Multiplication warm-up

___ ___ ___

x ___ ___

- Biggest product?
- Using five **different** digits?
- Visual? Numeric?
- Explain how you know your answer is the best.

A Math Teachers' Circle: Functions, Algebra, and Symmetry

Fawn Nguyen and Joshua Zucker
@fawnpnguyen and @joshuazucker
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A Teachers' Circle

- Interested participants
  - Usually K-12 teachers
- Problem-solving sessions
  - Focused on a topic or theme
- Led by a mathematician
- Interactive, participatory
- For the teachers, not directly for their students

Problem vs Exercise

- Exercise: already know how to do.
- Problem: don't know what to do.
- Algorithms and computations are almost always exercises.
- Know which tools to use, but not how to put them together? That's somewhere in between, perhaps.
- Learn the problem solving salute!
  - Our first strategy: Work backwards.

Problem Posing

- Common Core: "Students consider analogous problems" as a problem-solving technique.
- Creating your own problems also makes you more interested in solving them!
- Basic tool of problem posing: variations on a theme. "Add a knob"

A Strange Machine

First input	Second input	Output
2	8	26
1	4	9
4	1	9
8	2	26
0	0	0
0	8	8
3	6	27
3	2	11
5	10	65

First input	Second input	Output
1	1	3
4	7	39
-1	1	-1
2	1	5
4	5	29
2	5	17
100	1,000,000	101,000,100
10	100	1110
10	20	230

A Strange Machine

- A machine takes in two cards with numbers written on them, and produces a new card with ...
- The product plus the sum?
- One more than the first card, times the second, and then plus the first? (A-SSE2)
- $f(x,y) = xy + x + y$? (F-IF 1)
- You have 100 cards, labeled 1 through 100. What questions do you want to ask?

A Strange Machine

- What numbers can be outputs?
- What happens if you keep using the machine?
- How many cards do you end up with?
- What numbers might be on the final card?
- What other questions should I have anticipated?

A Strange Machine

- Where are the knobs?
- What are easier problems, not too scary, that you can work on?
- Does working backwards help?
- What other strategies can we come up with?
- What do we know about what happens?

A Strange Machine

- Turn down 100 to something easier.
- Replace “product plus sum” with something easier.
- Obvious result: *monovariant*
  (Be willing to state the simple stuff!)
- Less obvious strategies: symmetry? Invariant?

A Strange Machine

- If we have just "sum", there's only one answer, which is the total.
- Why? How do we know the answer is *invariant* no matter how we rearrange our use of the machine?
- So, now suddenly we ***care*** about whether the "product plus sum" operation is commutative and associative. (3.0A5, 6.EE3)

A Strange Machine

- $xy + x + y = yx + y + x$
- How about if we put in x and y, and then that with z, versus if we put in x with the result of y and z?
- $(xy + x + y)z + (xy + x + y) + z =$ $xyz + xz + yz + xy + x + y + z$
- And now we don't even need to check the other. ***Symmetry*!**
- So now what do we know?

A Strange Machine

- **Invariant!** There is only one possible answer.
- Also **wishful thinking**. It sure would be nice if we could factor $xy + x + y$.

xy	x
y	?

A Strange Machine

- **Wishful thinking**. It sure would be nice if we could factor $xy + x + y$.
- $xy + x + y = xy + x + y + 1$

	y	1
x	xy	x
1	y	1

A Strange Machine

- **Wishful thinking**. It sure would be nice if we could factor $xy + x + y$.
- $xy + x + y = xy + x + y + 1 - 1$
  $= (x + 1)(y + 1) - 1$ (F-IF8)

	y	1
x	xy	x
1	y	1

A Strange Machine

- Now $x,y \mapsto (x+1)(y+1)-1$
- And so

$$(x+1)(y+1)-1,\ z \mapsto ((x+1)(y+1)-1+1)(z+1)-1$$

Which is $(x+1)(y+1)(z+1)-1$

- And now it's really easy to see that the order of *x*, *y*, and *z* doesn't matter!
- So, do we have a final answer?

A Strange Machine

- Now we can come back to
  $(x - r)(x - s)(x - t) =$
  $x^3 - (r + s + t)x^2 + (rs + rt + st)x - rst$

- We don't want all that subtraction.
  $(x + r)(x + s)(x + t) =$
  $x^3 + (r + s + t)x^2 + (rs + rt + st)x + rst$

- We don't want x, either. So make it 1.
  $(1 + r)(1 + s)(1 + t) =$
  $1 + (r + s + t) + (rs + rt + st) + rst$

Is this a good problem?

- Practices fundamental skills in arithmetic and algebra.
- Easy entry: a student who can add and multiply can make substantial progress
- Depth: patterns emerge that lead to further questions. We have some unanswered questions!
- Multiple approaches
- Surprises!

Is this a good session?

- Dan Meyer: “Be less helpful”
- Time to think
- Curiosity, perplexity
- At least one fully understood punch line
- Problem solving as a way of generating motivation
- “Think deeply of simple things”