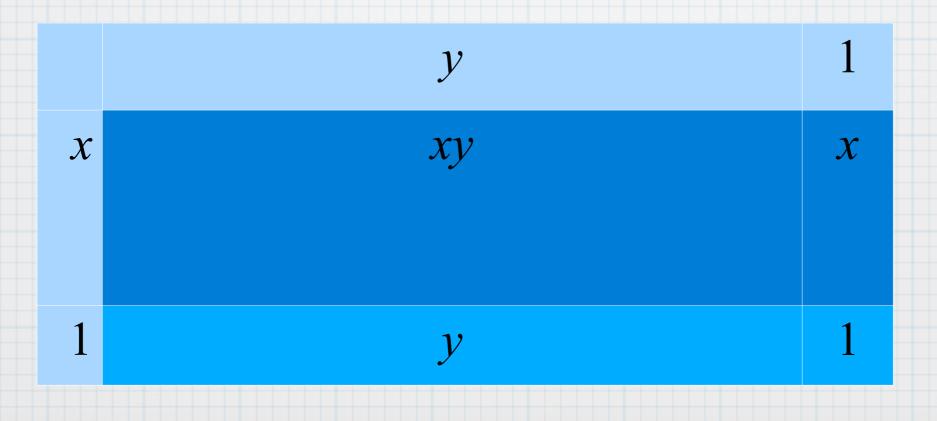
answer.

• Also wishful thinking. It sure would be nice if we could factor xy + x + y.

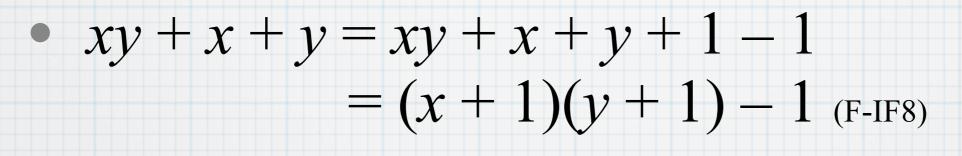
xy	X
${\mathcal Y}$?

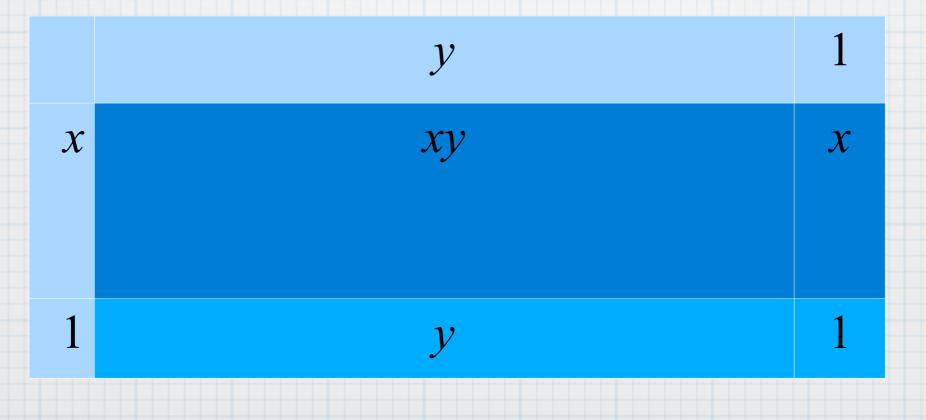
• Wishful thinking. It sure would be nice if we could factor xy + x + y.

• xy + x + y = xy + x + y + 1



• Wishful thinking. It sure would be nice if we could factor xy + x + y.





- Now x, y [x + 1 [y + 1 1]]
- And so $[x+1][y+1][-1, z \otimes [x+1][y+1][-1 \otimes 1[z+1][-1]]$ Which is [x+1][y+1][z+1][-1]
- And now it's really easy to see that the order of *x*, *y*, and *z* doesn't matter!
- So, do we have a final answer?

- Now we can come back to (x-r)(x-s)(x-t) = $x^{3} - (r+s+t)x^{2} + (rs+rt+st)x - rst$
- We don't want all that subtraction. (x + r)(x + s)(x + t) = $x^3 + (r + s + t)x^2 + (rs + rt + st)x + rst$
- We don't want x, either. So make it 1. (1+r)(1+s)(1+t) =1+(r+s+t)+(rs+rt+st)+rst

Is this a good problem?

- Practices fundamental skills in arithmetic and algebra.
- Easy entry: a student who can add and multiply can make substantial progress
- Depth: patterns emerge that lead to further questions. We have some unanswered questions!
- Multiple approaches
- Surprises!

Is this a good session?

- Dan Meyer: "Be less helpful"
- Time to think
- Curiosity, perplexity
- At least one fully understood punch line
- Problem solving as a way of generating motivation
- "Think deeply of simple things"